The compound binomial model revisited

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Abstract

In this paper we *re-cap* the discrete model and views by Gerber (1988), also re-taken by other authors. That is, we consider a discrete time risk model where the aggregate claim process is compound binomial. In each period there is a claim with probability p, or no claim with probability $1-p$ and that there is an independence in claim occurrrence in different time periods. The sequence of individual claims is a sequence of i.i.d. random variables, distributed on the positive integers, and independent of the claim number process. We follow the model formulation presented by Gerber (1988) and Dickson (1994).

Following the approach by Egídio dos Reis (2002) for the classical model, starting from a non-negative integer initial surplus, there is a positive probability that the risk process is ruined, *i.e.*, it drops to negative values. If ruin occurs, it happens at the instant of a claim, then we can address the ruin probability problem, either finite or infinite time, by considering the *number of claims necessary to get ruined*. We then consider the calculation of the distribution of the number of claims up to ruin, if it occurs. Besides, since the process once ruined will recover to positive levels some time in the future with probability one, we also consider the distribution of the number of claims ocurring during the recovery time period.

An interesting result is achieved concerning the particular case when the initial surplus is zero, which is the fact that the two discrete random variables above have the same distribution and that the distribution belongs to the *Lagrangian-type* family, and a closed form for the distribution is found.

From that, it is possible to find a recursion that allows the computation of the distribution of the number of claims up to ruin, considering any postive integer initial surplus. For these cases, we also find a formula for the distribution of the number of claim during a recovery time period.

Besides, with this model we will be able to compute approximations for the related quantities in the classical compound Poisson risk model.

Keywords: Ruin theory; compound binomial model; claim number up to ruin; claim number up to recovery; time to ruin; recursive calculation.

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1 Introduction

Consider the compound binomial model as like in Gerber (1988) and Dickson (1994). The discrete time risk model is defined as follows

$$U_t = u + t - S_{N_t}\ ,\ \ t = 0, 1, 2, ...,$$

where U_t is the insurer's surplus at time t, $U_0 = u$, $S_{N_t} = \sum_{i=0}^{N_t} X_i$ (with $X_0 \equiv 0$), N_t is the number of claims in the first t time periods. For simplicity we consider the premium income per unit time to be 1. It is assumed that this is a binomial process, in each period there is a claim with probability p, or no claim with probability $q = 1 - p$, and that there is an independence in claim occurrence in different time periods. The amount of the claim in period i is X_i, so that $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables, independent of the claim number process $\{N_t\}$, and with common probability function with support on the positive integers. This distribution is denoted as

$$F(x) = \Pr(X_i \leq x)\,,\ \ f(x) = \Pr(X_i = x)\,,\ x = 1, 2, ...$$

We denote its probability generating function (simply, p.g.f.) evaluated at z as $\hat{f}(z) = \sum_{i=1}^{\infty} z^i f(i)$.

We further assume that the mean $E(X_i) = \mu < \infty$. Finally, we assume that the premiums contain a loading, i.e. $p\mu < 1$, otherwise the probability of ruin would be one. The probability of ultimate ruin in this process is defined as $\psi(u) = \Pr(T < \infty | u)$, where

$$T = \min\{t \geq 1, U_t \leq 0 | U_0 = u\}\,.$$

Let, $\psi(u, n) = \Pr(T \leq n | u)$ denote the finite time ruin probability, $\delta(u, n) = 1 - \psi(u, n)$ denote the finite time survival probability and $\delta(u) = 1 - \psi(u)$ denote the ultimate survival probability. Note that under this definition having $U_t = 0$ for some t other than 0, it corresponds to a "ruin", however, the process can start from $u = 0$. The claim number process is a discrete time renewal process where the waiting time distribution is geometric. If we denote by $K(n)$ and $k(n)$ respectively the distribution and probability functions we have that $k(n) = pq^{n-1}$, $n = 1, 2, ...$ We will furthermore denote by $\phi(u, n) = \Pr(T = n | u)$, $n \epsilon \mathbb{N}$, the (defective) probability function of the time to ruin.

We know form Gerber (1988), formulae (5)-(7), that the probability of ultimate ruin can be computed recursively as follows

$$\begin{aligned}
\psi(0) &= p\mu \\
\psi(1) &= q^{-1}(\psi_d(0) - p) \\
\psi(u+1) &= q^{-1}\left(\psi_d(u) - p\sum_{i=1}^{u}\psi_d(u+1-i)f(i) - p[1 - F(u)]\right)
\end{aligned} \tag{1}$$

These formulae are similar to those in (Dickson & Waters 1991) [expression (7.2) with the modification in Section 8] developed for the compound Poisson discrete time model.

