

Tail Conditional Expectations for Exponential Dispersion Models*

Zinoviy Landsman
University of Haifa
Haifa, ISRAEL

Emiliano A. Valdez[‡]
University of New South Wales
Sydney, AUSTRALIA

February 2004

Abstract

There is a growing interest in the use of the tail conditional expectation as a measure of risk. For an institution faced with a random loss, the tail conditional expectation represents the conditional average amount of loss that can be incurred in a given period, given that the loss exceeds a specified value. This value is usually based on the quantile of the loss distribution, the so-called value-at-risk. The tail conditional expectation can therefore provide a measure of the amount of capital needed due to exposure to loss. This paper examines this risk measure for “exponential dispersion models,” a wide and popular class of distributions to actuaries which, on one hand, generalizes the Normal and shares some of its many important properties, but on the other hand, contains many distributions of non-negative random variables like the Gamma and the Inverse Gaussian.

*Keywords: tail value-at-risk, tail conditional expectations, exponential dispersion family.

[†]Acknowledgment: Dr. Valdez wishes to acknowledge financial support from the Institute of Actuaries of Australia through the UNSW Actuarial Foundation.

[‡]Presenter, XXXVth International ASTIN 2004 Colloquium, Bergen, Norway, 6-9 June 2004. [Tel. No. +61-2-9385-3391, Fax +61-2-9385-1883, Email: e.valdez@unsw.edu.au.]

1 Introduction

Insurance companies generally set aside amounts of capital from which it can draw from in the event that premium revenues become insufficient to pay out claims. Determining these amounts needed is not an obvious exercise. First, it must be able to determine with accuracy the probability distribution of the losses that it is facing. Next, it has to determine the best risk measure that can be used to determine the amount of loss to cover with a high degree of confidence. This risk measure which may be denoted for instance by ϑ , is technically defined to be a mapping from the loss random variable X to the set of real numbers \mathbf{R} . In effect, we have $\vartheta : X \rightarrow \mathbf{R}$. Artzner, Delbean, Eber, and Heath (1999) developed the axiomatic treatment for a “coherent” measure of risk and in that paper, the authors claimed that a risk measure must share four axioms: subadditivity, monotonicity, positive homogeneity, and translation invariance. See their paper for definitions of these axioms. The tail conditional expectation of a continuous loss random variable X shares these axioms.

Assume for the moment that an insurance company faces the risk of losing an amount X for some fixed period of time. This generally refers to the total claims for the insurance company. We denote its distribution function by $F_X(x) = \text{Prob}(X \leq x)$ and its tail function by $\overline{F}_X(x) = \text{Prob}(X > x)$. Note that although our setting applies to insurance companies, it is equally applicable for any institution confronted with any risky business. It may even refer to the loss faced by an investment portfolio. We define the *tail conditional expectation* of X as

$$TCE_X(x_q) = E(X | X > x_q) \quad (1)$$

and we can interpret this risk measure as the mean of worse losses. It gives an average amount of the tail of the distribution. This tail is usually based on the q -th quantile, x_q , of the loss distribution with the property

$$\overline{F}_X(x_q) = 1 - q,$$

where $0 < q < 1$. It is defined as

$$x_q = \inf(x | F_X(x) \geq q), \quad (2)$$

but for continuous random variable, it is uniquely defined as $x_q = F_X^{-1}(q)$. The formula used to evaluate this tail conditional expectation is

$$TCE_X(x_q) = \frac{1}{\overline{F}_X(x_q)} \int_{x_q}^{\infty} x dF_X(x), \quad (3)$$

provided that $\overline{F}_X(x_q) > 0$, where the integral is the Lebesgue-Stieltjes integral.

Tail conditional expectations for the univariate and multivariate Normal family have been well-developed in Panjer (2002). Landsman and Valdez (2003) extended

these results for the essentially larger class of elliptical distributions. Unfortunately, all members of the elliptical family are symmetric. In this paper, we develop formulas for tail conditional expectation for loss random variables belonging to the class of exponential dispersion models. This class of distributions has served as “error distributions” for generalized linear models in the sense developed by Nelder and Wedderburn (1972). This includes many well-known discrete distributions like Poisson and Binomial as well as continuous distributions like Normal, Gamma and Inverse Gaussian, which are except for Normal, not symmetric and many of which have non-negative support and provide excellent model fit for insurance losses. It is not surprising to find that they are becoming popular to actuaries. For example, credibility formulas for the class of exponential dispersion models preserve the property of a predictive mean; see Kaas, Dannenburg, Goovaerts (1997), Nelder and Verrall (1997), Landsman and Makov (1998) and Landsman (2002).

The rest of the paper is organized as follows. Section 2 recalls definition and main properties of the class of exponential dispersion models. We consider here both the reproductive and the additive versions of exponential dispersion models. In Section 3, we derive the main formula for computing tail conditional expectations for exponential dispersion models. In particular, we find that we can express the TCE in terms of a so-called *generalized* hazard form. Section 4 considers some familiar absolutely continuous distributions belonging to this class, such as the Normal, Gamma, and the Inverse Gaussian. In Section 5, we show that the “lack of memory” of the Exponential distribution is equivalent to a characteristic property expressed in terms of the tail conditional expectation. Section 6 provides some familiar discrete distributions belonging to the exponential dispersion family, and derive TCE formulas for these distributions. Section 7 briefly describes the computation formula for either the sum or the weighted sum of random variables within the class. We conclude this paper in Section 8.

2 Definition and Properties of Exponential Dispersion Models

The early development of exponential dispersion models is often attributed to Tweedie (1947) although a more thorough and systematic investigation of its statistical properties was done by Jorgensen (1986, 1987). This class of models together with their properties is now extensively discussed in Jorgensen (1997). For the properties discussed below and for an in-depth investigation of these properties, we ask the reader to consult this excellent source on dispersion models.

The random variable X is said to belong to the Exponential Dispersion family (EDF) of distributions if its probability measure $P_{\theta,\lambda}$ is absolutely continuous with respect to some measure Q_λ and can be represented as follows for some function $\kappa(\theta)$

called the cumulant:

$$dP_{\theta,\lambda} = e^{\lambda[\theta x - \kappa(\theta)]} dQ_\lambda(x). \quad (4)$$

The parameter θ is named the canonical parameter belonging to the set

$$\Theta = \{\theta \in \mathbf{R} \mid \kappa(\theta) < \infty\}.$$

The parameter λ is called the index parameter belonging to the set of positive real numbers $\Lambda = \{\lambda \mid \lambda > 0\} = \mathbf{R}_+$.

