The Prediction Error of the Chain Ladder Method Applied to Correlated Run-off Triangles

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Abstract

It is shown how the distribution-free method of Mack (1993) can be extended in order to estimate the prediction error of the Chain Ladder method for a portfolio of several correlated run-off triangles.

Keywords:

Chain Ladder, Prediction Error, Correlation of Run-offs, Segmented Portfolio

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1 Introduction

In Mack (1993), a distribution-free method was developed in order to estimate the prediction error of Chain Ladder reserve estimates. For claims reserving purposes, an insurance company usually subdivides its portfolio into several subportfolios such that the development behavior of each subportfolio can be assumed to be homogeneous. Then, for each subportfolio, the Chain Ladder method can be applied in order to estimate the appropriate claims reserves and their prediction error.

But what is finally needed, are the claims reserves for the whole portfolio of the insurance company and their prediction error. Whereas the estimates of the claims reserves of each subportfolio can simply be added together in order to arrive at an estimate for the claims reserves of the whole portfolio, this is only the case for the prediction variances if the subportfolios can be assumed to be independent. But in long tail business, the development of different subportfolios is influenced to a substantial degree by the development behavior of bodily injury claims (medical and nursing costs). Therefore, subportfolios in general can not be assumed to be independent. Then, the question arises how the prediction error of the aggregated portfolio can be arrived at.

In this situation, applying the Chain Ladder method to the overall triangle and taking the prediction error from this calculation is not a good solution because already the reserve estimates obtained in this way will not be identical to the aggregation of the reserve estimates of the individual subportfolios, see e.g. Ajne (1994). Moreover, the aggregation of run-off triangles with different development patterns is like mixing apples and oranges and will normally lead to invalid results.

Therefore in this paper, a new, more sensible approach is developed. We assume that the correlation between two run-off triangles finds its manifestation in a fixed correlation coefficient between the individual development factors of the two corresponding development periods of the triangles. This correlation coefficient may depend on the development period, but not on the accident year. This assumption fits very well to the basic assumption behind the Chain Ladder method that the individual development factors of each development period fluctuate randomly around a fixed, but unknown age-to-age factor.

In actuarial practice, this approach enables the actuary to set up a range and a prudential margin for the reserves of the whole portfolio as required e.g. by several national accounting standards. The reserving bounds described in this paper are solely based on stochastic assumptions and on the observed data and not on assumed correlations between lines of business - as often done - which do not refer to the peculiarities of the underlying portfolio.

The paper is organized as follows: Section 2.1 gives the basic notations and repeats the recursive formulae for the prediction error of a single accident year for one triangle. From this, the prediction error of the total claims amount of all accident years is derived in section 2.2. In section 3, a second run-off triangle is introduced as well as the decisive assumption on the correlation between both triangles. In section 4, the recursive formulae for the prediction error of the sum of the two triangles are derived. In section 5, a numerical example is given including the derivation of a range for the best estimate of the portfolio reserve.

2 The prediction error for one run-off triangle

2.1 The prediction error of the ultimate claims amount of one accident year

Let $C_{ik} > 0$ be the cumulative claims amount of accident year i, $1 \leq i \leq n$, after k years of development, $1 \leq k \leq n$, for a certain subportfolio. The amounts C_{ik} with $i + k \leq n + 1$ are observable and we are interested in predicting the amounts C_{in} for $i = 2, 3, \ldots, n$. The Chain Ladder method does this recursively by

$$\widehat{C}_{ik} = \widehat{C}_{i,k-1} \cdot \widehat{f}_k \tag{1}$$

with starting value $\widehat{C}_{i,n+1-i} = C_{i,n+1-i}$ and age-to-age factor

$$\widehat{f}_k = \frac{\sum_{i=1}^{n+1-k} C_{ik}}{C_{<,k-1}} = \sum_{i=1}^{n+1-k} \frac{C_{i,k-1}}{C_{<,k-1}} \cdot F_{ik} \tag{2}$$

which is a weighted average of individual development factors

$$F_{ik} := \frac{C_{ik}}{C_{i,k-1}} \text{ with } C_{<,k-1} := \sum_{i=1}^{n+1-k} C_{i,k-1}.$$

In the following we consider numerous conditional expectation values and variances. To avoid there lengthy expressions we introduce some notation. The condition "T_k" means that all variables $\{C_{ij} | 1 \leq i \leq n, 1 \leq j \leq k, i+j \leq n+1\}$ of the run-off triangle up to and including development year k are given. Especially, the condition "T_n" indicates that the whole triangle is given. Furthermore, we use "T_{ik}" when the variables $\{C_{ij} | 1 \leq j \leq k\}$ are given.

On the basis of the stochastic assumptions (see Mack (1993) and (1999), where the further results of this section can be found, too)

$$\mathrm{E}(F_{ik} | T_{i,k-1}) = f_k, \tag{3}$$

$$\mathrm{Var}(F_{ik} | T_{i,k-1}) = \frac{\sigma_k^2}{C_{i,k-1}}, \tag{4}$$

for all $1 \leq i \leq n$ and $2 \leq k \leq n$ where f_k and σ_k^2 are unknown parameters, the estimation procedure (1) and (2) can be shown to be reasonable and conditionally unbiased, i.e. $\mathrm{E}(\widehat{f}_k | T_{k-1}) = f_k$ and $\mathrm{E}(\widehat{C}_{in} | T_{n+1-i}) = C_{i,n+1-i} f_{n+2-i} \cdot \ldots \cdot f_n = \mathrm{E}(C_{in} | T_{n+1-i})$, if the accident years are independent.