Let

$$\begin{aligned}
f(u; x, y) &= \Pr(T < \infty, U_T = -y, U_{T-1} = x | u)\,,\ \ x = 1, 2, ...;\ y = 0, 1, 2, ... \\
f(u; x) &= \sum_y f(u; x, y),\ x = 1, 2, ... \\
g(u; y) &= \sum_x f(u; x, y),\ \ y = 0, 1, 2, ...
\end{aligned}$$

The first function above defines the joint probability function of the amount of the surplus immediately before and at ruin and the subsequent functions are their respective marginal functions, respectively the probability of the amount of the surplus immediately prior to ruin and of the severity of ruin. We note that these are defective functions, that is, $\sum_y \sum_x f(u;x,y) = \sum_x f(u;x) = \sum_y f(u;y) = \psi(u)$. Gerber (1988) shows that, for $u = 0$,

$$f(0;x,y) = pf(x+y+1),\; x = 1,2,...;\; y = 0,1,2,...$$

and notes that *this formula is also true for* $x = 0$ *(then it simply gives the probability that there is a claim of size* $y+1$ *in the first period).* A similar comment applies to the formula $f(0;x)$ next. Then, for $u = 0$,

$$\begin{aligned} f(0;x) &= p\sum_{y=0}^{\infty} f(x+y+1) = p\left[1 - F(x)\right] \\ g(0;y) &= p\sum_{x=0}^{\infty} f(x+y+1) = p\left[1 - F(x)\right],\; y = 0,1,2,... \end{aligned}$$

We denote the p.g.f. of $g(u;y)$ as $\hat{g}(u;z) = \sum_{y=0}^{\infty} z^y g(u;y)$. It's easy to show that

$$\hat{g}(0;z) = \frac{p}{1-z}\left[1 - \hat{f}(z)\right] \tag{2}$$

Similar to Egídio dos Reis (2002) for the classical risk model, instead of evaluating finite time ruin probabilities we can think of evaluating how many claims the surplus takes to get ruined, if it happens. Besides, we will also consider the number of claims during a period of negative surplus, provided that ruin occurs. Let M be the random variable that represents the number of claims up to ruin (from initial surplus u). We denote its p.f. and d.f respectively by $b(u;n)$ and $B(u;n)$, $n = 1,2,...$ (at least one claim is needed to get ruined, if that effectively occurs). Let K denote the random variable of the number of claims that occur until the process recovers to positive levels once the process has been ruined (from initial surplus u). We denote its p.f. and d.f., respectively by $v(u;n)$ and $V(u;n)$, $n = 0,1,2,...$ (once ruined the process can recover with no claims, besides, recovery has probability one).

As noted by Gerber (1988) and later examined by Dickson (1994) this binomial model can be used to find approximations for related quantities, like ruin probabilities, in the classical continuous time compound Poisson model. Furthermore, Dickson (1994) concluded that the approximations obtained for ultimate ruin probabilities are not as good as other approximations based on discretizations, using for instance other discrete approaches like the one by Dickson & Waters (1991), unless you use much higher discretization units. However, for our main problem and addressing approximations, the use of the binomial model to approximate the distribution of the number of claims up to ruin, or to recovery, in the classical model, this is a suitable approach. The reasoning is the following: In the discrete compound Poisson model, as used by Dickson & Waters (1991), in each discretization period the number of claims can be multiple since its distribution is Poisson. Of course you can make the period length very small so that the probability of having more than one claim is very small, but still positive. In our main problem we want to consider the claim number up to ruin, in other words, the claim number that causes ruin, the model by Dickson & Waters (1991) will give the time period that causes ruin, if it happens. Hence, this latter is suitable for approximating the distribution for the time to ruin and not for the distribution of the claim number up to ruin.

The discrete model and the results presented in this paper not only can be used as approximations to related quantities in the classical model, but also have value on its own. Besides, as we use a renewal approach, by considering that the distribution of the waiting time is geometric, we could extend the method presented to other renewal models, for instance when the waiting time is negative binomial distributed, and use this model to approximate quantities in continuous time Erlang renewal risk models. Like in the model presented where the geometric can be viewed as the *discrete version* of the exponential distribution as the waiting time of the Poisson process, the negative binomial will be viewed as the *discrete version* of an Erlang distribution, considering an appropriate parameter choice.

In the following section we deal with the distribution of the number of claims up to ruin. In Section 3 we work the probability function of the number of claims occurring during a recovery time period, if ruin occurs. In Section 4, using a renewal argument, we work recursion formulae for finite time survival probabilities and in the last section we provide some formulae with an example, when single claims are geometric.

2 On the number of claims up to ruin

Considering the first claim occurrence and that the waiting time for a claim has a geometric distribution, we have that for $u = 1, 2, ...$

$$
\begin{aligned}
B(u;1) &= b(u;1) = \sum_{t=1}^{\infty} pq^{t-1} \sum_{x=u+t}^{\infty} f(x) && (3)\\
&= p\sum_{t=1}^{\infty} q^{t-1}\left[1 - F(u+t-1)\right] = p\sum_{t=0}^{\infty} q^{t}\left[1 - F(u+t)\right] && (4)\\
&= pq^{-u}\sum_{t=0}^{\infty} q^{t+u}\left[1 - F(u+t)\right] = pq^{-u}\sum_{t=u}^{\infty} q^{t}\left[1 - F(t)\right] && (5)
\end{aligned}
$$

For $u = 1, 2, ...$ and $n = 1, 2, ...$, we have

$$B(u;n+1) = B(u;1) + \sum_{t=1}^{\infty} pq^{t-1} \sum_{x=1}^{u+t-1} f(x)B(u+t-x;n).$$

Hence, $b(u;n+1) = B(u;n+1) - B(u;n)$,

$$b(u;n+1) = \sum_{t=1}^{\infty} pq^{t-1} \sum_{x=1}^{u+t-1} f(x)b(u+t-x;n) \qquad (6)$$

The formula above can be worked out to find a recursion formula useful for computation purposes. Before we work it out we will need an auxiliary expression to be presented next.