The representation in (4) is called the *reproductive* form of EDF and we shall denote by $X \sim ED(\theta, \lambda)$ for a random variable belonging to this family. Another form of EDF is called the *additive* form which can be obtained by the transformation $Y = \lambda X$. Its probability measure $P_{\theta,\lambda}^*$ is absolutely continuous with respect to some measure Q_λ^* which can be represented as

$$dP_{\theta,\lambda}^* = e^{[\theta y - \lambda \kappa(\theta)]} dQ_\lambda^*(y). \quad (5)$$

If the measure Q_λ in (4) is absolutely continuous with respect to a Lebesgue measure, then the density of X has the form

$$f_X(x) = e^{\lambda[\theta x - \kappa(\theta)]} q_\lambda(x). \quad (6)$$

The same can be said about additive model, $ED^*(\theta, \lambda)$, and Y has the density

$$f_Y(y) = e^{[\theta y - \lambda \kappa(\theta)]} q_\lambda^*(y). \quad (7)$$

We now briefly describe some basic and important properties of this class.

Consider the reproductive form of EDF. We note that its cumulant generating function can be derived as follows:

$$\begin{aligned} K_X(t) &= \log E(e^{Xt}) = \log \left\{ \int_{\mathbf{R}} e^{xt} e^{\lambda[\theta x - \kappa(\theta)]} dQ_\lambda(x) \right\} \\ &= \log \left\{ \int_{\mathbf{R}} e^{\{\lambda[(\theta+t/\lambda)x - \kappa(\theta)]\}} dQ_\lambda(x) \right\} \\ &= \log \left\{ e^{\{\lambda[\kappa(\theta+t/\lambda) - \kappa(\theta)]\}} \int_{\mathbf{R}} e^{\{\lambda[(\theta+t/\lambda)x - \kappa(\theta+t/\lambda)]\}} dQ_\lambda(x) \right\} \\ &= \lambda [\kappa(\theta + t/\lambda) - \kappa(\theta)]. \end{aligned}$$

It follows that its moment generating function can be written as

$$M_X(t) = \exp \{ \lambda [\kappa(\theta + t/\lambda) - \kappa(\theta)] \} \quad (8)$$

From these generating functions, it becomes straightforward to derive the mean and the variance. For example, it can readily be shown that its mean is

$$\mu = E(X) = \left. \frac{\partial K_X(t)}{\partial t} \right|_{t=0} = \kappa'(\theta) \quad (9)$$

and its variance is

$$Var(X) = \left. \frac{\partial^2 K_X(t)}{\partial t^2} \right|_{t=0} = \lambda^{-1} \kappa''(\theta). \quad (10)$$

Notice that we can view the mean (9) as a function of θ

$$\tau(\theta) = \mu = \kappa'(\theta)$$

so that

$$\theta = \tau^{-1}(\mu).$$

By defining the unit variance function

$$V(\mu) = \kappa''(\theta) = \kappa''(\tau^{-1}(\mu)), \quad (11)$$

the variance in (10) can be expressed as

$$Var(X) = \sigma^2 V(\mu)$$

where $\sigma^2 = \lambda^{-1}$ is called the dispersion parameter. This reparameterization leads us to write $X \sim ED(\mu, \sigma^2)$ in terms of the mean and dispersion parameters.

Suppose X has an additive form, i.e. $dP_{\theta, \lambda}^* = e^{[\theta x - \lambda \kappa(\theta)]} dQ_{\lambda}^*(x)$. Then

$$K_X(t) = \lambda [\kappa(\theta + t) - \kappa(\theta)]$$

and

$$M_X(t) = \exp \{ \lambda [\kappa(\theta + t) - \kappa(\theta)] \}. \quad (12)$$

It immediately follows that for the additive version of the ED family, we have the mean

$$\mu = E(X) = \lambda \kappa'(\theta) \quad (13)$$

and variance

$$Var(X) = \lambda \kappa''(\theta) = V(\mu) / \sigma^2 \quad (14)$$

where the unit variance function is

$$V(\mu) = \kappa''(\tau^{-1}(\mu))$$

with again

$$\tau(\theta) = \kappa'(\theta).$$

We will similarly write $X \sim ED^*(\mu, \sigma^2)$ whenever X is represented in the additive EDF form. Provided no confusion arises, we will sometimes write $ED(\mu, \lambda)$ or $ED(\theta, \lambda)$ or $ED(\theta, \sigma^2)$ for the family of reproductive exponential dispersion models. Similar notations can be used for the additive form, except that a superscript $*$ is used to emphasize the form.

The EDF has been established as a rich model with wide potential applications. It extends the natural exponential family (NEF) and includes many standard models such as Normal, Gamma, Inverse Gaussian, Poisson, Binomial and the Negative Binomial. We consider these members of the EDF in a later section.

3 TCE Formula for the Exponential Dispersion Family

Consider the loss random variable X belonging to the family of exponential dispersion models in reproductive or additive form. Let q be such that $0 < q < 1$ and let x_q denote the q -th quantile of the distribution of X . To keep the notation simple, we denote the tail probability function as $\bar{F}(\cdot | \theta, \lambda)$ emphasizing the parameters θ and λ .

Theorem 1 *Suppose that the NEF which generates the EDF (see Jorgensen (1997), Sect. 3.1) is regular (see Brown (1986), Ch. 3) or at least steep. Then the tail conditional expectation of X is*

- For $X \sim ED(\mu, \lambda)$, the reproductive form of EDF,

$$TCE_X(x_q) = \mu + \sigma^2 h, \quad (15)$$

where $\sigma^2 = 1/\lambda$ and

$$h = \frac{\partial}{\partial \theta} \log \bar{F}(x_q | \theta, \lambda)$$

is a generalized hazard function.

