

Explicit solutions for survival probabilities in the classical risk model

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Abstract

The purpose of this paper is to show that, for the classical risk model, explicit expressions for survival and ruin probabilities in a finite time horizon can be obtained through the inversion of the double Laplace transform of the distribution of time to ruin. To do this, we follow Gerber and Shiu (1998). Although other methods for such question do exist, we found this approach more direct and simple. The method for determination of the transform follows Dickson (2001). For the analitic inversion, we have applied twice, after some algebra, the Laplace complex inversion formula.

Keywords: Classical risk model; double Laplace transform; Time to ruin; Survival probability

1 Introduction

In this paper we assume that the counting claims process is an homogeneous Poisson process and that the Laplace transform of the density function of a single claim is known. In the following, we shall use the model and notation of Bowers *et al.* (1987, Chapter 12) with slight modifications. Thus, we consider that the company has an initial reserve of $U(0) = u$, whose surplus at time t is

$$U(t) = u + ct - Y(t), \quad t > 0, \quad (1)$$

where c is the constant premium income per unit of time and $\{Y(t)\}_{t>0}$ is a compound Poisson process with parameter λ , where $Y(t)$ is the aggregate claim amount up to time t .

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The density function of the time between two consecutive claims will be denoted by $g(t) = \lambda \exp(-\lambda t)$.

A single claim amount is denoted by the random variable X with distribution $F(x)$ and density $f(x) = F'(x)$.

The distribution and density functions of the aggregate claim amount up to time t are denoted by $F_t(x)$ and $f_t(x)$, respectively.

The Laplace transform of a non-negative function $h(y)$ will be denoted by $\widehat{h}(s)$, defined in the complex plane by

$$\widehat{h}(s) = \int_0^{\infty} \exp(-sy)h(y)dy. \quad (2)$$

Otherwise specified, the abscissa of convergence will be denoted by σ .

If the function has two independent variables (x, y) , we will maintain the previous definition with respect to each of them, that is,

$$\widehat{h}(x, s) = \int_0^{\infty} \exp(-sy)h(x, y)dy,$$

and

$$\widehat{h}(\delta, y) = \int_0^{\infty} \exp(-\delta x)h(x, y)dx.$$

The double Laplace transform will be defined by

$$\widehat{\widehat{h}}(\delta, s) = \int_0^{\infty} \int_0^{\infty} \exp(-\delta x - sy)h(x, y)dx dy.$$

The finite time ruin of the company, up to time t , considering an initial reserve u , will be denoted by $\psi(u, t)$ and the survival probability by $\sigma(u, t) = 1 - \psi(u, t)$. The ultimate ruin probability will then be $\psi(u) = \psi(u, \infty)$ and the non-ruin probability $\sigma(u) = 1 - \psi(u)$.

Following Gerber and Shiu (1998) and Dickson (2001), we denote by T the time to ruin and we define a function ϕ by

$$\phi(u) = E [e^{-\delta T} I(T < \infty) | U(0) = u], \quad (3)$$

where I denotes the usual indicator function and δ is a parameter with a non-negative real part. $\phi(u)$ can then be considered the Laplace transform corresponding to the random variable T . For $\delta = 0$, we get $\phi(u) = \psi(u)$.

We notice that ϕ can be written as

$$\phi(u) = \int_0^{\infty} e^{-\delta t} \frac{\partial}{\partial t} \psi(u, t) dt, \quad (4)$$

and, integrating by parts we get

$$\phi(u) = \psi(u, 0) + \delta \widehat{\psi}(u, \delta).$$

In Section 2 of this paper we derive the Laplace transform of $\phi(u)$. In Section 3 we get the double Laplace transform $\widehat{\widehat{\sigma}}(\delta, s)$ of the survival probability $\sigma(u, t)$.

In Section 4 we proceed directly to the complex inversion of the transform $\widehat{\sigma}(\delta, s)$ and explicit results are achieved for exponential, mixed exponential and Erlang distributed claims.

For the complex inversion, we suppose that the reader is familiarized with the *residue* theorem and with the Laurent series expansion for complex functions.

At the end of that section some numerical results are presented and, for comparison purposes only, we have computed the exact formulas obtained and the known Seal's formula,

$$\sigma(u, t) = F_t(u + ct) - c \int_0^t f_\tau(u + c\tau) \sigma(0, t - \tau) d\tau,$$

where

$$\sigma(0, y) = \frac{1}{cy} \int_0^{cy} F_y(x) dx.$$

The numerical evaluation of the integrals has been approximated using a *dichotomic* algorithm (see Lima *et al.* 2002).

2 The Laplace transform of ϕ

Considering the instant and the amount of the first claim we may write

$$\begin{aligned} \phi(u) &= \int_0^\infty g(t) [1 - F(u + ct)] e^{-\delta t} dt \\ &\quad + \int_0^\infty g(t) e^{-\delta t} \int_0^{u+ct} f(x) \phi(u + ct - x) dx dt. \end{aligned} \quad (5)$$

Changing the variable of integration t considering $s = u + ct$, we get

$$\begin{aligned} \phi(u) &= c^{-1} \int_u^\infty g\left(\frac{s-u}{c}\right) [1 - F(s)] e^{-\delta \frac{s-u}{c}} ds \\ &\quad + c^{-1} \int_u^\infty g\left(\frac{s-u}{c}\right) e^{-\delta \frac{s-u}{c}} \int_0^s f(x) \phi(s-x) dx ds \end{aligned} \quad (6)$$

Differentiation with respect to u gives

$$\begin{aligned} c\phi'(u) &= -g(0) [1 - F(u)] - g(0) \int_0^u f(x) \phi(u-x) dx \\ &\quad - c^{-1} \int_u^\infty g'\left(\frac{s-u}{c}\right) [1 - F(s)] e^{-\delta \frac{s-u}{c}} ds \\ &\quad - c^{-1} \int_u^\infty g'\left(\frac{s-u}{c}\right) e^{-\delta \frac{s-u}{c}} \int_0^s f(x) \phi(s-x) dx ds + \delta \phi(u). \end{aligned} \quad (7)$$