It is simple to understand that formula (3) is also valid for $u = 0$. However, it is not the case concerning formula (6). We cannot simply extend formula (6) for $u = 0$. This is because if we want to consider ruin on the $(n+1)$-th claim, $n = 1, 2, ...$, we cannot have any claim in the first time period as any claim amount would necessarily imply ruin (on the first claim occurrence!). Recall that at the end of this period we will have one premium income of one unit, so a claim occurrence would lead the surplus to a non positive level, which is *ruin*. Then, any claim occurrence would be *allowed* to happen from the second period on. Hence,

$$\begin{aligned}
b(0;n+1) &= \sum_{t=2}^{\infty} pq^{t-1} \sum_{x=1}^{t-1} f(x) b(t-x;n) \\
&= \sum_{t=2}^{\infty} pq^{t-1} \sum_{x=1}^{t-1} f(t-x) b(x;n) \\
&= \sum_{x=1}^{\infty} pb(x;n) q^{x-1} \sum_{t=x+1}^{\infty} q^{t-x} f(t-x) \\
&= \sum_{x=1}^{\infty} pb(x;n) q^{x-1} \sum_{t=1}^{\infty} q^{t} f(t) \\
&= \sum_{x=1}^{\infty} pb(x;n) q^{x-1} \hat{f}(q) \\
&= pq^{-1} \hat{f}(q) \sum_{x=1}^{\infty} q^{x} b(x;n), \qquad (7)
\end{aligned}$$

Note that the series $\sum_{x=1}^{\infty} q^x b(x;n) < \infty$, since $b(x;n)$ is a probability then $0 \leq q^x b(x;n) < q^x$ and $\sum_{x=1}^{\infty} q^x$ is a geometric series with ratio $0 < q < 1$.

Result 1 *For* $n = 2, 3, ...$

$$b(1;n) = q^{-1} b(0;n) \qquad (8)$$

$$b(u;n) = q^{-1} \left(b(u-1;n) - p \sum_{x=1}^{u-1} b(x;n-1) f(u-x) \right), \; u = 2, 3, ... \qquad (9)$$

Proof. Consider equation (6), we can re-write it as

$$
\begin{aligned}
b(u;n) &= \sum_{t=1}^{\infty} pq^{t-1} \sum_{x=1}^{u+t-1} f(u+t-x)b(x;n-1) \\
&= p\sum_{x=1}^{u} b(x;n-1)\sum_{t=1}^{\infty} q^{t-1} f(u+t-x) + p\sum_{x=u+1}^{\infty} b(x;n-1)\sum_{t=x+1-u}^{\infty} q^{t-1} f(u+t-x) \\
&= pq^{-(u+1)}\left(\sum_{x=1}^{u} q^x b(x;n-1)\sum_{t=1}^{\infty} q^{u+t-x} f(u+t-x)\right. \\
&\quad \left. + \sum_{x=u+1}^{\infty} q^x b(x;n-1)\sum_{t=x+1-u}^{\infty} q^{u+t-x} f(u+t-x)\right. \\
&= pq^{-(u+1)}\left(\sum_{x=1}^{u} q^x b(x;n-1)\sum_{t=u+1-x}^{\infty} q^t f(t) + \sum_{x=u+1}^{\infty} q^x b(x;n-1)\sum_{t=1}^{\infty} q^t f(t)\right) \\
&= pq^{-(u+1)}\left(\sum_{x=1}^{u} q^x b(x;n-1)\sum_{t=u+1-x}^{\infty} q^t f(t) + \hat{f}(q)\sum_{x=u+1}^{\infty} q^x b(x;n-1)\right) \qquad (10)
\end{aligned}
$$

changing the summations order and re-arranging.

For $u = 1$ we have

$$
\begin{aligned}
b(1;n) &= pq^{-2}\left(qb(1;n-1)\sum_{t=1}^{\infty} q^t f(t) + \hat{f}(q)\sum_{x=2}^{u} q^x b(x;n-1)\right) \\
&= pq^{-2}\left(qb(1;n-1)\hat{f}(q) + \hat{f}(q)\sum_{x=2}^{u} q^x b(x;n-1)\right) \\
&= pq^{-2}\left(\hat{f}(q)\sum_{x=1}^{u} q^x b(x;n-1)\right) \\
&= q^{-1}b(0;n),
\end{aligned}
$$

using (7).