- For $X \sim ED^*(\mu, \lambda)$, the additive form of EDF,

$$TCE_X(x_q) = \mu + h. \quad (16)$$

Proof. To prove (15), first note that because EDF is regular or steep, $\kappa(\theta)$ is a differentiable function and one can differentiate in θ under the integral sign the tail function

$$\bar{F}(x_q | \theta, \lambda) = \int_{x_q}^{\infty} e^{\lambda[\theta x - \kappa(\theta)]} dQ_\lambda(x)$$

and to write

$$\begin{aligned} \frac{\partial}{\partial \theta} \log \bar{F}(x_q | \theta, \lambda) &= \frac{1}{\bar{F}(x_q | \theta, \lambda)} \int_{x_q}^{\infty} \frac{\partial}{\partial \theta} \{e^{\lambda[\theta x - \kappa(\theta)]} dQ_\lambda(x)\} \\ &= \frac{1}{\bar{F}(x_q | \theta, \lambda)} \int_{x_q}^{\infty} \lambda [x - \kappa'(\theta)] e^{\lambda[\theta x - \kappa(\theta)]} dQ_\lambda(x) \\ &= \frac{\lambda}{\bar{F}(x_q | \theta, \lambda)} \left[\int_{x_q}^{\infty} x dP_{\theta, \lambda} - \kappa'(\theta) \bar{F}(x_q | \theta, \lambda) \right] \\ &= \lambda [TCE_X(x_q) - \kappa'(\theta)] = [TCE_X(x_q) - \mu] / \sigma^2, \end{aligned}$$

after noting that the mean is $\kappa'(\theta)$. Re-arranging the terms and the result immediately follows. Suppose now $X \sim ED^*(\mu, \sigma^2)$, the additive case, it can similarly be proven that

$$\begin{aligned}
\frac{\partial}{\partial \theta} \log \bar{F}(x_q | \theta, \lambda) &= \frac{1}{\bar{F}(x_q | \theta, \lambda)} \int_{x_q}^{\infty} \frac{\partial}{\partial \theta} \{e^{[\theta x - \lambda \kappa(\theta)]} dQ_{\lambda}^*(x)\} \\
&= \frac{1}{\bar{F}(x_q | \theta, \lambda)} \int_{x_q}^{\infty} [x - \lambda \kappa'(\theta)] e^{\lambda[\theta x - \kappa(\theta)]} dQ_{\lambda}^*(x) \\
&= \frac{\lambda}{\bar{F}(x_q | \theta, \lambda)} \left[\int_{x_q}^{\infty} x dP_{\theta, \lambda}^* - \lambda \kappa'(\theta) \bar{F}(x_q | \theta, \lambda) \right] \\
&= TCE_X(x_q) - \lambda \kappa'(\theta) = TCE_X(x_q) - \mu,
\end{aligned}$$

with μ as defined in (13). A re-arrangement will lead us to the desired result. ■

The only difference with (16) and (15) is the additional factor of σ^2 or equivalently, $1/\lambda$. Let us notice that the value

$$h = \frac{\partial}{\partial \theta} \log \bar{F}(x_q | \theta, \lambda)$$

can be considered as a generalization of the hazard rate. In fact, if θ is a location parameter, i.e.,

$$\bar{F}(x_q | \theta, \lambda) = F\left(\frac{x - \theta}{\sigma}\right)$$

for some distribution function F having the density $f(x)$ (that happens for example, for normal distribution), then

$$h = \frac{\frac{1}{\sigma} f\left(\frac{x - \theta}{\sigma}\right)}{\bar{F}\left(\frac{x - \theta}{\sigma}\right)},$$

is exactly hazard rate.

4 Examples - Absolutely Continuous

We now consider some examples of distributions belonging to the class of exponential dispersion models. In particular, in this section, we give as examples the familiar Normal (an example of a symmetric distribution), Gamma and Inverse Gaussian (examples of non-symmetric and non-negative defined distributions), for the absolutely continuous case, and in Section 6, the Poisson, Binomial, and Negative Binomial for

the discrete case. For other examples, we suggest the reader to refer to Jorgensen (1997).

Example 4.1 Normal Let $X \sim N(\mu, \sigma^2)$ be normal with mean μ and variance σ^2 . Then we can express its density as

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x^2-2\mu x+\mu^2}{\sigma^2}\right)\right] \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}x^2\right) \exp\left[\frac{1}{\sigma^2}\left(\mu x - \frac{1}{2}\mu^2\right)\right]. \end{aligned} \quad (17)$$

Thus, we see that it belongs to the reproductive ED family by choosing $\theta = \mu$, $\lambda = 1/\sigma^2$, $\kappa(\theta) = \frac{1}{2}\theta^2$ and $q_\lambda(x) = (\sqrt{2\pi}\sigma)^{-1} \exp(-x^2/\sigma^2)$. Hence, the unit variance function for Normal distribution is

$$V(\mu) = \kappa''(\mu) = 1. \quad (18)$$

Now denoting by $\varphi(z) = (\sqrt{2\pi})^{-1} \exp(-\frac{1}{2}z^2)$ and $\Phi(z) = \int_{-\infty}^z \varphi(x) dx$, respectively, the density and distribution functions of a standard normal, we see that

$$\begin{aligned} \frac{\partial}{\partial \theta} \log \bar{F}(x_q | \theta, \lambda) &= \frac{\partial}{\partial \theta} \log \left[1 - \Phi\left(\sqrt{\lambda}(x_q - \theta)\right)\right] \\ &= \frac{\sqrt{\lambda}\varphi\left(\sqrt{\lambda}(x_q - \theta)\right)}{1 - \Phi\left(\sqrt{\lambda}(x_q - \theta)\right)}. \end{aligned}$$

Reparameterizing back to μ and σ^2 , we find from (15) that the TCE for the Normal distribution gives

$$\begin{aligned} TCE_X(x_q) &= \mu + \frac{1}{\lambda} \frac{\sqrt{\lambda}\varphi\left(\sqrt{\lambda}(x_q - \theta)\right)}{1 - \Phi\left(\sqrt{\lambda}(x_q - \theta)\right)} \\ &= \mu + \frac{(1/\sigma)\varphi[(x_q - \mu)/\sigma]}{1 - \Phi[(x_q - \mu)/\sigma]} \sigma^2. \end{aligned} \quad (19)$$

This gives the same formula as that derived in Panjer (2002) and Landsman and Valdez (2003).