Since that $g'(t) = -\lambda g(t)$, the above expression can be simplified to

$$c\phi'(u) = (\lambda + \delta) \phi(u) - \lambda [1 - F(u)] - \lambda \int_0^u f(x) \phi(u-x) dx. \quad (8)$$

For $\delta = 0$ we obtain the classical result

$$\psi'(u) = \frac{\lambda}{c}\psi(u) - \frac{\lambda}{c} \int_0^u f(x)\psi(u-x) dx - \frac{\lambda}{c} [1 - F(u)]. \quad (9)$$

Taking the Laplace transform of both sides of (8) and considering that

$$\widehat{f'(\cdot)} = s\widehat{f}(s) - f(0) \quad (10)$$

and that

$$[1 - \widehat{F}(x)] = [1 - \widehat{f}(s)] / s \quad (11)$$

we get

$$c [s\widehat{\phi}(s) - \phi(0)] = (\lambda + \delta)\widehat{\phi}(s) + \frac{\lambda}{s} [\widehat{f}(s) - 1] - \lambda\widehat{f}(s)\widehat{\phi}(s), \quad (12)$$

or

$$\widehat{\phi}(s) = \frac{c\phi(0) + \frac{\lambda}{s} [\widehat{f}(s) - 1]}{cs - \lambda - \delta + \lambda\widehat{f}(s)}. \quad (13)$$

Note that the denominator of (13) is the difference between both sides of Lundberg's fundamental equation written in the form

$$\lambda + \delta - cs = \lambda\widehat{f}(s). \quad (14)$$

Following Gerber and Shiu (1998), we verify that (14) has a non-negative root ρ , which is zero for $\delta = 0$ and, for a large class of claim amount distributions, also has a negative root denoted by R . Considering in (13) $s = \rho$ the denominator vanishes, so that ρ must also be a zero of the numerator, which results in

$$\phi(0) = \frac{\lambda [1 - \widehat{f}(\rho)]}{c\rho} \quad (15)$$

and

$$\widehat{\phi}(s) = \frac{\frac{\lambda}{s} [\widehat{f}(s) - 1] - \frac{\lambda}{\rho} [\widehat{f}(\rho) - 1]}{cs - \lambda - \delta + \lambda\widehat{f}(s)}. \quad (16)$$

3 The Laplace transform of $\sigma(u, t)$

If we consider the implicit and defective density under expression (4) we can write

$$\sigma(u, t) = 1 - \psi(u, t) = 1 - \int_0^t \frac{\partial}{\partial v} \psi(u, v) dv, \quad (17)$$

so that, considering relation (10), the double Laplace transform of $\sigma(u, t)$ can be written as

$$\widehat{\widehat{\sigma}}(s, \delta) = \frac{1}{\delta s} - \frac{\widehat{\phi}(s)}{\delta}. \quad (18)$$

Substituting (13) in (18) we get after some simplification

$$\widehat{\widehat{\sigma}}(s, \delta) = -\frac{-cs + \delta + cs\phi(0)}{s\delta (cs - \lambda - \delta + \lambda\widehat{f}(s))}. \quad (19)$$

Taking the same arguments used for expression (15) of last section, we get

$$\phi(0) = 1 - \frac{\delta}{c\rho}. \quad (20)$$

The substitution of (20) into (19) gives

$$\widehat{\widehat{\sigma}}(s, \delta) = \frac{s - \rho}{\rho s (cs - \lambda - \delta + \lambda\widehat{f}(s))}, \quad (21)$$

expression already obtained by Gerber and Shiu, starting from a different approach (1998, page 62).

4 The complex inversion of $\widehat{\widehat{\sigma}}$

4.1 Exponential distributed claims

If the single claim amount distribution is exponential with mean $1/\alpha$, we have

$$\widehat{f}(s) = \frac{\alpha}{\alpha + s}, \quad \text{Re}(s) > -\alpha,$$

the Lundberg equation becomes

$$\lambda + \delta - sc = \alpha \frac{\lambda}{s + \alpha}, \quad (22)$$

and

$$\widehat{\widehat{\sigma}}(s, \delta) = \frac{(-\rho + s)(\alpha + s)}{\rho s (-\delta\alpha - \delta s - \lambda s + cs\alpha + cs^2)}. \quad (23)$$

The first inversion with respect to s (the counterpart of u) is simple and the Laplace transform $\widehat{\sigma}(u, \delta)$ of $\sigma(u, t)$ with respect to t , can be determined by the complex inversion formula for Laplace transforms, so that,

$$\widehat{\sigma}(u, \delta) = \sum \left(\text{residues of } e^{us} \widehat{\widehat{\sigma}}(s, \delta) \text{ at each of its singularities in } C \right), \quad (24)$$

(Marsden 1999, page 471). We can see that the implicit integrand function in (24) is analytic, with respect to s , except for the poles $s = 0$ and $s = R$. Note

that $s = \rho$ is a removable singularity. The residues $r(s)$ will be then evaluated by the relation

$$r(s) = e^{us} \frac{(-\rho + s)(\alpha + s)}{-\rho\delta\alpha - 2\rho\delta s - 2\rho\lambda s + 2\rho c s\alpha + 3\rho c s^2}, \quad (25)$$

giving

$$\widehat{\sigma}(u, \delta) = \frac{1}{\delta} + e^{uR} \frac{(-\rho + R)(\alpha + R)}{-\rho\delta\alpha - 2\rho\delta R - 2\rho\lambda R + 2\rho c R\alpha + 3\rho c R^2}. \quad (26)$$