For $u = 2, 3, ...$, from (10) $p^{-1}q^{u+1}b(u;n)$ comes

$$\begin{aligned}
& \left(\sum_{x=1}^{u-1} q^x b(x;n-1) \sum_{t=u+1-x}^{\infty} q^t f(t) + q^u b(u;n-1) \sum_{t=1}^{\infty} q^t f(t) + \hat{f}(q) \sum_{x=u+1}^{\infty} q^x b(x;n-1)\right) \\
= & \left(\sum_{x=1}^{u-1} q^x b(x;n-1) \sum_{t=u+1-x}^{\infty} q^t f(t) + \hat{f}(q) q^u b(u;n-1) + \hat{f}(q) \sum_{x=u+1}^{\infty} q^x b(x;n-1)\right) \\
= & \left(\sum_{x=1}^{u-1} q^x b(x;n-1) \sum_{t=u+1-x}^{\infty} q^t f(t) + \hat{f}(q) \sum_{x=u}^{\infty} q^x b(x;n-1)\right) \\
= & \left(\sum_{x=1}^{u-1} q^x b(x;n-1) \sum_{t=u-x}^{\infty} q^t f(t) + \hat{f}(q) \sum_{x=u}^{\infty} q^x b(x;n-1) - \sum_{x=1}^{u-1} q^x b(x;n-1) q^{u-x} f(u-x)\right) \\
= & \left(\sum_{x=1}^{u-1} q^x b(x;n-1) \sum_{t=u-x}^{\infty} q^t f(t) + \hat{f}(q) \sum_{x=u}^{\infty} q^x b(x;n-1) - q^u \sum_{x=1}^{u-1} b(x;n-1) f(u-x)\right).
\end{aligned}$$

Now, the first two terms in the formula above correspond to $p^{-1}q^u b(u-1;n)$ and we get the recursion formula

$$\begin{aligned}
b(u;n) &= q^{-1}\left(b(u-1;n) - p\sum_{x=1}^{u-1} b(x;n-1) f(u-x)\right) \\
&= q^{-1}\left(b(u-1;n) - p\sum_{x=1}^{u-1} b(u-x;n-1) f(x)\right)
\end{aligned}$$

□

We see from the recursion above that the computation of $b(u;n)$ we will need the figures of $b(x;m)$ for the previous values of $(x = 0, 1, ..., u-1;\ m = 1, 2, ..., n-1)$. We still need a workable formula for $b(0;n)$. This is coming next. Yet, it is easy to check that recursion (1) can be obtained from the recursion above for $b(u+1;n+1)$, since $\psi(u+1) = \sum_{n=0}^{\infty} b(u+1;n+1)$, as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty} b(u+1;n) &= b(u+1;1) + \sum_{n=2}^{\infty} b(u+1;n) \\
&= b(u+1;1) + q^{-1}\left(\sum_{n=2}^{\infty} b(u;n) - p\sum_{n=2}^{\infty}\sum_{x=1}^{u} b(x;n-1) f(u+1-x)\right) \\
&= b(u+1;1) + q^{-1}\left(\psi(u) - b(u,1) - p\sum_{x=1}^{u} f(u+1-x)\sum_{n=1}^{\infty} b(x;n)\right) \\
&= b(u+1;1) + q^{-1}\left(\psi(u) - b(u,1) - p\sum_{x=1}^{u} f(u+1-x)\psi(x)\right).
\end{aligned}$$

Since $q^{-1}b(u,1) = b(u+1;1) + q^{-1}g(0;u)$, then recursion (1) follows.

In the following result we work closed formulae for $b(0;n+1)$, $n = 0, 1, 2, ...$

Result 2 *For $u = 0$*

$$
\begin{aligned}
b(0;1) &= \hat{g}(0;q) &(11)\\
b(0;2) &= p\hat{f}(q)\hat{g}'(0;q) &(12)\\
b(0;n+1) &= \frac{1}{n!}p^n \frac{d^{n-1}}{dz^{n-1}}\left[\hat{f}(z)^n \hat{g}'(0;z)\right]\Big|_{z=q}, \quad n = 2,3,... &(13)
\end{aligned}
$$

Proof. Formula (11) is immediate from (5). Let's now consider (12). From (7) we have

$$b(0;2) = pq^{-1}\hat{f}(q)\sum_{x=1}^{\infty} q^x b(x;1)$$

and from (5) we have that

$$
\begin{aligned}
\sum_{x=1}^{\infty} q^x b(x;1) &= \sum_{x=1}^{\infty} q^x \sum_{i=x}^{\infty} q^{i-x} g(0;i) = \sum_{x=1}^{\infty}\sum_{i=x}^{\infty} q^i g(0;i)\\
&= \sum_{i=1}^{\infty} q^i g(0;i) \sum_{x=1}^{i} 1\\
&= \sum_{i=1}^{\infty} i q^i g(0;i) = q\sum_{i=1}^{\infty} i q^{i-1} g(0;i)\\
&= q\hat{g}'(0;q),
\end{aligned}
$$