Example 4.2 Gamma Let $X \sim Ga(\alpha, \beta)$ be Gamma distributed with parameters α and β . We express its density as follows:

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \text{ for } x > 0. \quad (20)$$

A re-arrangement of the terms we get

$$f(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta x + \alpha \log \beta).$$

Thus, we see that it belongs to the additive ED family by choosing $\theta = -\beta$, $\lambda = \alpha$, $\kappa(\theta) = -\log(-\theta)$ and $q_\lambda(x) = x^{\alpha-1} (\Gamma(\alpha))^{-1}$. Because $\tau(\theta) = \kappa'(\theta) = -1/\theta$, the unit variance function for the Gamma distribution is

$$V(\mu) = \kappa''(\tau^{-1}(\mu)) = \mu^2. \quad (21)$$

Note that both the Exponential and the Chi-Square distributions are special cases of the Gamma distribution. By choosing $\alpha = 1$ in (20), we have the Exponential and by choosing $\alpha = n/2$ and $\beta = 1/2$, we end up with the Chi-Square distribution with n degrees of freedom. From the parameterization

$$\alpha = \lambda \text{ and } \beta = -\theta,$$

the Gamma density in (20) becomes

$$f(x|\theta, \lambda) = \frac{x^{\lambda-1}}{\Gamma(\lambda)} \exp[\theta x + \lambda \log(-\theta)],$$

here emphasizing the parameters θ and λ . Therefore, we have

$$\begin{aligned} \frac{\partial}{\partial \theta} \bar{F}(x_q|\theta, \lambda) &= \int_{x_q}^{\infty} \frac{\partial}{\partial \theta} \left\{ \frac{x^{\lambda-1}}{\Gamma(\lambda)} \exp[\theta x + \lambda \log(-\theta)] \right\} dx \\ &= \int_{x_q}^{\infty} f(x|\theta, \lambda) (x + \lambda/\theta) dx \\ &= -\frac{1}{\theta} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)} \int_{x_q}^{\infty} f(x|\theta, \lambda+1) dx - (\lambda/\theta) \bar{F}(x_q|\theta, \lambda) \\ &= -\frac{\lambda}{\theta} [\bar{F}(x_q|\theta, \lambda+1) + \bar{F}(x_q|\theta, \lambda)] \end{aligned}$$

and

$$\frac{\partial}{\partial \theta} \log \bar{F}(x_q|\theta, \lambda) = -\frac{\lambda}{\theta} \left[\frac{\bar{F}(x_q|\theta, \lambda+1)}{\bar{F}(x_q|\theta, \lambda)} + 1 \right].$$

Since $\mu = -\lambda/\theta$, we have the TCE formula for a Gamma distribution:

$$TCE_X(x_q) = \frac{\alpha \bar{F}(x_q|\alpha+1, \beta)}{\beta \bar{F}(x_q|\alpha, \beta)} = \mu \frac{\bar{F}(x_q|\alpha+1, \beta)}{\bar{F}(x_q|\alpha, \beta)} \quad (22)$$

after reparameterizing back to the original parameters. In the special case where $\alpha = 1$, we have the Exponential distribution and

$$\begin{aligned} TCE_X(x_q) &= \frac{1 \bar{F}(x_q|2, \beta)}{\beta \bar{F}(x_q|1, \beta)} = \frac{1}{\beta} \frac{\beta x_q e^{-\beta x_q} + e^{-\beta x_q}}{e^{-\beta x_q}} \\ &= x_q + (1/\beta) = \mu + x_q, \end{aligned}$$

which interestingly represents the lack of memory, a property well-known about the Exponential distribution.

Example 4.3 Inverse Gaussian Let $X \sim IG(\lambda, \mu)$ be an Inverse Gaussian with parameters λ and μ . The density for an Inverse Gaussian is normally written as

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right], \text{ for } x > 0. \quad (23)$$

Clearly, we can write this density as

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[\lambda\left(-\frac{1}{2\mu^2}x - \frac{x^{-1}}{2} + \frac{1}{\mu}\right)\right].$$

See Jorgensen (1997) for this form of the density. Thus, we see that it belongs to the reproductive ED family by choosing $\theta = -1/(2\mu^2)$, $\kappa(\theta) = -1/\mu = -(-2\theta)^{1/2}$ and $q_\lambda(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp(-\frac{\lambda}{2x})$. Now observing that the μ parameter can be expressed as

$$\mu = (-2\theta)^{-1/2},$$

then

$$\frac{\partial \mu}{\partial \theta} = (-2\theta)^{-3/2} = \mu^3,$$

and so the variance function

$$V(\mu) = \mu^3. \quad (24)$$

It can be shown for the Inverse Gaussian that its distribution function can be expressed as

$$F(x|\mu, \lambda) = \Phi\left(\frac{1}{\mu}\sqrt{\lambda x} - \sqrt{\lambda/x}\right) + e^{2\lambda/\mu}\Phi\left(-\frac{1}{\mu}\sqrt{\lambda x} - \sqrt{\lambda/x}\right),$$

where $\Phi(\cdot)$ denotes the cdf of a standard Normal. See, for example, Jorgensen (1997), p. 137 and Klugman, et al. (1998). Thus, we have

$$\begin{aligned} \frac{\partial}{\partial \theta} \bar{F}(x_q|\mu(\theta), \lambda) &= -\frac{\partial}{\partial \mu} \left[\Phi\left(\frac{1}{\mu}\sqrt{\lambda x_q} - \sqrt{\lambda/x_q}\right) + e^{2\lambda/\mu}\Phi\left(-\frac{1}{\mu}\sqrt{\lambda x_q} - \sqrt{\lambda/x_q}\right) \right] \cdot \frac{\partial \mu}{\partial \theta} \\ &= \mu\sqrt{\lambda x_q}\varphi\left(\frac{1}{\mu}\sqrt{\lambda x_q} - \sqrt{\lambda/x_q}\right) + \mu e^{2\lambda/\mu} \cdot 2\lambda\Phi\left(-\frac{1}{\mu}\sqrt{\lambda x_q} - \sqrt{\lambda/x_q}\right) \\ &\quad - \mu e^{2\lambda/\mu} \cdot \sqrt{\lambda x_q}\varphi\left(-\frac{1}{\mu}\sqrt{\lambda x_q} - \sqrt{\lambda/x_q}\right). \end{aligned}$$

Denoting by $z_q^* = \frac{1}{\mu}\sqrt{\lambda x_q} - \sqrt{\lambda/x_q}$ so that $-z_q^* - 2\sqrt{\lambda/x_q} = -\frac{1}{\mu}\sqrt{\lambda x_q} - \sqrt{\lambda/x_q}$, we have

$$\frac{\partial}{\partial \theta} \log \bar{F}(x_q|\mu, \lambda) = \frac{\mu}{\bar{F}(x_q|\mu, \lambda)} \left\{ \sqrt{\lambda x_q}\varphi(z_q^*) + e^{2\lambda/\mu} \left[\begin{array}{c} 2\lambda\Phi(-z_q^* - 2\sqrt{\lambda/x_q}) \\ -\sqrt{\lambda x_q}\varphi(-z_q^* - 2\sqrt{\lambda/x_q}) \end{array} \right] \right\}$$

Using the reproductive version of Theorem 1, we have

$$TCE_X(x_q) = \mu + \frac{\mu/\lambda}{\bar{F}(x_q|\mu, \lambda)} \left\{ \sqrt{\lambda x_q} \varphi(z_q^*) + e^{2\lambda/\mu} \begin{bmatrix} 2\lambda\Phi(-z_q^* - 2\sqrt{\lambda/x_q}) \\ -\sqrt{\lambda x_q} \varphi(-z_q^* - 2\sqrt{\lambda/x_q}) \end{bmatrix} \right\}.$$

Let us notice that from (18), (21) and (24), the Normal, Gamma, and the Inverse Gaussian distributions have unit variance function of the form

$$V(\mu) = \mu^p, \quad (25)$$

for $p = 0, 2, 3$, respectively. Members of the EDF with variance function of the form in (25), where $p \in \mathbf{R}$, are sometimes called *Tweedie* models. In Figure 1, we compare the resulting tail conditional expectations of these three members of the Tweedie family. The parameters in each distribution have been selected so that they all have mean $E(X) = 10$ and variance $Var(X) = 100$. We can see from the graph that starting from some level q , larger p in the Tweedie model leads to larger TCE.