The second and last inversion is not so simple. The integrand function being

$$e^{\delta t} \widehat{\sigma}(u, \delta) = e^{\delta t} \frac{1}{\delta} + e^{\delta t + uR} \frac{(-\rho + R)(\alpha + R)}{-\rho\delta\alpha - 2\rho\delta R - 2\rho\lambda R + 2\rho c R\alpha + 3\rho c R^2}, \quad (27)$$

the integral of the first part is simply 1, but, in the second, if we try the substitution of R and ρ by their expressions on δ extracted from equation (22), that is,

$$R = \frac{1}{2c} \left(\delta + \lambda - c\alpha - \sqrt{(\delta^2 + 2\delta\lambda + 2c\delta\alpha + \lambda^2 - 2\lambda c\alpha + c^2\alpha^2)} \right), \quad (28)$$

and

$$\rho = \frac{1}{2c} \left(\delta + \lambda - c\alpha + \sqrt{(\delta^2 + 2\delta\lambda + 2c\delta\alpha + \lambda^2 - 2\lambda c\alpha + c^2\alpha^2)} \right), \quad (29)$$

the integrand function is not analytic in any half complex plane. So, let us change the variable of integration through the natural relation (from equation (22))

$$\delta = R \frac{-\lambda + c\alpha + cR}{\alpha + R}. \quad (30)$$

The new integrand function for the second part of (27), multiplied by $\frac{d\delta}{dR}$, will then be

$$-\exp \left(R \frac{-t\lambda + tc\alpha + tcR + u\alpha + uR}{\alpha + R} \right) \frac{-\lambda\alpha + c\alpha^2 + 2c\alpha R + cR^2}{\alpha(-\lambda + c\alpha + cR)R}, \quad (31)$$

where ρ has been substituted by $-R - (-\delta - \lambda + c\alpha)/c$ (from the relation between the two roots of equation 22). The new integrand is analytic, except for three singularities: $\{0\}$, $\{r\}$ and $\{-\alpha\}$, respecting the relations

$$-\alpha < r < 0. \quad (32)$$

The first two singularities are simple poles, but $\{-\alpha\}$ is an essential singularity. However, notice that when, under Lundberg's fundamental equation, δ progresses from 0 to $+\infty$, R decreases from r , the symmetric of the adjustment coefficient, until $-\alpha$. For $\delta = 0$, the integral that defines $\widehat{\sigma}(u, \delta)$ is divergent,

so that this transform and the first inversion already made, is only valid for $\text{Re}(\delta) > 0$. We conclude that $\{r\}$ and $\{0\}$ that correspond to $\delta = 0$ must be excluded for the second inversion. Besides, we can verify that the consideration of any of the other poles mentioned, would result in an expression that would not give probabilities at all. The analysis of the new contour of integration presents some additional difficulties, once that $\{-\alpha\}$ corresponds to $\delta = \infty$. However, in our opinion, it can be avoided, once that the limit of the contour can precede the limit of the improper integral considered, so that the singularity $\{-\alpha\}$ shall be inside the contour.

To evaluate the residue at $\{-\alpha\}$ it is necessary to develop the integrand function in a Laurent series expanded about the point $R = -\alpha$, that is, in powers of $(R + \alpha)$. Considering $R + \alpha = z$, expression (31) becomes

$$\frac{\lambda\alpha - cz^2}{\alpha(\lambda - cz)(-z + \alpha)} \exp\left((-z + \alpha) \frac{t\lambda - tcz - uz}{z}\right), \quad (33)$$

or, using partial fractions, we get

$$\frac{1}{\alpha} e^{-(t\lambda + u\alpha + tc\alpha)} \left(-1 + \frac{\lambda}{\lambda - cz} + \frac{\alpha}{-z + \alpha}\right) e^{(u+tc)z} e^{\left(\frac{1}{z}\alpha t\lambda\right)}. \quad (34)$$

Considering that

$$\frac{\lambda}{\lambda - cz} = 1 + \frac{c}{\lambda}z + \frac{c^2}{\lambda^2}z^2 + \frac{c^3}{\lambda^3}z^3 + \dots, \quad (35)$$

$$\frac{\alpha}{-z + \alpha} = 1 + \frac{1}{\alpha}z + \frac{1}{\alpha^2}z^2 + \frac{1}{\alpha^3}z^3 + \dots, \quad (36)$$

$$e^{(u+tc)z} = 1 + (u+tc)z + \frac{(u+tc)^2}{2!}z^2 + \frac{(u+tc)^3}{3!}z^3 + \dots, \quad (37)$$

$$e^{\frac{1}{z}\alpha t\lambda} = 1 + \alpha t\lambda z^{-1} + \frac{(\alpha t\lambda)^2}{2!}z^{-2} + \frac{(\alpha t\lambda)^3}{3!}z^{-3} + \dots, \quad (38)$$

and making the product of these series, we obtain the residue at $z = 0$, which is the coefficient of z^{-1} in the product obtained. After simplification, we get

$$\begin{aligned} \sigma(u, t) &= 1 + \frac{e^{-[(\lambda+c\alpha)t+\alpha u]}}{\alpha} \sum_{k=0}^{\infty} \frac{(u+ct)^k (\lambda\alpha t)^{k+1}}{k!(k+1)!} \\ &\quad - \frac{e^{-[(\lambda+c\alpha)t+\alpha u]}}{\alpha} \sum_{j=0}^{\infty} \left[\left(\frac{c}{\lambda}\right)^j + \left(\frac{1}{\alpha}\right)^j \right] \sum_{k=0}^{\infty} \frac{(u+ct)^k (\lambda\alpha t)^{j+k+1}}{k!(j+k+1)!}. \end{aligned} \quad (39)$$

4.2 Gama distributed claims

If a single claim amount has gama distribution of the simple type $\Gamma(2, \alpha)$, with Laplace transform

$$\hat{f}(s) = \frac{\alpha^2}{(\alpha + s)^2}, \quad \text{Re}(s) > -\alpha \quad (40)$$

the Lundberg fundamental equation will then be

$$\delta + \lambda - cs = \lambda \frac{\alpha^2}{(\alpha + s)^2}. \quad (41)$$

This equation has three roots - Q , R and ρ respecting the relations

$$Q < -\alpha < R < 0 \leq \rho. \quad (42)$$

Note that $\rho = 0$ if and only if $\delta = 0$. Substituting (41) in (21) we get

$$\begin{aligned} \widehat{\sigma}(s, \delta) &= \frac{\rho - s}{\rho s \left(\lambda \left(1 - \widehat{f}(s) \right) + \delta - cs \right)} \\ &= \frac{(\rho - s)(\alpha + s)^2}{\rho s (2\lambda\alpha s + \lambda s^2 + \delta\alpha^2 + 2\delta\alpha s + \delta s^2 - cs\alpha^2 - 2cs^2\alpha - cs^3)}. \end{aligned} \quad (43)$$

The first inversion of (43) with respect to s (the counterpart of u in the complex plane), gives the first Laplace transform $\widehat{\sigma}(u, \delta)$ of $\sigma(u, t)$ with respect to t and can be obtained through the inversion formula (24).