from what follows (12). For expression (13), we can work it our recursively to find it for a general n in the following way. Consider $n = 2$. Retrieving from (7) and introducing (6) we have

$$\sum_{x=1}^{\infty} q^x b(x;2) = p\sum_{x=1}^{\infty}\sum_{t=1}^{\infty} q^{x+t-1}\sum_{i=1}^{x+t-1} f(i) b(x+t-i;1) \tag{14}$$

with, by (5), $b(x+t-i;1) = \sum_{j=x+t-i}^{\infty} q^{j-(x+t-i)} g(0;j)$. Then

$$q^{x+t-1}\sum_{i=1}^{x+t-1} f(i) \sum_{j=x+t-i}^{\infty} q^{j-(x+t-i)} g(0;j) = \sum_{i=1}^{x+t-1} q^{i-1} f(i) \sum_{j=x+t-i}^{\infty} q^j g(0;j)\ .$$

Consider now, from (14), the summation on t, introduce in the above expression and change the sums order

$$
\begin{aligned}
&\left(\sum_{i=1}^{x}\sum_{t=1}^{\infty} + \sum_{i=x+1}^{\infty}\sum_{t=i-x+1}^{\infty}\right) q^{i-1} f(i) \sum_{j=x+t-i}^{\infty} q^j g(0;j)\\
&= \sum_{i=1}^{x} q^{i-1} f(i) \sum_{j=x+1-i}^{\infty} q^j g(0;j) \sum_{t=1}^{\infty} 1 + \sum_{i=x+1}^{\infty} q^{i-1} f(i) \sum_{j=1}^{\infty} q^j g(0;j) \sum_{t=i-x+1}^{i+j-x} 1\\
&= \sum_{i=1}^{x} q^{i-1} f(i) \sum_{j=x+1-i}^{\infty} q^j g(0;j)\,(i+j-x) + \sum_{i=x+1}^{\infty} q^{i-1} f(i) \sum_{j=1}^{\infty} j q^j g(0;j)\ .
\end{aligned}
$$

In the first part of the expression introduce the outter summation on x, change orders to get

$$\begin{aligned}
&\sum_{x=1}^{\infty}\sum_{i=1}^{x} q^{i-1} f(i) \sum_{j=x+1-i}^{\infty} q^{j} g(0;j)\,(i+j-x) \\
=\;& \sum_{i=1}^{\infty} q^{i-1} f(i) \sum_{j=1}^{\infty} q^{j} g(0;j) \sum_{x=i}^{i+j-1} (i+j-x) \\
=\;& \sum_{i=1}^{\infty} q^{i-1} f(i) \sum_{j=1}^{\infty} q^{j} g(0;j) \frac{j\,(j+1)}{2} \\
=\;& \frac{q^{-1}}{2} \hat{f}(q) \left(\sum_{j=1}^{\infty} j\,(j-1)\, q^{j} g(0;j) + \sum_{j=1}^{\infty} 2 j q^{j} g(0;j) \right) \\
=\;& \frac{q}{2} \hat{f}(q) \hat{g}''(0;q) + \hat{f}(q)\hat{g}'(0;q) \ .
\end{aligned}$$

Do a similar procedure for the 2nd part and simplify

$$\begin{aligned}
\sum_{x=1}^{\infty}\sum_{i=x+1}^{\infty} q^{i} f(i) \hat{g}'(0;q) &= \hat{g}'(0;q) \sum_{i=2}^{\infty} q^{i} f(i) \sum_{x=1}^{i-1} 1 \\
&= q\hat{f}'(q)\hat{g}'(0;q) - \hat{g}'(0;q)\hat{f}(q) \ .
\end{aligned}$$

Put now the two parts together to get

$$\begin{aligned}
b(0;3) &= pq^{-1}\hat{f}(q) \sum_{x=1}^{\infty} q^{x} b(x;2) = p^{2} q^{-1} \hat{f}(q) \left[q\hat{f}(q)\hat{g}'(0;q) + \frac{q}{2}\hat{f}(q)\hat{g}''(0;q) \right] \\
&= \frac{1}{2} p^{2} \frac{d}{dz} \left[\hat{f}(z)^{2} \hat{g}'(0;z) \right] \Big|_{z=q} \ .
\end{aligned}$$

□

In the following section we will get the same result by a completely different approach.

3 On the number of claims before recovery

In this section we will work the probability function of K, the number of of claims occurring during a recovery time period, if ruin occurs. First, consider $u = 0$. Similar results for a positive initial surplus will follow easily. Consider the following result that relates the distribution of K and M with for an initial surplus zero.