The prediction error $\mathrm{mse}(\widehat{C}_{in})$ for the ultimate claims amount of an accident year is defined as

$$\mathrm{mse}(\widehat{C}_{in}) := \mathrm{E}((C_{in} - \widehat{C}_{in})^2 | T_n)$$

because for reserving purposes only the future variability given the observable data is of interest. This can be written in the form

$$\mathrm{mse}(\widehat{C}_{in}) \quad = \quad \mathrm{Var}(C_{in} | T_{n+1-i}) + (\mathrm{E}(\widehat{C}_{in} | T_{n+1-i}) - \widehat{C}_{in})^2$$

which for estimation purposes is approximated by

$$\mathrm{mse}(\widehat{C}_{in}) \approx \mathrm{Var}(C_{in} | T_{n+1-i}) + \mathrm{Var}(\widehat{C}_{in} | T_{n+1-i}). \tag{5}$$

In (5) $\mathrm{Var}(C_{in}|T_{n+1-i})$ is called the random error and $\mathrm{Var}(\widehat{C}_{in}|T_{n+1-i})$ the estimation error. To keep the notation as simple as possible we omit from now on the conditions in the random error and the estimation error. So, whenever expectations like $\mathrm{E}(C_{ik}), \mathrm{E}(\widehat{C}_{ik})$ and variances like $\mathrm{Var}(C_{ik})$ or $\mathrm{Var}(\widehat{C}_{ik})$ are considered, in the strict sense $\mathrm{E}(C_{ik}|T_{n+1-i}), \mathrm{E}(\widehat{C}_{ik}|T_{n+1-i}), \mathrm{Var}(C_{ik}|T_{n+1-i})$ and $\mathrm{Var}(\widehat{C}_{ik}|T_{n+1-i})$ are meant. The exact formulations of the following derivations can be found in Mack(1993).

Now, we deduce recursions for the random error and for the estimation error. For this purpose, the equations (3) and (4) are used in the form

$$\mathrm{E}(C_{ik}|T_{i,k-1}) = C_{i,k-1} f_k,$$
$$\mathrm{Var}(C_{ik}|T_{i,k-1}) = C_{i,k-1}\sigma_k^2.$$

Then we have for $i + k > n + 1$

$$\begin{aligned}\mathrm{Var}(C_{ik}) &= \mathrm{E}(\mathrm{Var}(C_{ik}|T_{i,k-1})) + \mathrm{Var}(\mathrm{E}(C_{ik}|T_{i,k-1})) \\ &= \mathrm{E}(C_{i,k-1})\sigma_k^2 + \mathrm{Var}(C_{i,k-1}) f_k^2.\end{aligned}$$

This yields for the estimator $\widehat{Var}(C_{in})$ of the random error $\mathrm{Var}(C_{in})$ of the ultimate claims amount the recursion

$$\widehat{\mathrm{Var}}(C_{ik}) = \widehat{\mathrm{Var}}(C_{i,k-1}) \cdot \widehat{f_k^2} + \widehat{C}_{i,k-1}\widehat{\sigma}_k^2 \tag{6}$$

with the starting value

$$\widehat{\mathrm{Var}}(C_{i,n+1-i}) = 0$$

as $C_{i,n+1-i}$ is already known. An unbiased estimator of $\widehat{\sigma}_k^2$ is given by

$$\widehat{\sigma}_k^2 = \frac{1}{n-k}\sum_{i=1}^{n+1-k} C_{i,k-1}(F_{ik} - \widehat{f}_k)^2. \tag{7}$$

Similarly, $\widehat{C}_{ik} = \widehat{C}_{i,k-1}\widehat{f}_k$ yields

$$\begin{aligned}\mathrm{Var}(\widehat{C}_{ik}) &= \mathrm{E}(\mathrm{Var}(\widehat{C}_{i,k-1}\widehat{f}_k|T_{k-1})) + \mathrm{Var}(\mathrm{E}(\widehat{C}_{i,k-1}\widehat{f}_k|T_{k-1})) \\ &= \mathrm{E}(\widehat{C}_{i,k-1}^2\mathrm{Var}(\widehat{f}_k|T_{k-1})) + \mathrm{Var}(\widehat{C}_{i,k-1}) f_k^2.\end{aligned}$$