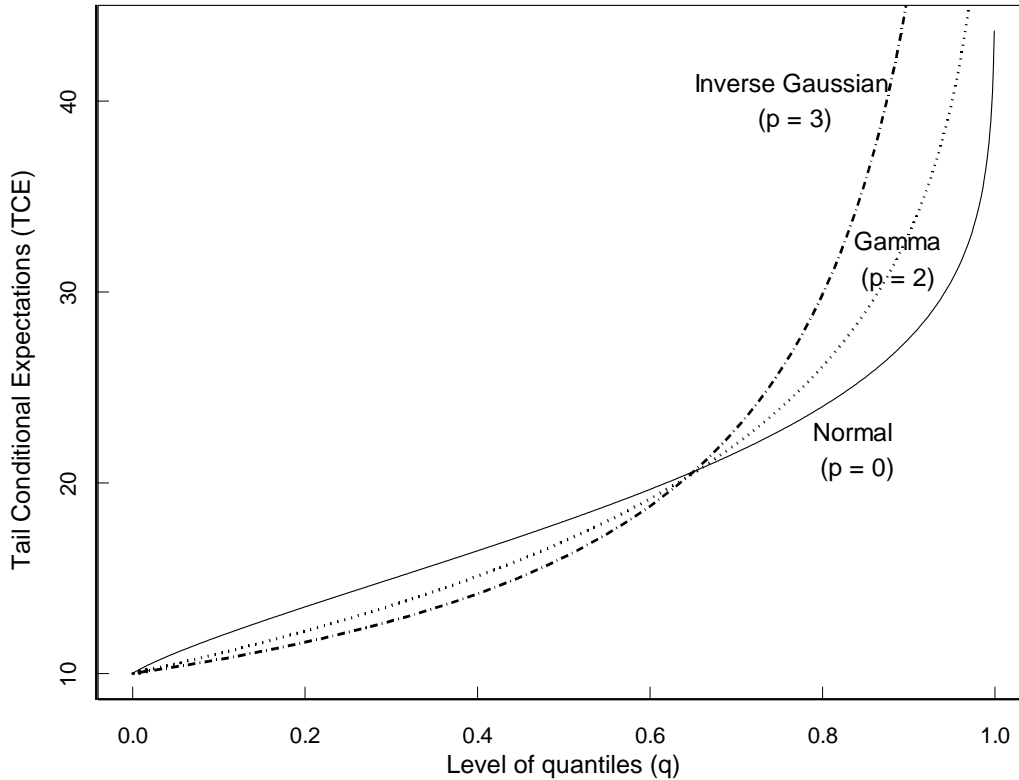


Figure 1: Tail Conditional Expectations of Normal, Gamma, and Inverse Gaussian.

5 A Characteristic Property of the Exponential Distribution

In the previous section, we have shown that for the exponential distribution, the tail conditional expectation can be expressed as

$$TCE_X(x_q) = x_q + \mu \quad (26)$$

and we associated this property as a lack of memory. In this section, we show that (26) is indeed equivalent to the “lack of memory” property because (26) holds only for exponential distribution. As a matter of fact, the existence of the expectation guarantees the following representation for the TCE:

$$\begin{aligned} TCE_X(x_q) &= \frac{\int_{x_q}^{\infty} x dF_X(x)}{\overline{F}_X(x_q)} \\ &= \frac{-x\overline{F}_X(x_q)|_{x_q}^{\infty} + \int_{x_q}^{\infty} \overline{F}_X(x) dx}{\overline{F}_X(x_q)} \\ &= x_q + \frac{\int_{x_q}^{\infty} \overline{F}_X(x) dx}{\overline{F}_X(x_q)}, \end{aligned} \quad (27)$$

provided that $\overline{F}_X(x_q) > 0$.

Theorem 2 *Suppose $F_X(x)$ is a continuous distribution function in the internal points of its support, which is finite or infinite open interval (a, b) . Then the representation*

$$TCE_X(x_q) = x_q + \alpha, \quad \text{for any } q \in (0, 1) \quad (28)$$

where $\alpha \neq 0$ is some constant not depending on x_q , holds if and only if X has a shifted exponential distribution and then

$$\alpha = E(X) = \mu. \quad (29)$$

Proof. First we notice that condition (28) automatically requires that b , the right end of support of distribution $F_X(x)$, should equal $b = \infty$. In fact, if $b < \infty$, by the definition of quantile (2) there exists $0 < q_0 < 1$ that

$$x_q = b, \quad \text{for } q \geq q_0.$$

Then, strongly speaking, $TCE_X(x_q)$ is not defined for such q . Of course, one can naturally extend the definition for $q \geq q_0$ as

$$\lim_{p \rightarrow q_0-0} \frac{\int_{x_p}^{\infty} x dF_X(x)}{\overline{F}_X(x_p)} = \lim_{p \rightarrow q_0-0} \frac{\int_{x_p}^b x dF_X(x-0) + x_{q_0}(F_X(b) - F_X(b-0))}{\overline{F}_X(x_p)} = x_{q_0} = b,$$

but then we have $\alpha = 0$. So, $b = \infty$ and define $U(z) = \int_z^\infty \bar{F}_X(x) dx$. As $\bar{F}_X(x)$ is continuous, $U(z)$ is differentiable. We can then write that for $z \in (a, \infty)$

$$TCE_X(z) = z - \frac{U(z)}{U'(z)}.$$

Condition (28) means that

$$\frac{U(z)}{U'(z)} = -\alpha \quad \text{for any } z \in (a, \infty),$$

that is,

$$\frac{U'(z)}{U(z)} = -\frac{1}{\alpha} = -\beta, \quad z \in (a, \infty).$$

This simple differential equation leads to the only solution of the form

$$U(z) = ke^{-\beta z}, \quad \text{for some constant } k, \quad z \in (a, \infty).$$