Result 3 *For* $u = 0$

$$v(0;n) = p(0;n+1), \quad n = 0,1,2,...$$

Consider first the paper by (Gerber 1988), Subsection 4.4 and the number of claims in a process starting from zero reaches an integer positive value x $(x = 1,2,...)$. Given x and $S_k = X_1 + X_2 +$

$X_3 + \ldots + X_k$, $S_0 \equiv 0$, the conditional probability that the level x, is visited for the first time between the k-th and the $(k+1)$-th claim is

$$x\frac{1}{k!}\left(\frac{p}{q}\right)^k \frac{(S_k+x-1)!}{(S_k+x-k)!}q^{S_k+x}.$$

This probability means that when the process upcrosses x, k claims have occurred. The probability function of S_k is the convoluted function $f^{*k}(.)$, with $f^{*0}(0)=1$.

From the formula above, the unconditional probability, considering the law of total probability, comes

$$x\frac{1}{k!}\left(\frac{p}{q}\right)^k \sum_{i=0}^{\infty}\frac{(i+x-1)!}{(i+x-k)!}q^{i+x}f^{*k}(i) = x\frac{1}{k!}\left(\frac{p}{q}\right)^k E\left[\frac{(S_k+x-1)!}{(S_k+x-k)!}q^{S_k+x}\right], \tag{15}$$

where the expectation is evaluated with respect to the p.f. $f^{*k}(.)$. Yet from ,Gerber (1988) formula (21), we know that for $k=1,2,...$

$$\left(\frac{p}{q}\right)^k E\left[\frac{(S_k+x-1)!}{(S_k+x-k)!}q^{S_k+x}\right] = p^k\,\frac{d^{k-1}}{dz^{k-1}}\left[\hat{f}(z)^k z^{x-1}\right]\Big|_{z=q},$$

where $\hat{f}(z)^k$ is the p.g.f. of $f^{*k}(.)$. Hence, the probability (15) comes

$$x\frac{1}{k!}p^k\,\frac{d^{k-1}}{dz^{k-1}}\left[\hat{f}(z)^k z^{x-1}\right]\Big|_{z=q},\ k=1,2,...;\ x=1,2,...$$

Now, consider that x is the deficit at ruin with (defective) distribution function $G(0;x)$. A deficit at ruin is considered to be recovered when the the surplus gets to a positive integer level. If zero was not ruin then recovery from x ($x=1,2,...$) to level zero would take at least one period (in this case a one period recovery will only be possible if the deficit is 1. However "zero" is already ruin and recovery is attained with level "1". Then we have only a change in scale, that is, we can consider in this case x to be non-negative. Thus, the probability of occurring k claims in *recovery time* is given by, using the law of total probability

$$\begin{aligned} v(0;k) &= \sum_{x=0}^{\infty}\frac{1}{k!}p^k\,\frac{d^{k-1}}{dz^{k-1}}\left[\hat{f}(z)^k\frac{d}{dz}z^x\right]\Big|_{z=q} g(0;x) \\ &= \sum_{x=0}^{\infty}\frac{1}{k!}p^k\left(\sum_{n=0}^{k-1}\binom{k-1}{n}\left(\frac{d^n}{dz^n}\hat{f}(z)^k\Big|_{z=q}\right)\left(\frac{d^{k-n}}{dz^{k-n}}z^x\Big|_{z=q}\right)\right)g(0;x), \end{aligned}$$

using Leibnitz's rule for derivatives of products. Changing the order of summations we get the above probability as

$$\begin{aligned} v(0;k) &= \frac{1}{k!}p^k\left(\sum_{n=0}^{k-1}\binom{k-1}{n}\left(\frac{d^n}{dz^n}\hat{f}(z)^k\Big|_{z=q}\right)\left(\frac{d^{k-n}}{dz^{k-n}}\hat{g}(0;z)\Big|_{z=q}\right)\right) \\ &= \frac{1}{k!}p^k\,\frac{d^{k-1}}{dz^{k-1}}\left[\hat{f}(z)^k\hat{g}'(0;z)\right]\Big|_{z=q},\ k=1,2,... \end{aligned}$$

re-applying Leibnitz's rule. Note that the symbol $\frac{d^0}{dz^0}\left[\hat{f}(z)\hat{g}'(0;z)\right] = \hat{f}(z)\hat{g}'(0;z)$

For $k = 0$, retrieve formula (15) to give q^x, recall that $f^{*0}(0) = 1$ and $S_0 \equiv 0$. Hence we have

$$v(0;0) = \sum_{x=0}^{\infty} q^x g(0;x) = \hat{g}(0;q)$$

$\square$

Formulae for $v(0;n)$ and $p(0;n+1)$ are similar to expression (13) from Egídio dos Reis (2002). It's not a surprise since this is the discrete analogue model to the classical compound Poisson risk model. Similar reasons apply in formula (3.8) by Dickson (1994) about the distribution of the deficit at ruin and the distribution of the surplus one time unit prior to ruin, with $u = 0$. The relation between the distributions of K and M is explained by a dual argument like in the classical continuous time compound Poisson risk process or even like in its discrete version, see Egídio dos Reis (1997).

This kind of probability function fits in the so called *general Lagrangian-type distributions*, see for instance Johnson, Kotz & Kemp (1992), pp.13 and 99.