From this the following recursion for the estimator $\widehat{Var}(\widehat{C}_{in})$ of the estimation error $\mathrm{Var}(\widehat{C}_{in})$ of the ultimate claims estimate $\widehat{C}_{in}$ can be deduced:

$$\widehat{\mathrm{Var}}(\widehat{C}_{ik}) = \widehat{\mathrm{Var}}(\widehat{C}_{i,k-1})\widehat{f}_k^2 + \widehat{C}_{i,k-1}^2 \cdot \frac{\widehat{\sigma}_k^2}{C_{<,k-1}} \tag{8}$$

because

$$\mathrm{Var}(\widehat{f}_k|T_{k-1})) = \frac{\sigma_k^2}{C_{<,k-1}}. \tag{9}$$

The starting value for this recursion is

$$\widehat{\mathrm{Var}}(\widehat{C}_{i,n+1-i}) = 0$$

because $\widehat{C}_{i,n+1-i}$ is already observed. This yields the joint recursion for the estimate of the prediction error:

$$\widehat{\mathrm{mse}}(\widehat{C}_{ik}) = \widehat{\mathrm{mse}}(\widehat{C}_{i,k-1}) \cdot \widehat{f}_k^2 + \widehat{C}_{i,k-1}^2\left(\frac{\widehat{\sigma}_k^2}{\widehat{C}_{i,k-1}} + \frac{\widehat{\sigma}_k^2}{C_{<,k-1}}\right). \tag{10}$$

2.2 The prediction error of the total ultimate claims amount of one run-off triangle

Annual reports of insurance companies usually disclose estimates only for reserves and claims amounts for all accident years together. To estimate a range of those aggregated amounts, we have to consider the estimation error and prediction error for all accident years together.

C_{1n} is already known and no estimate is necessary. Therefore the first accident year adds nothing to the random error and the estimation error for the whole run-off. Taking this into account, the prediction error $\mathrm{mse}(\sum_{i=2}^{n} \widehat{C}_{in})$ for all accident years is defined as

$$\mathrm{mse}\left(\sum_{i=2}^{n} \widehat{C}_{in}\right) := \mathrm{E}\left(\left(\sum_{i=2}^{n}(C_{in} - \widehat{C}_{in})\right)^2 \middle| T_n\right).$$

We have (Mack (1993))

$$\begin{aligned}\mathrm{mse}\left(\sum_{i=2}^{n} \widehat{C}_{in}\right) &= \mathrm{Var}\left(\sum_{i=2}^{n} C_{in} \middle| T_n\right) + \left(\sum_{i=2}^{n}(\mathrm{E}(\widehat{C}_{in}|T_{n+1-i}) - \widehat{C}_{in})\right)^2 \\ &= \mathrm{Var}\left(\sum_{i=2}^{n} C_{in} \middle| T_n\right) + \sum_{i=2}^{n}(\mathrm{E}(\widehat{C}_{in}|T_{n+1-i}) - \widehat{C}_{in})^2\end{aligned}$$

$$
\begin{aligned}
& +2 \sum_{2\le i<j\le n} (\mathrm{E}(\widehat{C}_{in}|T_{n+1-i}) - \widehat{C}_{in})(\mathrm{E}(\widehat{C}_{jn}|T_{n+1-j}) - \widehat{C}_{jn}) \\
\approx\ & \mathrm{Var}\left(\sum_{i=2}^{n} C_{in}\,\middle|\, T_n\right) + \sum_{i=2}^{n} \mathrm{Var}(\widehat{C}_{in}|T_{n+1-i}) \\
& +2 \sum_{2\le i<j\le n} \mathrm{Cov}(\widehat{C}_{in}, \widehat{C}_{jn}|T_{n+1-i})
\end{aligned}
$$

The random error of the total ultimate loss amount is $\mathrm{Var}(\sum_{i=2}^{n} C_{in}|T_n)$. The estimation error $\mathrm{Var}(\sum_{i=2}^{n} \widehat{C}_{in})$ of the ultimate claims amount of all accident years together is

$$
\mathrm{Var}\left(\sum_{i=2}^{n} \widehat{C}_{in}\right) := \sum_{i=2}^{n} \mathrm{Var}(\widehat{C}_{in}|T_{n+1-i}) + \sum_{2\le i<j\le n} 2\,\mathrm{Cov}(\widehat{C}_{in}, \widehat{C}_{jn}|T_{n+1-i}). \quad (11)
$$

It is important to note, that $\mathrm{Var}(\sum_{i=1}^{n} \widehat{C}_{in})$ is only a notation for the right-hand-side in (11) and that it is not a variance since the right-hand-side of the definition (11) can not be rewritten as one single conditional variance due to the different conditions of the variances and covariances in the sum. This yields the following approximation for $\mathrm{mse}(\sum_{i=1}^{n} \widehat{C}_{in})$ (which is analogous to (5)):

$$
\mathrm{mse}\left(\sum_{i=1}^{n} \widehat{C}_{in}\right) \approx \mathrm{Var}\left(\sum_{i=1}^{n} C_{in}\,\middle|\, T_n\right) + \mathrm{Var}\left(\sum_{i=1}^{n} \widehat{C}_{in}\right).
$$

Again, we omit the conditions for simplicity. The random error $\mathrm{Var}(\sum_{i=2}^{n} C_{in})$ fulfills due to the independence of the accident years (which here implies the conditional independence, Mack (2002)) the equation

$$
\mathrm{Var}\left(\sum_{i=2}^{n} C_{in}\right) = \sum_{i=2}^{n} \mathrm{Var}(C_{in}). \quad (12)
$$

Of course (12) can be generalized to

$$
\mathrm{Var}\left(\sum_{i=n+2-k}^{n} C_{in}\right) = \sum_{i=n+2-k}^{n} \mathrm{Var}(C_{in}). \quad (13)
$$