We can see that the integrand function in (43) is analytic with respect to s , except for the poles $s = 0$, $s = R$ and $s = Q$. Note that $s = \rho$ is a removable singularity and that Q is outside the domain of $\widehat{f}(s)$. The residues will then be evaluated by the relation

$$r(s) = e^{us} \frac{(\rho - s)(\alpha + s)^2}{4\rho\lambda\alpha s + 3\rho\lambda s^2 + \rho\delta\alpha^2 + 4\rho\delta\alpha s + 3\rho\delta s^2 - 2\rho cs\alpha^2 - 6\rho cs^2\alpha - 4\rho cs^3}, \quad (44)$$

giving $\widehat{\sigma}(u, \delta)$ in the form

$$\frac{1}{\delta} + e^{uR} \frac{(\rho - R)(\alpha + R)^2}{\rho (4\lambda\alpha R + 3\lambda R^2 + \delta\alpha^2 + 4\delta\alpha R + 3\delta R^2 - 2cR\alpha^2 - 6cR^2\alpha - 4cR^3)}. \quad (45)$$

The inversion of (45) is more difficult and the correspondent integrand function, $e^{\delta t} \widehat{\sigma}(u, \delta)$, can be written in the form

$$e^{\delta t} \frac{1}{\delta} + e^{\delta t + uR} \frac{(\rho - R)(\alpha + R)^2}{\rho (4\lambda\alpha R + 3\lambda R^2 + \delta\alpha^2 + 4\delta\alpha R + 3\delta R^2 - 2cR\alpha^2 - 6cR^2\alpha - 4cR^3)}. \quad (46)$$

The integral of the first part of (46) is 1, but the integral of the other part must be discussed in more detail. We must note that when δ , the integration variable, progresses from 0 to ∞ , R decreases from the symmetric of the adjustment coefficient until $-\alpha$.

To proceed with the inversion, we must change the integration variable δ by its natural relation with R , that is

$$\delta = -\lambda + cR + \lambda \frac{\alpha^2}{(\alpha + R)^2}, \quad (47)$$

and multiply the result obtained by the derivative of δ with respect to R . After simplification we get

$$\begin{aligned} & - \exp \left(-R \frac{2t\lambda\alpha + t\lambda R - tc\alpha^2 - 2tcR\alpha - tcR^2 - u\alpha^2 - 2u\alpha R - uR^2}{(\alpha + R)^2} \right) \frac{1}{R} \\ & + \exp \left(-R \frac{2t\lambda\alpha + t\lambda R - tc\alpha^2 - 2tcR\alpha - tcR^2 - u\alpha^2 - 2u\alpha R - uR^2}{(\alpha + R)^2} \right) \frac{1}{\rho}. \end{aligned} \quad (48)$$

Using the same arguments as in the exponential distribution example, we can conclude that, again, $-\alpha$ is the only singular point to consider. To determine the residue of the first part of (48) let us expand it into a Laurent series of powers of $(R + \alpha)$. For that purpose we take $R = z - \alpha$ obtaining

$$\begin{aligned} & \exp \left(-t\lambda - tc\alpha - u\alpha + ztc + zu + \frac{1}{z^2} t\lambda\alpha^2 \right) \frac{1}{-z + \alpha} \\ & = \exp(-t\lambda - tc\alpha - u\alpha) \left\{ \exp[(u + ct)z] \exp[(t\lambda\alpha^2)z^{-2}] \frac{1}{-z + \alpha} \right\} \end{aligned} \quad (49)$$

As we can see, each of the last three factors can be expanded in a series of powers of z . Multiplying this series and collecting the coefficient of z^{-1} we get

$$\begin{aligned} & \exp(-t\lambda - tc\alpha - u\alpha) \sum_{j=1}^{\infty} \left\{ \left(\frac{1}{\alpha} \right)^{2j-1} \sum_{k=1}^{\infty} \frac{a^{2k-1} b^{k+j-1}}{(2k-1)!(k+j-1)!} \right. \\ & \left. + \left(\frac{1}{\alpha} \right)^{2j} \sum_{k=1}^{\infty} \frac{a^{2k-2} b^{k+j-1}}{(2k-2)!(k+j-1)!} \right\}, \end{aligned} \quad (50)$$

expression where a denotes $u + ct$ and b denotes $t\lambda\alpha^2$.

To determine the residue of the second part of (48) it is necessary to express $1/\rho$ as a function of R . Writing Lundberg's fundamental equation (41) in the form

$$s^3 + \left(2\alpha - \frac{1}{c}\delta - \frac{1}{c}\lambda \right) s^2 + \left(\alpha^2 - \frac{2}{c}\alpha\delta - \frac{2}{c}\alpha\lambda \right) s - \frac{1}{c}\alpha^2\delta = s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

and considering that the sum of the three roots is $-a_1$ and that its product is $-a_3$, we have

$$\rho = -a_1 - R - R_1 = -a_1 - R + \frac{a_3}{\rho R} = -a_1 - (z - \alpha) + \frac{a_3}{\rho(z - \alpha)}.$$

Considering that δ expressed on z is

$$\delta = -\frac{\lambda z^2 - cz^3 + cz^2\alpha - \lambda\alpha^2}{z^2},$$

we get

$$\rho = -(-\lambda\alpha^2 - \alpha\rho\lambda + cz^2\alpha - \lambda\alpha z + 2\rho cz^2) \frac{\alpha}{\rho cz^2},$$

or

$$\frac{1}{\rho} = \frac{\rho cz^2}{-\alpha(-\lambda\alpha^2 - \alpha\rho\lambda + cz^2\alpha - \lambda\alpha z + 2\rho cz^2)}. \quad (51)$$