As

$$-U'(z) = k\beta e^{-\beta z} = \bar{F}_X(z), \quad z \in (a, \infty), \quad (30)$$

it immediately follows that $\beta > 0$. From the restriction on the left end of support, $F_X(a-0) = 0$, and (30) follows that $a > -\infty$, i.e. a is finite, and then

$$k\beta = e^{\beta a}.$$

This gives

$$\bar{F}_X(x) = e^{-\beta(x-a)}, \quad \text{for all } x \geq a,$$

which implies X has a shifted exponential distribution. Equation (29) automatically follows. ■

Of course, among nonnegative random variables, the only distribution satisfying condition (28) is the traditional exponential

$$\bar{F}_X(x) = e^{-\beta x}, \quad x \geq 0.$$

6 Examples - Discrete

In this section, we consider some examples of discrete distributions belonging to the class of exponential dispersion models.

Example 6.1 Poisson Let $X \sim \text{Poisson}(\mu)$ be Poisson distributed with mean parameter μ . We express its probability function as follows:

$$p(x) = \frac{e^{-\mu} \mu^x}{x!}, \quad \text{for } x = 0, 1, \dots$$

A re-arrangement of the terms gives us

$$p(x) = \frac{1}{x!} \exp(x \log \mu - \mu).$$

Thus, we see that it belongs to the additive ED family by choosing $\theta = \log \mu$, $\lambda = 1$, $\kappa(\theta) = e^\theta$ and $q_\lambda(x) = 1/x!$. Therefore, we have

$$\begin{aligned} \frac{\partial}{\partial \theta} \bar{F}(x_q | \theta, 1) &= \frac{\partial}{\partial \theta} \left[\sum_{x=x_q+1}^{\infty} \frac{1}{x!} \exp(\theta x - e^\theta) \right] \\ &= \sum_{x=x_q+1}^{\infty} \frac{\partial}{\partial \theta} \left[\frac{1}{x!} \exp(\theta x - e^\theta) \right] \\ &= \sum_{x=x_q+1}^{\infty} \left[\frac{1}{x!} \exp(\theta x - e^\theta) \cdot (x - e^\theta) \right] \\ &= e^\theta \left[\sum_{y=x_q}^{\infty} \frac{1}{y!} \exp(\theta y - e^\theta) - \bar{F}_X(x_q | \theta, 1) \right] \\ &= e^\theta [\bar{F}(x_q - 1 | \theta, 1) - \bar{F}(x_q | \theta, 1)] \\ &= e^\theta p(x_q | \theta, 1) \end{aligned}$$

and

$$\frac{\partial}{\partial \theta} \log \bar{F}(x_q | \theta, 1) = \frac{1}{\bar{F}(x_q | \theta, 1)} e^\theta p(x_q | \theta, 1).$$

Since $\mu = e^\theta$, we have the TCE formula for a Poisson distribution:

$$TCE_X(x_q) = \mu \left[1 + \frac{p(x_q)}{\bar{F}(x_q)} \right]. \quad (31)$$

where $\bar{F}(x_q) = \bar{F}(x_q | \theta, 1)$. Notice also that for the Poisson distribution, $\mu = \text{Var}(X)$, so we can also write

$$TCE_X(x_q) = \mu + \frac{p(x_q)}{\bar{F}(x_q)} \times \text{Var}(X).$$

In that sense, the Poisson distribution is analogue of Normal for a discrete case.

Example 6.2 Binomial Let $X \sim \text{binomial}(p, n)$ be binomial with parameters p and n . Then we can express its probability function as

$$p(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} \left(\frac{p}{1-p} \right)^x (1-p)^n.$$

By defining $\theta = \log [p/1 - p]$, we can write this as

$$\begin{aligned} p(x) &= \binom{n}{x} e^{\theta x} (1 + e^\theta)^{-n} \\ &= \binom{n}{x} \exp [\theta x - n \log (1 + e^\theta)], \end{aligned} \quad (32)$$

for $x = 0, 1, \dots, n$. Thus, we see that it belongs to the additive ED family with $\lambda = n$, $\kappa(\theta) = \log (1 + e^\theta)$ and $q_\lambda(x) = \binom{n}{x}$. Notice that we can actually write $q_\lambda(x) = \binom{n}{x} = \frac{\Gamma(n+1)}{\Gamma(n+1-x)x!}$. We therefore see that for $x_q < n$,

$$\begin{aligned} \frac{\partial}{\partial \theta} \bar{F}(x_q | p(\theta), n) &= \sum_{x=x_q+1}^n \frac{\Gamma(n+1)}{\Gamma(n+1-x)x!} \frac{\partial}{\partial \theta} \{ \exp [\theta x - n \log (1 + e^\theta)] \} \\ &= \sum_{x=x_q+1}^n \frac{\Gamma(n+1)}{\Gamma(n+1-x)x!} \left(x - n \frac{e^\theta}{1 + e^\theta} \right) \exp [\theta x - n \log (1 + e^\theta)] \\ &= \sum_{x=x_q+1}^n \frac{\Gamma(n+1)}{\Gamma(n+1-x)x!} (x - np) p^x (1 - p)^{n-x} \\ &= \sum_{x=x_q+1}^n \frac{\Gamma(n+1)}{\Gamma(n+1-x)(x-1)!} p^x (1 - p)^{n-x} - np \bar{F}(x_q | p, n) \\ &= np \sum_{y=x_q}^{n-1} \frac{\Gamma(n)}{\Gamma(n-y)y!} p^y (1 - p)^{n-1-y} - np \bar{F}(x_q | p, n) \\ &= np \bar{F}(x_q - 1 | p, n - 1) - np \bar{F}(x_q | p, n). \end{aligned}$$

Noting that $\mu = np$ for the binomial and reparameterizing back to μ and σ^2 , we find from (16) that the TCE for the binomial distribution gives

$$\begin{aligned} TCE_X(x_q) &= \mu + \frac{1}{\bar{F}(x_q | p, n)} \frac{\partial}{\partial \theta} \bar{F}(x_q | p, n) \\ &= \mu \frac{\bar{F}(x_q - 1 | p, n - 1)}{\bar{F}(x_q | p, n)}. \end{aligned} \quad (33)$$