(13) and the recursion (6) for the random error of one accident year yield the recursion

$$
\widehat{\mathrm{Var}}\left(\sum_{i=n+2-k}^{n} C_{ik}\right) = \widehat{\mathrm{Var}}\left(\sum_{i=n+3-k}^{n} C_{i,k-1}\right)\widehat{f_k^2} + \widehat{C}_{\ge,k-1}\widehat{\sigma}_k^2,
$$

with

$$\widehat{C}_{\geq,k-1} := \sum_{i=n+2-k}^{n} \widehat{C}_{i,k-1}. \tag{14}$$

Note, $\widehat{C}_{\geq,k-1}$ is the sum of the estimated claims amounts of development period $k-1$ plus the known amount $C_{n+2-k,k-1}$ of the actual calendar year. This recursion starts with $k=2$ since for the first development year all claims amounts $C_{i1}, 1 \leq i \leq n$, are already known. Here and in the following we use the convention that an empty summation is equal to 0.

For the estimation error $\mathrm{Var}(\sum_{i=2}^{n} \widehat{C}_{in})$ such a simple relation as (12) does not hold since all correlations between the ultimate claims amount estimates of different accident years have to be considered. A recursion for $\widehat{\mathrm{Cov}}(\widehat{C}_{in}, \widehat{C}_{jn})$ can be achieved by (with $k > n+1-i$ and $i < j$)

$$\begin{aligned}
\mathrm{Cov}(\widehat{C}_{ik}, \widehat{C}_{jk}) &= \mathrm{E}(\mathrm{Cov}(\widehat{C}_{i,k-1}\widehat{f}_k, \widehat{C}_{j,k-1}\widehat{f}_k | T_{k-1})) + \\
&\quad + \mathrm{Cov}(\mathrm{E}(\widehat{C}_{i,k-1}\widehat{f}_k | T_{k-1}), \mathrm{E}(\widehat{C}_{j,k-1}\widehat{f}_k | T_{k-1})) \\
&= \mathrm{E}(\widehat{C}_{i,k-1}\widehat{C}_{j,k-1}\mathrm{Var}(\widehat{f}_k | T_{k-1})) + \mathrm{Cov}(\widehat{C}_{i,k-1}, \widehat{C}_{j,k-1}) f_k^2
\end{aligned} \tag{15}$$

and using (9)

$$\widehat{\mathrm{Cov}}(\widehat{C}_{ik}, \widehat{C}_{jk}) = \widehat{\mathrm{Cov}}(\widehat{C}_{i,k-1}, \widehat{C}_{j,k-1}) \widehat{f_k^2} + \widehat{C}_{i,k-1}\widehat{C}_{j,k-1} \frac{\widehat{\sigma}_k^2}{C_{<,k-1}} \tag{16}$$

starting with $\widehat{\mathrm{Cov}}(\widehat{C}_{i,n+1-i}, \widehat{C}_{j,n+1-i}) = 0$ since $i < j$ and $C_{i,n+1-i}$ is known. (16) and (8) yield the following recursion for the estimation error:

$$\widehat{\mathrm{Var}}\left(\sum_{i=n+2-k}^{n} \widehat{C}_{ik} \right) = \widehat{\mathrm{Var}}\left(\sum_{i=n+3-k}^{n} \widehat{C}_{i,k-1} \right) \widehat{f_k^2} + (\widehat{C}_{\geq,k-1})^2 \frac{\widehat{\sigma}_k^2}{C_{<,k-1}}.$$

For the same reason as before, this recursion starts with $k=2$.

The recursions for the random error $\mathrm{Var}(\sum_{i=2}^{n} C_{in})$ and the estimation error $\mathrm{Var}(\sum_{i=2}^{n} \widehat{C}_{in})$ yield the recursion for the prediction error $\mathrm{mse}(\sum_{i=2}^{n} C_{in})$ of the total claims amounts for all accident years:

$$\begin{aligned}
\widehat{\mathrm{mse}}\left(\sum_{i=n+2-k}^{n} \widehat{C}_{ik} \right) &= \widehat{\mathrm{mse}}\left(\sum_{i=n+3-k}^{n} \widehat{C}_{i,k-1} \right) \widehat{f_k^2} \\
&\quad + (\widehat{C}_{\geq,k-1})^2 \left(\frac{\widehat{\sigma}_k^2}{\widehat{C}_{\geq,k-1}} + \frac{\widehat{\sigma}_k^2}{C_{<,k-1}} \right).
\end{aligned} \tag{17}$$

The recursion starts with $k = 2$. Using (4) and (9) it can be shown that (17) is the same recursion as the one already given in Mack (1999) for the prediction error. Structure of recursion (17) is the same as in (10). The only difference between the two recursions are the estimated claims amounts $\widehat{C}_{\geq,k-1}$ instead of the claims amount $\widehat{C}_{i,k-1}$ for one accident year in (10).