Solving the last equation for $1/\rho$, we can see that it has two roots but, only one represents $1/\rho$, which is

$$h(z) = \frac{1}{\rho} = \frac{1}{2(cz^2 - \lambda z - \alpha\lambda)} \frac{-2cz^2 + \alpha\lambda - \sqrt{(\alpha^2\lambda^2 + 4cz^3\lambda)}}{\alpha}. \quad (52)$$

Expanding this expression into a Maclaurin's series, we can write

$$\frac{1}{\rho} = \sum_{k=0}^{\infty} c_k z^k,$$

where

$$c_k = \frac{1}{k!} \left[\frac{d^{(k)}}{dz} h(z) \right]_{z=0}.$$

Considering now the second part of (48), it can be written as

$$\exp(-t\lambda - tc\alpha - u\alpha) \left\{ \exp[(u + ct)z] \exp(t\lambda\alpha^2) z^{-2} \sum_{k=0}^{\infty} c_k z^k \right\}.$$

If we expand the exponential functions between brackets in the above expression in powers of z and make the product of the three series, the coefficient of z^{-1} is

$$\sum_{j=1}^{\infty} c_{(2j-2)} \sum_{k=1}^{\infty} \frac{a^{2k-1} b^{k+j-1}}{(2k-1)!(k+j-1)!} + \sum_{j=1}^{\infty} c_{(2j-1)} \sum_{k=1}^{\infty} \frac{a^{2k-2} b^{k+j-1}}{(2k-2)!(k+j-1)!},$$

expression where $a = u + ct$ and $b = t\lambda\alpha^2$ as before. Finally, for this part of the residue, we get

$$\exp(-t\lambda - tc\alpha - u\alpha) \sum_{j=1}^{\infty} \left\{ c_{(2j-2)} \sum_{k=1}^{\infty} \frac{a^{2k-1} b^{k+j-1}}{(2k-1)!(k+j-1)!} + c_{(2j-1)} \sum_{k=1}^{\infty} \frac{a^{2k-2} b^{k+j-1}}{(2k-2)!(k+j-1)!} \right\}.$$

Putting all the residues together, we finally have,

$$\sigma(u, t) = 1 - \exp(-t\lambda - tc\alpha - u\alpha) \times \sum_{j=1}^{\infty} \left\{ \left(\frac{1}{\alpha}\right)^{2j-1} \sum_{k=1}^{\infty} \frac{a^{2k-1} b^{k+j-1}}{(2k-1)!(k+j-1)!} + \left(\frac{1}{\alpha}\right)^{2j} \sum_{k=1}^{\infty} \frac{a^{2k-2} b^{k+j-1}}{(2k-2)!(k+j-1)!} + c_{(2j-2)} \sum_{k=1}^{\infty} \frac{a^{2k-1} b^{k+j-1}}{(2k-1)!(k+j-1)!} + c_{(2j-1)} \sum_{k=1}^{\infty} \frac{a^{2k-2} b^{k+j-1}}{(2k-2)!(k+j-1)!} \right\}.$$

Substituting a and b and simplifying, we get

$$\sigma(u, t) = 1 - \exp(-t\lambda - tc\alpha - u\alpha) \times \sum_{j=1}^{\infty} \left\{ \left[\left(\frac{1}{\alpha}\right)^{2j-1} + c_{(2j-2)} \right] \sum_{k=1}^{\infty} \frac{(u+ct)^{2k-1} (t\alpha^2\lambda)^{j+k-1}}{(2k-1)!(k+j-1)!} + \left[\left(\frac{1}{\alpha}\right)^{2j} + c_{(2j-1)} \right] \sum_{k=1}^{\infty} \frac{(u+ct)^{2k-2} (t\alpha^2\lambda)^{j+k-1}}{(2k-2)!(k+j-1)!} \right\}. \quad (53)$$

4.3 Mixed exponential distributed claims

If the individual claim amount has a mixed exponential distribution with two weights, b and $(1-b)$, the Laplace transform is

$$\widehat{f}(s) = b \frac{\alpha}{\alpha + s} + (1-b) \frac{\beta}{\beta + s}. \quad (54)$$

Supposing that, without loss of generality, $\beta > \alpha$, the transform will be defined only for $s > -a$.

It is interesting to see the graphic corresponding to both sides of the Lundberg's equation

$$\delta + \lambda - cs = \lambda \widehat{f}(s),$$

Considering $\delta = .5$, $\lambda = 1$, $\alpha = 1/2$, $\beta = 2$, $c = 1.1$ and $b = 1/3$, the Laplace transform of $f(\cdot)$ is

$$\widehat{f}(s) = \frac{2 + 3s}{(1 + 2s)(2 + s)},$$

and the graphic has the following shape:

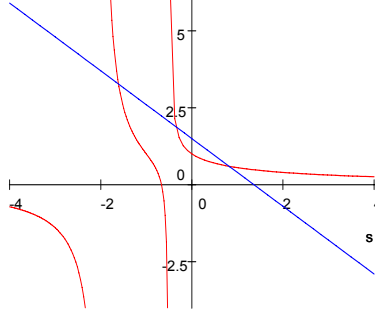


Figure 1

The substitution of $\widehat{f}(s)$ in (21) gives, after simplification,

$$\widehat{\sigma}(s, \delta) = \frac{-(\rho - s)(\alpha + s)(\beta + s)}{h(s)}, \quad (55)$$

Where

$$\frac{h(s)}{\rho s} = cs^3 + (-\lambda - \delta + c\alpha + c\beta)s^2 + [\alpha(b\lambda - \delta - \lambda + c\beta) - b\beta\lambda - \beta\delta]s - \delta\beta\alpha \quad (56)$$

As it can be seen, the equation $h(s) = 0$ has a null root and three roots more : Q , R and ρ , each of them depending on δ and respecting the inequalities

$$Q < -\beta < -\alpha < R < 0 \leq \rho. \quad (57)$$

We can see that when δ moves from zero to infinity, ρ also goes from zero to infinity and R varies from right to left starting from the symmetric of the adjustment coefficient until $-\alpha$. The root Q is outside the domain of $\hat{f}(s)$ and consequently must not be considered for inversion purposes.