Example 6.3 Negative Binomial Let $X \sim NB(p, \alpha)$ belong to the Negative Binomial family with parameters λ and p . Its probability function has the form

$$p(x) = \frac{\Gamma(\alpha + x)}{\Gamma(\alpha)x!} p^\alpha (1 - p)^x, \text{ for } x = 0, 1, \dots$$

A re-arrangement of the terms leads us to

$$p(x) = \frac{\Gamma(\alpha + x)}{\Gamma(\alpha) x!} \exp [x \log (1 - p) + \alpha \log p].$$

By choosing $\theta = \log (1 - p)$, $\lambda = \alpha$, $\kappa(\theta) = -\log (1 - e^\theta)$ and $q_\lambda(x) = \frac{\Gamma(\alpha + x)}{\Gamma(\alpha) x!}$, we see that it belongs to the additive ED family. We therefore see that

$$\begin{aligned} \frac{\partial}{\partial \theta} \bar{F}(x_q | p(\theta), \alpha) &= \sum_{x=x_q+1}^{\infty} \frac{\Gamma(\alpha + x)}{\Gamma(\alpha) x!} \frac{\partial}{\partial \theta} \left\{ \exp [\theta x + \lambda \log (1 - e^\theta)] \right\} \\ &= \sum_{x=x_q+1}^{\infty} \frac{\Gamma(\alpha + x)}{\Gamma(\alpha) x!} \left(x - \lambda \frac{e^\theta}{1 - e^\theta} \right) \exp [\theta x + \lambda \log (1 - e^\theta)] \\ &= \sum_{x=x_q+1}^{\infty} \frac{\Gamma(\alpha + x)}{\Gamma(\alpha) x!} \left(x - \alpha \frac{1 - p}{p} \right) p^\alpha (1 - p)^x \\ &= \sum_{x=x_q+1}^{\infty} \frac{\Gamma(\alpha + x)}{\Gamma(\alpha) (x - 1)!} x p^\alpha (1 - p)^x - \alpha \frac{1 - p}{p} \bar{F}(x_q | p, \alpha) \\ &= \alpha \frac{1 - p}{p} \sum_{y=x_q}^{\infty} \frac{\Gamma(\alpha + y + 1)}{\Gamma(\alpha + 1) y!} p^{\alpha+1} (1 - p)^y - \alpha \frac{1 - p}{p} \bar{F}(x_q | p, \alpha) \\ &= \alpha \frac{1 - p}{p} \left[\bar{F}(x_q - 1 | p, \alpha + 1) - \bar{F}(x_q | p, \alpha) \right]. \end{aligned}$$

Note that for a $NB(p, \alpha)$ random variable, its mean is $\mu = \alpha(1 - p)/p$ so that reparameterizing back to μ and σ^2 , we find from (16)

$$TCE_X(x_q) = \mu \frac{\bar{F}(x_q - 1 | p, \alpha + 1)}{\bar{F}(x_q | p, \alpha)}. \quad (34)$$

Comparing the $TCE_X(x_q)$ for Negative Binomial with Gamma as shown in (22), we may conclude that the Negative Binomial is the discrete analogue of Gamma.

7 Tail Conditional Expectation for Sums

Consider the case where we have n independent random variables X_1, X_2, \dots, X_n coming from the same EDF family having a common parameter θ but different λ 's. Consider first the additive case, that is, $X_k \sim \text{ED}^*(\mu_k, \lambda_k)$ for $k = 1, 2, \dots, n$. The density is thus

$$p_k(x | \theta, \lambda_k) = \exp [\theta x - \lambda_k \kappa(\theta)] q_{\lambda_k}(x), \text{ for } k = 1, 2, \dots, n. \quad (35)$$

Denote the sum by

$$S = X_1 + X_2 + \dots + X_n$$

and denote by $TCE_X(x_q | \mu, \lambda)$ the tail conditional expectation of X belonging to the EDF family either in the reproductive or additive form with mean parameter μ and index parameter λ .

Theorem 3 *Suppose X_1, X_2, \dots, X_n are n independent random variables from the additive family (35). Then the tail conditional expectation of the sum is*

$$TCE_S(s_q) = TCE_{X_1}(s_q | \mu_S, \lambda_S)$$

where $\mu_S = \sum_{k=1}^n \mu_k$, $\lambda_S = \sum_{k=1}^n \lambda_k$, and s_q is the q -th quantile of the distribution of S .

Proof. By independence and using (12), the moment generating function of the sum can be expressed as

$$M_S(t) = \exp \{ \lambda_S [\kappa(\theta + t) - \kappa(\theta)] \}$$

where we have expressed $\lambda_S = \sum_{k=1}^n \lambda_k$. Thus, we see that the sum also belongs to the additive Exponential Dispersion family with

$$S \sim \text{ED}^*(\mu_S, \lambda_S).$$

It becomes straightforward from Theorem 1 to prove that the tail conditional expectation for the sum can be expressed as

$$TCE_S(s_q) = \mu_S + h \tag{36}$$

where the generalized hazard function $h = \frac{\partial}{\partial \theta} \log \bar{F}(s_q | \theta, \lambda_S)$ and s_q is the q -th quantile of the distribution of S . ■

To illustrate, consider the Gamma distribution case in Example (4.2). Let X_1, \dots, X_n be n independent random variables such that

$$X_k \sim \text{Ga}(\alpha_k, \beta) \text{ for } k = 1, 2, \dots, n.$$

From (22) and Theorem 3, taking into account $\lambda_k = \alpha_k$, we see that the TCE for the sum is of the form

$$TCE_S(s_q) = \mu_S \frac{\bar{F}(s_q | \alpha_S + 1, \beta)}{\bar{F}(s_q | \alpha_S, \beta)},$$

where $\alpha_S = \sum_{k=1}^n \alpha_k$.

Next, we consider the reproductive form of the EDF family, that is, $X_k \sim \text{ED}(\mu, \lambda_k)$ for $k = 1, 2, \dots, n$. The density is thus

$$p_k(x | \theta, \lambda_k) = \exp[\lambda_k(\theta x - \kappa(\theta))] q_{\lambda_k}(x), \text{ for } k = 1, 2, \dots, n.$$

Unfortunately, we can derive explicit form only for a weighted sum of X_i 's. As a matter of fact, we have the following result.