The prediction error $\text{mse}(\sum_{i=1}^{n} \widehat{C}_{in})$ gives the mean squared deviation between the estimated ultimate claims amount $\sum_{i=1}^{n} \widehat{C}_{in}$ and the true ultimate claims amount $\sum_{i=1}^{n} C_{in}$. The estimation error $\text{Var}(\sum_{i=1}^{n} \widehat{C}_{in})$ gives the mean squared deviation between the estimated ultimate claims amount $\sum_{i=1}^{n} \widehat{C}_{in}$ and the expected ultimate claims amount $\text{E}(\sum_{i=1}^{n} C_{in}) = \text{E}(\sum_{i=1}^{n} \widehat{C}_{in})$. Whereas the prediction error has to be used for the variability loading for a loss portfolio transfer, it is the estimation error which has to be used when assessing a confidence interval (range) around $\sum_{i=1}^{n} \widehat{C}_{in}$ for the best estimate $\text{E}(\sum_{i=1}^{n} C_{in})$ of $\sum_{i=1}^{n} C_{in}$.

3 A Chain ladder-type model for the correlation between two run-off triangles

Now assume we have another subportfolio with cumulative run-off data $\{D_{ik}\}$ in addition to the data $\{C_{ik}\}$ considered in section 2. Denote with g_k and τ_k^2 its Chain-Ladder parameters corresponding to f_k and σ_k^2, respectively, i.e. we have the following assumptions (20),(21) and the estimators

$$\widehat{g}_k = \frac{\sum_{i=1}^{n+1-k} D_{ik}}{D_{<,k-1}} \tag{18}$$

$$\widehat{\tau}_k^2 = \frac{1}{n-k} \sum_{i=1}^{n-k+1} D_{i,k-1}(G_{ik} - \widehat{g}_k)^2 \tag{19}$$

with

$$G_{ik} := \frac{D_{ik}}{D_{i,k-1}},$$

$$D_{<,k-1} := \sum_{i=1}^{n+1-k} D_{i,k-1}.$$

As before, the stochastic assumptions behind these estimators are

$$\mathrm{E}(G_{ik}|T_{i,k-1}) = g_k \tag{20}$$

$$\mathrm{Var}(G_{ik}|T_{i,k-1}) = \frac{\tau_k^2}{D_{i,k-1}}. \tag{21}$$

Here, $T_{i,k-1}$ is used for the set $\{D_{ij}|1 \leq j \leq k-1\}$. Again, the accident years $i = 1, \ldots, n$ are assumed to be independent.

For each of the data sets $\{C_{ik}\}$ and $\{D_{ik}\}$ the stochastic model for the Chain Ladder consists of an own submodel for each development period $k, 2 \leq k \leq n$. In order to arrive at formulae for expectation and variance of the ultimate claims D_{in} in terms of the observable amounts $\{D_{ik}, i+k \leq n+1\}$, the submodels are simply chained together.

Therefore it seems natural to restrict any assumptions regarding the correlation between the arrays $\{C_{ik}, 1 \leq i, k \leq n\}$ and $\{D_{ik}, 1 \leq i, k \leq n\}$ to each of the pairwise corresponding development years $k, 2 \leq k \leq n$, if we want to stay within the chain ladder world. In this sense, the natural generalization of (4) and (21) is the assumption

$$\mathrm{Cov}(F_{ik}, G_{ik}|T_{i,k-1}) = \frac{\rho_k}{\sqrt{C_{i,k-1}D_{i,k-1}}} \tag{22}$$

which is equivalent to assuming that the correlation coefficient between the individual development factors F_{ik} and G_{ik}

$$\frac{\mathrm{Cov}(F_{ik}, G_{ik}|T_{i,k-1})}{\sqrt{\mathrm{Var}(F_{ik}|T_{i,k-1}) \cdot \mathrm{Var}(G_{ik}|T_{i,k-1})}} = \frac{\rho_k}{\sigma_k \tau_k}$$

is constant for k fixed. Of course, we assume

$$\mathrm{Cov}(F_{ik}, G_{jk}|T_{i,k-1}) = 0 \text{ for } i \neq j,$$

because different accident years are assumed to be globally independent.

In these formulae "$T_{i,k-1}$" means, both sets of observable variables $\{C_{ij}|1 \leq j \leq k-1\}$ and $\{D_{ij}|1 \leq j \leq k-1\}$ are given. Moreover, we assume (3), (4), (20) and (21) to hold for this "$T_{i,k-1}$". Note, in this case (3), (4), (20) and (21) with the "$T_{i,k-1}$" as before still hold, being just a consequence of the new assumption, i.e. we have

$$\begin{aligned} \mathrm{E}(F_{ik}|C_{k-1}) &= \mathrm{E}(\mathrm{E}(F_{ik}|C_{k-1}, D_{k-1})|C_{k-1}) = f_k, && (23) \\ \mathrm{Var}(F_{ik}|C_{k-1}) &= \mathrm{E}(\mathrm{Var}(F_{ik}|C_{k-1}, D_{k-1})|C_{k-1}) \\ &\quad + \mathrm{Var}(\mathrm{E}(F_{ik}|C_{k-1}, D_{k-1})|C_{k-1}) \\ &= \frac{\sigma_k^2}{C_{i,k-1}}. && (24) \end{aligned}$$

Here, the notation C_{k-1} is used for the set $\{C_{ij}|1 \leq j \leq k-1\}$ and D_{k-1} for $\{D_{ij}|1 \leq j \leq k-1\}$. Aside, (23) and (24) justify actuarial practice using the Chain Ladder method for a subportfolio without considering in addition the observables of all other segments of the portfolio. Also, the assumption allows us to use for one run-off the condition "$T_{i,k-1}$" without specifying whether "$T_{i,k-1}$" relates to one or two given sets of observables since it is without relevance. Considering variables related to both run-offs, then "$T_{i,k-1}$" and also "T_k" are related of course to both sets.