The first inversion of expression (55) with respect to s (the counterpart of u in the complex plane), gives the Laplace transform $\hat{\sigma}(u, \delta)$ of $\sigma(u, t)$ with respect to t and can be obtained through the inversion formula (24), already used for the previous distributions. We can see that the implicit function defined in relation (55) is analytic with respect to s , except for the singularities $s = 0$ and $s = R$. Notice once again that $s = \rho$ is a removal singularity.

Considering that the above singular points are simple poles, the corresponding residues $r(\cdot)$ can be determined by the relation

$$r(s) = \frac{-(\rho - s)(\alpha + s)(\beta + s)}{h'(s)}, \quad (58)$$

and considering the complex inversion formula we get

$$\hat{\sigma}(u, \delta) = \frac{1}{\delta} + e^{uR}r(R).$$

The inversion of this transform implicitly needs the evaluation of an integral with respect to δ of the function

$$e^{\delta t}\hat{\sigma}(u, \delta) = \frac{e^{\delta t}}{\delta} + e^{\delta t + uR}r(R), \quad (59)$$

in a special contour (following the inversion theorem, it is sufficient to evaluate the integral in the interval $(\sigma_0 - i\infty, \sigma_0 + i\infty)$, where σ_0 represents any real number greater than the abscissa of convergence $-\alpha$).

The integral corresponding to the first part of relation (59) gives simply 1, but the integral of $\exp(\delta t + uR)r(R)$, is not that simple. It is not desirable to express R as a function of δ , once the resulting function would not be analytic in any half complex plane. So, the solution is to express δ as a function of R , that is,

$$\delta = -\lambda + cR + \lambda\hat{f}(R).$$

After the multiplication by the derivative of δ with respect to R and simplifying, $\exp\left[\left(-\lambda + cR + \lambda\widehat{f}(R)\right)t + uR\right] r(R)\delta'(R)$ becomes

$$\left(\frac{1}{\rho} - \frac{1}{R}\right) \exp\left[R\frac{-t\lambda\alpha - tR\lambda + tc\beta\alpha + tRc\alpha + tRc\beta + tcR^2 + t\lambda b\alpha - t\lambda\beta b + u\beta\alpha + uR\alpha + uR\beta + uR^2}{(\alpha+R)(\beta+R)}\right] \quad (60)$$

As in the previous examples, the singularities to be considered are in the exponential part of the integrand function and as we can see they are two essential singular points: one corresponding to $R = -\alpha$ and the other resulting from $R = -\beta$. Once the order does not matter, we will start with $R = -\alpha$ and with the second part of (60)

$$-\frac{1}{R} \exp\left[R\frac{-t\lambda\alpha - tR\lambda + tc\beta\alpha + tRc\alpha + tRc\beta + tcR^2 + t\lambda b\alpha - t\lambda\beta b + u\beta\alpha + uR\alpha + uR\beta + uR^2}{(\alpha+R)(\beta+R)}\right]$$

If we write $R = z - \alpha$, the above expression becomes

$$-\frac{1}{z - \alpha} \exp\left[(z - \alpha) \frac{-t\lambda z - tc\alpha z + tc\beta z + tcz^2 + t\lambda b\alpha - t\lambda\beta b - u\alpha z + u\beta z + uz^2}{z(\beta + z - \alpha)}\right].$$

Expanding the exponent in simple fractions we get

$$-\frac{1}{z - \alpha} \exp(-t\lambda - tc\alpha - u\alpha) \exp[(ct + u)z] \exp\left(t\lambda b \frac{\alpha}{z}\right) \exp\left(-\beta t\lambda \frac{-1 + b}{\beta + z - \alpha}\right).$$

In order to evaluate the residue we must expand the expression in a Laurent series about the point $z = 0$. For that purpose it is necessary to expand each exponential function into a Mac-Laurin series about $z = 0$ and make the product of these series. The residue will then be the coefficient of z^{-1} in the product obtained. So, we have

$$\begin{aligned} \frac{-1}{z - \alpha} &= \sum_{k=0}^{\infty} \left(\frac{1}{\alpha}\right)^k z^k, \\ e^{(ct+u)z} &= \sum_{k=0}^{\infty} \frac{(ct+u)^k}{k!} z^k, \\ e^{t\lambda b \frac{\alpha}{z}} &= \sum_{k=0}^{\infty} \frac{(t\lambda b\alpha)^k}{k!} z^{-k}, \\ e^{-\beta t\lambda \frac{-1+b}{\beta+z-\alpha}} &= \sum_{k=0}^{\infty} c_k z^k. \end{aligned}$$

We must notice that the last exponential does not have a development as simple as the others, though the coefficients c_k may be obtained through the corresponding derivatives,

$$c_k = \frac{1}{k!} \left[\frac{d^k}{dz^k} \exp\left(-\beta t\lambda \frac{-1+b}{\beta+z-\alpha}\right) \right]_{z=0}.$$

After the product of the four series, the coefficient of z^{-1} is

$$\sum_{j=0}^{\infty} c_j \sum_{k=1}^{\infty} \left(\frac{1}{\alpha}\right)^k \sum_{l=0}^{\infty} \frac{(t\lambda b\alpha)^{j+k+l-1}}{(j+k+l-1)!} \times \frac{(ct+u)^{l-1}}{(l-1)!},$$

so, the first component of the residue corresponding to the root α is

$$\exp(-t\lambda - tc\alpha - u\alpha) \sum_{j=0}^{\infty} c_j \sum_{k=1}^{\infty} \left(\frac{1}{\alpha}\right)^k \sum_{l=1}^{\infty} \frac{(t\lambda b\alpha)^{j+k+l-1}}{(j+k+l-1)!} \times \frac{(ct+u)^{l-1}}{(l-1)!}. \quad (61)$$