Theorem 4 Suppose $\lambda_S = \sum_{k=1}^n \lambda_k$ and $Y_k = w_k X_k$ where $X_k \sim ED(\mu, \lambda_k)$ and $w_k = \lambda_k / \lambda_S$ for $k = 1, 2, \dots, n$. Define the weighted sum

$$\tilde{S} = \sum_{k=1}^n Y_k = \sum_{k=1}^n w_k X_k.$$

Then its tail conditional expectation is given by

$$TCE_{\tilde{S}}(\tilde{s}_q) = TCE_{X_1}(\tilde{s}_q | \mu, \lambda_S)$$

where \tilde{s}_q is the q -th quantile of the distribution of \tilde{S} .

Proof. From (8), we have the moment generating function of Y_k

$$M_{Y_k}(t) = M_{X_k}(w_k t) = \exp\{\lambda_k [\kappa(\theta + t/\lambda_S) - \kappa(\theta)]\}$$

so that the moment generating function of \tilde{S} is given by

$$M_S(t) = \exp\{\lambda_S [\kappa(\theta + t/\lambda_S) - \kappa(\theta)]\}.$$

Thus by taking into account (9), we see $\tilde{S} \sim ED(\mu, \lambda_S)$ and its tail conditional expectation immediately follows from Theorem 1. ■

To illustrate, consider the Normal distribution case in Example (4.1). Let X_1, \dots, X_n be n independent random variables such that

$$X_k \sim N(\mu, \sigma_k^2) \text{ for } k = 1, 2, \dots, n.$$

From Theorem 4 and the first item of Theorem 1, we see that the TCE for the weighted sum \tilde{S} has the form

$$TCE_{\tilde{S}}(\tilde{s}_q) = \mu + \frac{(1/\sigma_{\tilde{S}}) \varphi[(\tilde{s}_q - \mu)/\sigma_{\tilde{S}}]}{1 - \Phi[(\tilde{s}_q - \mu)/\sigma_{\tilde{S}}]} \sigma_{\tilde{S}}^2,$$

where $\sigma_{\tilde{S}}^2 = 1/\lambda_S = 1/\sum_{k=1}^n (1/\sigma_k^2)$, the harmonic mean of the individual variances.

Note that for Normal distribution, we can also derive result for sum S since the Normal distribution $N(\mu_k, \sigma_k^2)$ can also be considered as a member of $ED^*(\mu_k, \lambda_k)$ where $\lambda_k = \sigma_k^2$. The well-known TCE formula for sum S (see for example, Panjer, 2002) given by

$$TCE_S(s_q) = \mu + \frac{(1/\sigma_S) \varphi[(s_q - \mu)/\sigma_S]}{1 - \Phi[(s_q - \mu)/\sigma_S]} \sigma_S^2$$

immediately follows from Theorem 3.

8 Concluding Remarks

This paper examines tail conditional expectations for loss random variables that belong to the class of exponential dispersion models. This class of distributions has served as “error distributions” for generalized linear models in the sense developed by Nelder and Wedderburn (1972). This class extends many of the properties and ideas developed for natural exponential families. It also includes several standard and well-known discrete distributions like Poisson and Binomial as well as continuous distributions like Normal and Gamma, and this paper develops tail conditional expectations for these members. We find an appealing way to express the tail conditional expectation for the class of Exponential Dispersion models; this TCE is equal to the expectation plus an additional term which is the partial derivative of the logarithm of the tail of the distribution with respect to the canonical parameter θ . We observe that this partial derivative is a generalization of the hazard rate function.

References

- [1] Artzner, P., Delbaen, F., Eber, J.M., and Heath, D. (1999) Coherent Measures of Risk. *Mathematical Finance* **9**, 203-228.
- [2] Barndorff-Nielsen, O.E. (1978) Hyperbolic Distributions and Distributions on Hyperbolae. *Scandinavian Journal of Statistics* **5**, 151-157.
- [3] Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A., and Nesbitt, C.J. (1997) *Actuarial Mathematics* 2nd edition. Society of Actuaries, Schaumburg, Illinois.
- [4] Brown, L.D. (1987) *Fundamentals of Statistical Exponential Families*. IMS Lecture Notes - Monograph Series: Vol. 9, Hayward, California.
- [5] Feller, W. (1971) *An Introduction to Probability Theory and its Applications* Vol. 2. John Wiley, New York.
- [6] Joe, H. (1997) *Multivariate Models and Dependence Concepts*. Chapman & Hall, London.
- [7] Jorgensen, B. (1986) Some Properties of Exponential Dispersion Models. *Scandinavian Journal of Statistics* **13**, 187-198.
- [8] Jorgensen, B. (1987) Exponential Dispersion Models (with discussion). *Journal of the Royal Statistical Society Series B* **49**, 127-162.
- [9] Jorgensen, B. (1997) *The Theory of Dispersion Models*. Chapman & Hall, London.

- [10] Kaas, R., Dannenburg, D., Goovaerts, M. (1997). Exact Credibility for weighted observations. *Astin Bulletin*, **27**, 2, 287-295.
- [11] Klugman, S., Panjer, H. and Willmot, G. (1998) *Loss Models: From Data to Decisions*. John Wiley, New York.
- [12] Landsman, Z. (2002) Credibility Theory: A New View from the Theory of Second Order Optimal Statistics. *Insurance: Mathematics & Economics* **30**, 351-362.
- [13] Landsman, Z. and Makov, U. (1998). Exponential dispersion models and Credibility. *Scandinavian Actuarial Journal*, 1, 89-96
- [14] Landsman, Z. and Valdez, E.A. (2003) Tail Conditional Expectations for Elliptical Distributions. *North American Actuarial Journal*, **7**, 55-71.
- [15] Nelder, J.A. and Verrall, R.J. (1997) Credibility Theory and Generalized Linear Models. *Astin Bulletin* **27**, 71-82.
- [16] Nelder, J.A. and Wedderburn, R.W.M. (1972) Generalized Linear Models. *Journal of the Royal Statistical Society Series A* **135**, 370-384.
- [17] Panjer, H.H. (2002) Measurement of Risk, Solvency Requirements, and Allocation of Capital within Financial Conglomerates. Institute of Insurance and Pension Research, University of Waterloo Research Report 01-15.
- [18] Tweedie, M.C.K. (1947) Functions of a Statistical Variate with Given Means, with Special Reference to Laplacian Distributions. *Proceedings of the Cambridge Philosophical Society* **49**, 41-49.

Zinoviy Landsman, PhD
Actuarial Research Center
Department of Statistics
University of Haifa
Mount Carmel, Haifa 31905, Israel.
E-mail: landsman@stat.haifa.ac.il

Emiliano A. Valdez, PhD, FSA, FIAA
School of Actuarial Studies
Faculty of Commerce & Economics
University of New South Wales
Sydney, Australia 2052
E-mail: e.valdez@unsw.edu.au