In analogy of the estimation of σ_k^2 and τ_k^2, the new parameter ρ_k can be estimated by

$$\widehat{\rho}_k = \frac{1}{n-k-1+w_k^2} \sum_{i=1}^{n+1-k} \sqrt{C_{i,k-1} D_{i,k-1}} (F_{ik} - \widehat{f}_k)(G_{ik} - \widehat{g}_k) \qquad (25)$$

with

$$w_k^2 := \frac{(\sum_{i=1}^{n+1-k} \sqrt{C_{i,k-1} D_{i,k-1}})^2}{C_{<,k-1} \cdot D_{<,k-1}}.$$

The factor $\frac{1}{n-k-1+w_k^2}$ instead of $\frac{1}{n-k}$ as for $\widehat{\sigma}_k^2$ and $\widehat{\tau}_k^2$ ensures that the estimator $\widehat{\rho}_k$ for ρ_k is unbiased. Note, that w_k^2 is positive and ≤ 1 (Cauchy-Schwarz inequality).

4 Estimation of the prediction error of the sum of two run-off triangles

First of all, we have to define the prediction error $\mathrm{mse}(\widehat{C}_{in} + \widehat{D}_{in})$ for the ultimate claims amount of an accident year of the portfolio. It is defined analogously as for one run-off:

$$\mathrm{mse}(\widehat{C}_{in} + \widehat{D}_{in}) := \mathrm{E}((C_{in} + D_{in} - (\widehat{C}_{in} + \widehat{D}_{in}))^2 | T_n).$$

This can be approximated by

$$\mathrm{mse}(\widehat{C}_{in} + \widehat{D}_{in}) \approx \mathrm{Var}(C_{in} + D_{in} | T_{n+1-i}) + \mathrm{Var}(\widehat{C}_{in} + \widehat{D}_{in} | T_{n+1-i}).$$

Here, $\mathrm{Var}(C_{in} + D_{in} | T_{n+1-i})$ is the random error and $\mathrm{Var}(\widehat{C}_{in} + \widehat{D}_{in} | T_{n+1-i})$ is the estimation error. Again, we omit these conditions in the following.

Based on the assumption (22) which can be rewritten as

$$\mathrm{Cov}(C_{ik}, D_{ik} | T_{i,k-1}) = \sqrt{C_{i,k-1} D_{i,k-1}} \rho_k,$$

we now can calculate the random error $\mathrm{Var}(C_{in} + D_{in})$ and the estimation error $\mathrm{Var}(\widehat{C}_{in} + \widehat{D}_{in})$ of the combined triangle $\{C_{ik} + D_{ik} | i + k \leq n + 1\}$. We have

$$\mathrm{Var}(C_{in} + D_{in}) = \mathrm{Var}(C_{in}) + 2\mathrm{Cov}(C_{in}, D_{in}) + \mathrm{Var}(D_{in})$$

and therefore, in addition to the recursions considered before, we need only a recursion for $\mathrm{Cov}(C_{in}, D_{in})$, too. From

$$\begin{aligned} \mathrm{Cov}(C_{ik}, D_{ik}) &= \mathrm{E}(\mathrm{Cov}(C_{ik}, D_{ik} | T_{i,k-1})) \\ &\quad + \mathrm{Cov}(\mathrm{E}(C_{ik} | T_{i,k-1}), \mathrm{E}(D_{ik} | T_{i,k-1})) \\ &= \mathrm{E}(\sqrt{C_{i,k-1} D_{i,k-1}}) \rho_k + \mathrm{Cov}(C_{i,k-1}, D_{i,k-1}) f_k g_k \end{aligned}$$

we deduce the recursion (for $i + k > n + 1$)

$$\widehat{\mathrm{Cov}}(C_{ik}, D_{ik}) = \widehat{\mathrm{Cov}}(C_{i,k-1}, D_{i,k-1}) \widehat{f}_k \widehat{g}_k + \sqrt{\widehat{C}_{i,k-1} \widehat{D}_{i,k-1}} \widehat{\rho}_k \tag{26}$$

for the estimated covariance between C_{ik} and D_{ik}. The starting value is

$$\widehat{\mathrm{Cov}}(C_{i,n+1-i}, D_{i,n+1-i}) = 0$$

as both variables have already been observed. Similarly, for the estimation error we have

$$\mathrm{Var}(\widehat{C}_{in} + \widehat{D}_{in}) = \mathrm{Var}(\widehat{C}_{in}) + 2\mathrm{Cov}(\widehat{C}_{in}, \widehat{D}_{in}) + \mathrm{Var}(\widehat{D}_{in})$$