For the other component of (60), we must express $1/\rho$ as a function of R , followed by the development in a Laurent series as we have just made. For the purpose, we start with the Lundberg's fundamental equation written as

$$\begin{aligned} 0 &= s^3 + a_1 s^2 + a_2 s + a_3 \\ &= s^3 - \frac{\lambda + \delta - c\alpha - c\beta}{c} s^2 - \frac{\delta\alpha - \lambda b\alpha + \lambda\beta b + \lambda\alpha + \delta\beta - c\alpha\beta}{c} s - \delta\alpha \frac{\beta}{c}. \end{aligned}$$

Considering that the sum of the three roots is $-a_1$ and its product is $-a_3$, we have

$$\rho = -a_1 - R - R_1 = -a_1 - (z - \alpha) + \frac{a_3}{\rho(z - \alpha)},$$

or,

$$\rho = \frac{\lambda + \delta - c\alpha - c\beta}{c} - (z - \alpha) - \frac{\delta\beta\alpha}{c\rho(z - \alpha)}. \quad (62)$$

Expressing δ as a function of z through the relation

$$\begin{aligned} \delta &= -\lambda + c(z - \alpha) + \lambda \hat{f}(z - \alpha) \\ &= (z - \alpha) \frac{-\lambda z - cz\alpha + cz\beta + cz^2 + \lambda b\alpha - \lambda\beta b}{z(\beta + z - \alpha)}, \end{aligned}$$

substituting in (62) and considering $\rho = 1/x$, we obtain a second degree equation in x ,

$$b_0 x^2 + b_1 x + b_2 = 0,$$

Where

$$\begin{aligned} b_0 &= \beta\alpha cz^2 + (-\beta\alpha^2 c + \beta^2\alpha c - \beta\alpha\lambda) z + \beta\alpha^2\lambda b - \beta^2\alpha\lambda b \\ b_1 &= (c\beta + c\alpha) z^2 + (c\beta^2 - \lambda\beta + \lambda\beta b - c\alpha^2 - \lambda b\alpha) z + \lambda b\alpha^2 - \alpha\lambda\beta b \\ b_2 &= -cz\alpha + cz\beta + cz^2. \end{aligned}$$

Considering the two roots of the last equation, we conclude that only one represents $1/\rho$, which must be that one that gives $x = 0$ for $z = 0$, once for $R = -\alpha$, we know that $\rho = \infty$. Solving for instance, for the example above, we would have,

$$x(z) = \frac{-105z + 10 - 110z^2 - \sqrt{(5481z^2 + 3180z + 6908z^3 + 100 + 4356z^4)}}{2(-20 + 44z^2 + 26z)}. \quad (63)$$

Expanding $x(z)$ in a Mac-Laurin series we get,

$$x(z) = \sum_{k=0}^{\infty} d_k z^k, \text{ where } d_k = \frac{1}{k!} \left[\frac{d^k}{dz^k} x(z) \right]_{z=0},$$

and, as in the first part of the residue, making the product of the four series and isolating the coefficient of z^{-1} , we get an expression similar to the previous one, just with the changes resulting from the series corresponding to $1/\rho$ instead $1/R$.

After simplification we get

$$\exp(-t\lambda - tc\alpha - u\alpha) \sum_{j=0}^{\infty} c_j \sum_{k=1}^{\infty} d_{k-1} \sum_{l=1}^{\infty} \frac{(t\lambda b\alpha)^{j+k+l-1}}{(j+k+l-1)!} \times \frac{(ct+u)^{l-1}}{(l-1)!}. \quad (64)$$

Putting both components together, the value of the residue corresponding to $-\alpha$ comes:

$$\exp(-t\lambda - tc\alpha - u\alpha) \sum_{j=0}^{\infty} c_j \sum_{k=1}^{\infty} \left[\left(\frac{1}{\alpha} \right)^k + d_{k-1} \right] \sum_{l=1}^{\infty} \frac{(t\lambda b\alpha)^{j+k+l-1}}{(j+k+l-1)!} \times \frac{(ct+u)^{l-1}}{(l-1)!}. \quad (65)$$

For the root β , taking similar steps we get

$$\exp(-t\lambda - tc\beta - u\beta) \sum_{j=0}^{\infty} e_j \sum_{k=1}^{\infty} \left[\left(\frac{1}{\beta} \right)^k + f_{k-1} \right] \sum_{l=1}^{\infty} \frac{(t\lambda(1-b)\beta)^{j+k+l-1}}{(j+k+l-1)!} \times \frac{(ct+u)^{l-1}}{(l-1)!}, \quad (66)$$

expression where the coefficients $\{e_k\}$ are obtained from the following Mac-Laurin development:

$$e^{\alpha t \lambda \frac{b}{\alpha+z-\beta}} = \sum_{k=0}^{\infty} e_k z^k,$$

and the coefficients $\{f_k\}$ from the new development of $1/\rho$ in powers of z .

Summing up the three residues we finally have for $\sigma(u, t)$ the expression

$$\begin{aligned} & 1 - e^{-t\lambda - tc\alpha - u\alpha} \sum_{j=0}^{\infty} c_j \sum_{k=1}^{\infty} \left[\left(\frac{1}{\alpha} \right)^k + d_{k-1} \right] \sum_{l=1}^{\infty} \frac{(t\lambda b\alpha)^{j+k+l-1}}{(j+k+l-1)!} \times \frac{(ct+u)^{l-1}}{(l-1)!} \\ & - e^{-t\lambda - tc\beta - u\beta} \sum_{j=0}^{\infty} e_j \sum_{k=1}^{\infty} \left[\left(\frac{1}{\beta} \right)^k + f_{k-1} \right] \sum_{l=1}^{\infty} \frac{(t\lambda(1-b)\beta)^{j+k+l-1}}{(j+k+l-1)!} \times \frac{(ct+u)^{l-1}}{(l-1)!}. \end{aligned} \quad (67)$$

4.4 Some numerical results

In the tables below, column (1) represents exact formulae and column (2) the integral approximation.