For positive initial surplus u, the distribution of the number of claims up to recovery $v(u;n)$ will have a similar expression, just substitute in $v(0;n)$ $\hat{g}(0;z)$ by $\hat{g}(u;z)$. To get $g(u;x)$ just retrieve it from (Dickson 1994).

4 On the time to ruin

We can work out recursion formulae for finite time survival probabilities by stating a renewal equation. Let $u = 1, 2, ...$ and $n = 2, 3...$, can state easily that

$$\begin{aligned} \delta(u,n) &= 1 - K(n) + \sum_{i=1}^{n-1} k(i) \sum_{j=1}^{u+i-1} f(j)\delta(u+i-j;n-i) \\ &= q^n + p\sum_{i=1}^{n-1} q^{i-1} \sum_{j=1}^{u+i-1} f(j)\delta(u+i-j;n-i) \end{aligned}$$

by considering that if the process is going to survive to time period n it can happen either with no claims at all or at least one claim and at the most maximum of occur it does with at the most with n claims, and by considering the first claim occurrence. For $n = 1$ we have

$$\delta(u,1) = 1 - K(1) + k(1)\sum_{j=1}^{u} f(j) = q + pF(u), \quad u = 1, 2, ...$$

With the formulae above we can compute recursively the finite time survival probability.

For $u = 0$, we can get explicit formulae for $\phi(0,n)$ from Li & Garrido (2002). First, and following Cheng, Gerber & Shiu (2000) or Li & Garrido (2002), Sections 3 and 2 respectively, we need to define $\rho(v)\epsilon\,(0,1)$ as the solution of the equation, for $v\epsilon\,(0,1)$:

$$\frac{p}{z}\hat{f}(z) + \frac{1-p}{z} = \frac{1}{v},$$

then from Li & Garrido (2002), Section 3, we have

$$\phi(0,n) = \frac{1}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}v^n}\left(\frac{v-1}{1-\rho(v)}\right)\bigg|_{v=0}, \quad n = 1, 2, ... \tag{16}$$

However, following a similar reasoning as above, we can also build a recursion for the survival probabilities noting that any claim on the first period will necessarily imply ruin. Hence, we have

$$\begin{aligned}
\delta(0,1) &= q \\
\delta(0,2) &= 1-K(2)+(1-K(1))\,k(1)f(1)=q^2+pqf(1) \\
\delta(0,n) &= 1-K(n)+(1-K(1))\sum_{i=1}^{n-2}k(i)\sum_{j=1}^{i}f(j)\delta(i+1-j;n-1-i),\ \ n=3,4,... \\
&= q^n+p\sum_{i=1}^{n-2}q^i\sum_{j=1}^{i}f(j)\delta(i+1-j;n-1-i)
\end{aligned}$$

5 The Binomial/Geometric model

Throughout this section we assume that the distribution of the individual claim amount distribution is geometric which probability and distribution functions are given by, respectively,

$$f(x)=(1-\alpha)\alpha^{x-1};\quad F(x)=1-\alpha^x,\ \ x=1,2,...$$

For this case, see Gerber (1988) and/or Dickson (1994),

$$\begin{aligned}
\psi(0) &= p/\left(1-\alpha\right) \\
f(0;x,y) &= p(1-\alpha)\alpha^{x+y},\ x=1,2,...;\ y=0,1,2,... \\
g(0;x) &= p\alpha^x,\ \ x=0,1,2,...
\end{aligned}$$

From this we get

$$\begin{aligned}
\hat{f}(z) &= \frac{(1-\alpha)z}{1-\alpha z} \\
\hat{g}(0;z) &= \frac{p}{1-\alpha z}\ .
\end{aligned}$$

Then using formulae (11)-(13) we can easily compute the (defective) probability function $b(0;n+1)$, using an analytical software like `Mathematica`, which we print out as an example the following:

$$
\begin{aligned}
b(0;1) &= \frac{1-q}{1-\alpha q} \\
b(0;2) &= \frac{\alpha q\,(1-\alpha)\,(1-q)^2}{(1-\alpha q)^3} \\
b(0;3) &= \frac{\alpha q(1-\alpha)^2\,(1-q)^3\,(1+\alpha q)}{(1-\alpha q)^5} \\
b(0;4) &= \frac{\alpha q(1-\alpha)^3(1-q)^4\,\left(1+3\,\alpha q+\alpha^2\,q^2\right)}{(1-\alpha q)^7} \\
b(0;5) &= \frac{\alpha q(1-\alpha)^4(1-q)^5\left(1+6\alpha q+6\,\alpha^2\,q^2+\alpha^3\,q^3\right)}{(1-\alpha\,q)^9} \\
b(0;6) &= \frac{\alpha q(1-\alpha)^5(1-q)^6\left(1+10\,\alpha\,q+20\,\alpha^2\,q^2+10\,\alpha^3\,q^3+\alpha^4\,q^4\right)}{(1-\alpha\,q)^{11}} \\
b(0;7) &= \frac{\alpha q(1-\alpha)^6\,(1-q)^7\left(1+15\,\alpha\,q+50\,\alpha^2\,q^2+50\,\alpha^3\,q^3+15\,\alpha^4\,q^4+\alpha^5\,q^5\right)}{(1-\alpha\,q)^{13}} \\
b(0;8) &= \frac{\alpha q(1-\alpha)^7(1-q)^8\left(1+21\,\alpha\,q+105\,\alpha^2\,q^2+175\,\alpha^3\,q^3+105\,\alpha^4\,q^4+21\,\alpha^5\,q^5+\alpha^6\,q^6\right)}{(1-\alpha\,q)^{15}} \\
b(0;9) &= \frac{\alpha q(1-\alpha)^8(1-q)^9\left(1+28\,\alpha\,q+196\,\alpha^2\,q^2+490\,\alpha^3\,q^3+490\,\alpha^4\,q^4+196\,\alpha^5\,q^5+28\,\alpha^6\,q^6+\alpha^7\,q^7\right)}{(1-\alpha\,q)^{17}} \\
b(0;10) &= (1-\alpha\,q)^{-19}\Big(\alpha q\,(1-\alpha)^9\,(1+q)^{10}\left(1+36\,\alpha q+336\alpha^2q^2\right) \\
&\qquad + 1176\alpha^3q^3+1764\,\alpha^4q^4+1176\alpha^5q^5+336\alpha^6q^6+36\alpha^7\,q^7+\alpha\,q^8\Big)
\end{aligned}
$$

Once computed the probabilities $b(0;n)$ we can compute the probabilities $b(u;n)$ for a positive integer u, using formulae (5) and recursion formulae (8) and (9). For that, just note that formula (4) can be in this case easily simplified. The sum

$$
\sum_{t=0}^{u-1} q^t\,[1-F\,(t)] = \sum_{t=0}^{u-1} (\alpha q)^t = \frac{1-(\alpha q)^u}{1-\alpha q}\ ,
$$

then

$$
b(u;1) = q^{-u}\left(\frac{p}{1-\alpha q} - p\frac{1-(\alpha q)^u}{1-\alpha q}\right) = \frac{p\alpha^u}{1-\alpha q} = \alpha^u b(0;1).
$$

For other values of $b(u;n)$, $n=2,3,...$, just use the recursion (8) and (9). Let us now consider the distribution of the time to ruin with initial surplus zero. First consider the equation

$$
\frac{p}{z}\hat{f}(z) + \frac{1-p}{z} = \frac{1}{v} \Leftrightarrow \frac{p}{z}\frac{(1-a)\,z}{1-az} + \frac{1-p}{z} = \frac{1}{v}
$$

We find $\rho(v)$ to be (there's another positive solution outside the range $(0,1)$).

$$
\begin{aligned}
\rho(v) &= \frac{1}{2\alpha}\left((\alpha-p)\,v+1-\sqrt{-2\alpha v-2pv+4\alpha pv-2\alpha pv^2+\alpha^2v^2+p^2v^2+1}\right) \\
&= \frac{1}{2\alpha}\left((\alpha-p)\,v+1-\sqrt{((\alpha-p)\,v+1)^2-4\alpha vq}\right)
\end{aligned}
$$

Using formulae (16) from Section 4 we can get formulae for $\phi(0,n)$ Like Li & Garrido (2002) refer *it is difficult to obtain $\phi(0,n)$ in closed form. Only for some special distributions of the claim amounts can explicit expressions be found.* Examples of such distributions are given. From the above formula we find

$$
\begin{aligned}
\phi(0,1) &= p \\
\phi(0,2) &= \alpha pq \\
\phi(0,3) &= \alpha pq\left(\alpha+p-2\alpha p\right) \\
\phi(0,4) &= \alpha pq\left(\alpha^2+p^2-5\alpha p\left(\alpha+p-\alpha p\right)+3\alpha p\right) \\
\phi(0,5) &= \alpha pq\left(\alpha+p-2\alpha p\right)\left(\alpha^2+p^2-7\alpha p\left(p+\alpha-\alpha p\right)+5\alpha p\right) \\
\phi(0,6) &= \alpha pq\left(\begin{array}{c} \alpha^4+p^4+10\alpha p^3+10\alpha^3 p-14\alpha p^4-14\alpha^4 p+20\alpha^2p^2-70\alpha^2p^3-70\alpha^3p^2 \\ +56\alpha^2p^4+140\alpha^3p^3+56\alpha^4p^2-84\alpha^3p^4-84\alpha^4p^3+42\alpha^4p^4 \end{array}\right) \\
\phi(0,7) &= \alpha pq\left(\alpha+p-2\alpha p\right)\left(\begin{array}{c} \alpha^4+p^4+14\alpha p^3+14\alpha^3 p-18\alpha p^4-18\alpha^4 p+36\alpha^2p^2-114\alpha^2p^3- \\ 114\alpha^3p^2+84\alpha^2p^4+228\alpha^3p^3+84\alpha^4p^2-132\alpha^3p^4-132\alpha^4p^3+66\alpha^4p^4 \end{array}\right)
\end{aligned}
$$

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