and

$$\begin{aligned}\mathrm{Cov}(\widehat{C}_{ik}, \widehat{D}_{ik}) &= \mathrm{E}(\mathrm{Cov}(\widehat{C}_{i,k-1}\widehat{f}_k, \widehat{D}_{i,k-1}\widehat{g}_k | T_{k-1})) \\ &\quad + \mathrm{Cov}(\mathrm{E}(\widehat{C}_{i,k-1}\widehat{f}_k | T_{k-1}), \mathrm{E}(\widehat{D}_{i,k-1}\widehat{g}_k | T_{k-1})) \\ &= \mathrm{E}(\widehat{C}_{i,k-1}\widehat{D}_{i,k-1}\mathrm{Cov}(\widehat{f}_k, \widehat{g}_k | T_{k-1})) \\ &\quad + \mathrm{Cov}(\widehat{C}_{i,k-1}, \widehat{D}_{i,k-1}) f_k g_k\end{aligned}$$

as well as

$$\begin{aligned}\mathrm{Cov}(\widehat{f}_k, \widehat{g}_k | T_{k-1}) &= \mathrm{Cov}\left(\sum_{j=1}^{n+1-k} \frac{C_{j,k-1}}{C_{<,k-1}} F_{jk}, \sum_{j=1}^{n+1-k} \frac{D_{j,k-1}}{D_{<,k-1}} G_{jk} \middle| T_{k-1} \right) \\ &= \sum_{j=1}^{n+1-k} \frac{C_{j,k-1}}{C_{<,k-1}} \frac{D_{j,k-1}}{D_{<,k-1}} \mathrm{Cov}(F_{jk}, G_{jk} | T_{k-1}) \\ &= \sum_{j=1}^{n+1-k} \frac{\sqrt{C_{j,k-1} D_{j,k-1}}}{C_{<,k-1} D_{<,k-1}} \rho_k\end{aligned}$$

Taken together, we have the recursion

$$\begin{aligned}\widehat{\mathrm{Cov}}(\widehat{C}_{ik}, \widehat{D}_{ik}) &= \widehat{\mathrm{Cov}}(\widehat{C}_{i,k-1}, \widehat{D}_{i,k-1}) \cdot \widehat{f}_k \widehat{g}_k \\ &\quad + \frac{\widehat{C}_{i,k-1}\widehat{D}_{i,k-1}}{C_{<,k-1} \cdot D_{<,k-1}} \widehat{\rho}_k \sum_{j=1}^{n+1-k} \sqrt{C_{j,k-1} D_{j,k-1}}\end{aligned} \tag{27}$$

with starting value

$$\widehat{\mathrm{Cov}}(\widehat{C}_{i,n+1-i}, \widehat{D}_{i,n+1-i}) = 0.$$

This completes the derivation of formulae for the random error, for the estimation error and taken together for the prediction error for the ultimate claims amount of one accident year in a portfolio consisting of two correlated subportfolios.

For actuarial evaluation of the liabilities of a whole portfolio and their potential adverse development the errors of the ultimate claims amount for all accident years of the portfolio are important quantities. The prediction error of the total ultimate claims amount $\sum_{i=2}^{n}(\widehat{C}_{in} + \widehat{D}_{in})$ is

$$\mathrm{mse}\left(\sum_{i=2}^{n} (\widehat{C}_{in} + \widehat{D}_{in}) \right) := \mathrm{E}\left(\left(\sum_{i=2}^{n} (C_{in} + D_{in} - (\widehat{C}_{in} + \widehat{D}_{in})) \right)^2 \middle| T_n \right)$$

$$
\begin{aligned}
&= \mathrm{Var}\bigg(\sum_{i=2}^{n}(C_{in}+D_{in})\bigg|\, T_n\bigg) \\
&\quad + \bigg(\sum_{i=2}^{n}(\mathrm{E}(\widehat{C}_{in}+\widehat{D}_{in}|T_{n+1-i}) - (\widehat{C}_{in}+\widehat{D}_{in}))\bigg)^2 \\
&= \mathrm{Var}\bigg(\sum_{i=2}^{n}(C_{in}+D_{in})\bigg|\, T_n\bigg) \\
&\quad + \bigg(\sum_{i=2}^{n}(\mathrm{E}(\widehat{C}_{in}|T_{n+1-i}) - \widehat{C}_{in}) + \sum_{i=2}^{n}(\mathrm{E}(\widehat{D}_{in}|T_{n+1-i}) - \widehat{D}_{in})\bigg)^2 \\
&\approx \mathrm{Var}\bigg(\sum_{i=2}^{n}(C_{in}+D_{in})\bigg|\, T_n\bigg) \\
&\quad + \mathrm{Var}\bigg(\sum_{i=2}^{n}\widehat{C}_{in}\bigg) + \mathrm{Var}\bigg(\sum_{i=2}^{n}\widehat{D}_{in}\bigg) + \sum_{1\le i,j\le n} 2\,\mathrm{Cov}(\widehat{C}_{in},\widehat{D}_{jn}|T_{n+1-min(i,j)}),
\end{aligned}
$$

where $min(i,j)$ denotes the Minimum of i and j. The first term is the random error, the last three together are the estimation error. Note, here we used the notation $\mathrm{Var}(\sum_{i=2}^{n}\widehat{C}_{in})$ and $\mathrm{Var}(\sum_{i=2}^{n}\widehat{D}_{in})$ as introduced in section 2.2.