Table 1: Poisson (1) / Exponential (1) / $c = 1.1$

t	u=0		u=1		u=2		u=10	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
1	0,536599341	0,536599370	0,761944014	0,761944092	0,880294317	0,880294304	0,999691627	0,999691547
2	0,407136174	0,407136102	0,645431014	0,645430961	0,794327577	0,794327496	0,998650012	0,998649994
3	0,344789020	0,344789042	0,574022178	0,574022082	0,731540865	0,731540858	0,906770312	0,996770207
4	0,306693192	0,306693119	0,524715500	0,524715467	0,683592552	0,683592414	0,994104657	0,994104585
5	0,280402460	0,280402236	0,488107054	0,488106927	0,645580747	0,645580699	0,990767006	0,990766857
6	0,260881492	0,260881422	0,459570548	0,459570485	0,614551659	0,614551572	0,986885328	0,986885022
7	0,245661758	0,245661567	0,436536063	0,436535882	0,588632685	0,588632504	0,982580343	0,982577390
8	0,233373726	0,233373573	0,417448330	0,417448064	0,566579259	0,566578870	0,977957564	0,977932668
9	0,223188948	0,223188773	0,401304257	0,401304371	0,547530320	0,547530579	0,973105567	0,973070064
10	0,214573156	0,214572985	0,387424252	0,387424120	0,530869718	0,530869574	0,968096976	0,968031681

Table 2: Poisson (1) / Gamma (2, 2) / $c = 1.1$

t	u=0		u=1		u=2		u=10	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
1	0,488408329	0,488408329	0,751323376	0,751323571	0,897816146	0,897816210	0,999982332	0,999982321
2	0,364106058	0,364106058	0,635115445	0,635115467	0,814990211	0,814990259	0,999833305	0,999833287
3	0,307657089	0,307657089	0,565020725	0,565020650	0,753321999	0,753321990	0,999407099	0,999407088
4	0,273761441	0,273761441	0,517044703	0,517044698	0,705940773	0,705940754	0,998610508	0,998610486
5	0,250576368	0,250576368	0,481626155	0,481626066	0,668289058	0,668289121	0,997410280	0,997410231
6	0,233457960	0,233457960	0,454130089	0,454129892	0,637525634	0,637525569	0,995815550	0,995815446
7	0,220164930	0,220164930	0,432005863	0,432005642	0,611822732	0,611822549	0,993859724	0,993858203
8	0,209465851	0,209465851	0,413720023	0,413719840	0,589956294	0,589956068	0,991587492	0,991570632
9	0,200620565	0,200620562	0,398288708	0,398288104	0,571075787	0,571075558	0,989046753	0,989029026
10	0,193154096	0,193154049	0,385049822	0,385046060	0,554571624	0,554570268	0,986284040	0,986245934

Table 3: Poisson (1) / Mixed exponential ($\alpha = 1/2, b = 1/3, \beta = 2$)/ $c = 1.1$

t	u=0		u=1		u=2		u=10	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
1	0,5808459	0,5808459	0,7986258	0,7986259	0,8832519	0,8832520	0,9965719	0,9965719
2	0,4613693	0,4613693	0,6888200	0,6888201	0,7994853	0,7994854	0,9913266	0,9913266
3	0,3976509	0,3976509	0,6172846	0,6172846	0,7373643	0,7373643	0,9848518	0,9848517
4	0,3562979	0,3562979	0,5661029	0,5661030	0,6892756	0,6892756	0,9775631	0,9775625
5	0,3267453	0,3267453	0,5272508	0,5272508	0,6507489	0,6507484	0,9697607	0,9797420
6	0,3043222	0,3043222	0,4965127	0,4965132	0,6190425	0,6190417	0,9616615	0,9616539
7	0,2865857	0,2865857	0,4714373	0,4714368	0,5923864	0,5923853	0,9534211	0,9534167
8	0,2721181	0,2721181	0,4504924	0,4507775	0,5695864	0,5697266	0,9451512	0,9451518
9	0,2600340	0,2600340	0,4326665	0,4329839	0,5498062	0,5500083	0,9369309	0,9369279
10	0,2497496	0,2497496	0,4172624	0,4175560	0,5324411	0,5326497	0,9288161	0,9288266

Some differences between the first column and the second are due, in our opinion, to the complexity of the integral approximation using Seal's formulae and the implicit error committed with this type of approximation. The results obtained from the exact formulae, should be in principle more accurate, once we have used some auxiliary functions such as Bessel functions and *generalized hypergeometric* functions that simplify significantly the necessary programs and that are built-in functions in the software used.

5 Additional remarks

In this paper we have shown that, in the classical risk model, it is possible to obtain explicit expressions for survival or ruin probabilities in a finite time horizon when the individual claims are mixed exponential or Erlang distributed. The same techniques used could also be extended to other particular claim amount distributions.

The extension of these techniques to other risk models depends, in first place, on the possibility of obtaining an explicit or implicit formula for the double Laplace transform correspondent to the time of ruin and, in second place, on the analytic properties of this transform. For renewal models, for instance, the double Laplace transform obtained by Dickson and Hipp (2001) for the Erlang (2) risk process is, for that purpose, a challenge and a good investigation starting point.

References

- Dickson, D.C.M., Hipp,C., (2001). On the time to ruin for Erlang(2) risk processes: *Insurance: Mathematics and Economics* 29 (2001) 333-344.
- Garcia, J.M.A., (1999). Other properties of the cosine transform. Cemapre working paper, ISEG, Technical University of Lisbon.
- Gerber, H.U., (1979). *An Introduction to Mathematical Risk Theory*. S.S. Huebner Foundation Monograph series No.8. Irwin, Homewood, IL.
- Gerber, H.U., Shiu, E.S.W., (1998). On the time value of ruin. *North American Actuarial Journal* 2, 48-78.
- Marsden, J.E., Hoffman, M.J., (1999). *Basic complex analysis*. W. H. Freeman, New York.
- Lima, F.D.P., Garcia, J.M.A., Reis, A.D.E., (2002). Fourier/Laplace transforms and ruin probabilities. *Astin Bulletin* (printing).
- Poularikas, A. D.(1996). *The Transforms and Applications Handbook*, CRC Press.