The random error $\mathrm{Var}(\sum_{i=2}^{n}(C_{in}+D_{in}))$ - omitting conditions - can be written as

$$
\begin{aligned}
&\mathrm{Var}\bigg(\sum_{i=2}^{n}(C_{in}+D_{in})\bigg) \\
&= \mathrm{Var}\bigg(\sum_{i=2}^{n}C_{in}\bigg) + 2\,\mathrm{Cov}\bigg(\sum_{i=2}^{n}C_{in},\sum_{i=2}^{n}D_{in}\bigg) + \mathrm{Var}\bigg(\sum_{i=2}^{n}D_{in}\bigg)
\end{aligned}
$$

For the random errors $\mathrm{Var}(\sum_{i=2}^{n}C_{in})$ and $\mathrm{Var}(\sum_{i=2}^{n}D_{in})$ we have already derived recursions in section 2. Therefore, only a recursion for the covariance of $\sum_{i=2}^{n}C_{in}$ and $\sum_{i=2}^{n}D_{in}$ is needed. Due to the (conditional) independence of the accident years we have

$$
\mathrm{Cov}\bigg(\sum_{i=2}^{n}C_{in},\sum_{i=2}^{n}D_{in}\bigg) = \sum_{i=2}^{n}\mathrm{Cov}(C_{in},D_{in}).
$$

Using the recursions for $\widehat{\mathrm{Cov}}(C_{ik},D_{ik}), 2\le i\le n$ yields the recursion

$$
\begin{aligned}
&\widehat{\mathrm{Cov}}\bigg(\sum_{i=n+2-k}^{n}C_{ik},\sum_{i=n+2-k}^{n}D_{ik}\bigg) \\
&= \widehat{\mathrm{Cov}}\bigg(\sum_{i=n+3-k}^{n}C_{i,k-1},\sum_{i=n+3-k}^{n}D_{i,k-1}\bigg)\widehat{f}_k\widehat{g}_k + \widehat{\rho}_k\sum_{i=n+2-k}^{n}\sqrt{\widehat{C}_{i,k-1}\widehat{D}_{i,k-1}}
\end{aligned}
$$

starting with $k = 2$ since for the first development year all C_{i1} and D_{i1} are known.

For the covariances $\mathrm{Cov}(\widehat{C}_{in}, \widehat{D}_{jn})$ in the estimation error we proceed as in (15) and for (26). This leads to the recursion

$$\begin{aligned} \widehat{\mathrm{Cov}}(\widehat{C}_{ik}, \widehat{D}_{jk}) &= \widehat{\mathrm{Cov}}(\widehat{C}_{i,k-1}, \widehat{D}_{j,k-1}) \widehat{f}_k \widehat{g}_k + \\ &+ \frac{\widehat{C}_{i,k-1} \widehat{D}_{j,k-1}}{C_{<,k-1} \cdot D_{<,k-1}} \widehat{\rho}_k \sum_{m=1}^{n+1-k} \sqrt{C_{m,k-1} \cdot D_{m,k-1}} \end{aligned} \tag{28}$$

with starting value $k = n + 1 - min(i, j)$. Recursion (27) is a special case of (28). The recursion for $\sum_{i,j} \widehat{\mathrm{Cov}}(\widehat{C}_{in}, \widehat{D}_{jn})$ is then

$$\begin{aligned} \sum_{i,j=n+2-k}^{n} \widehat{\mathrm{Cov}}(\widehat{C}_{ik}, \widehat{D}_{jk}) = & \sum_{i,j=n+3-k}^{n} \widehat{\mathrm{Cov}}(\widehat{C}_{i,k-1}, \widehat{D}_{j,k-1}) \widehat{f}_k \widehat{g}_k \\ & + \widehat{C}_{\geq,k-1} \widehat{D}_{\geq,k-1} \widehat{\rho}_k \frac{\sum_{i=1}^{n+1-k} \sqrt{C_{i,k-1} \cdot D_{i,k-1}}}{C_{<,k-1} \cdot D_{<,k-1}} \end{aligned}$$

starting with $k = 2$ (cf. definition of $\widehat{C}_{\geq,k-1}$ in (14)). This recursion completes the derivation of the recursions for the estimation error and the prediction error for the ultimate claims amounts estimates of the sum of two correlated subportfolios. The extension to more than two subportfolios is obvious.

5 Numerical example

In our numerical example we use data published by the Reinsurance Association of America (RAA) in their historical loss development study (RAA (2001)). Cumulative incurred losses $\{C_{ik}\}$ of General Liability (GL) reinsurance business are given in Table 1. Table 2 contains the corresponding data $\{D_{ik}\}$ for Auto Liability (AL) reinsurance business. For details see RAA (2001). For a demonstration of our approach with these runoffs we assume that the claims development comprised in each of these triangles is homogeneous so that we can limit our analysis to the two given triangles and we have not to perform any analysis of subtriangles. Moreover, we assume for simplicity that the development stops after the fourteenth year for both run-offs. Therefore we dispense with any extrapolation beyond the fourteenth development year.

The Chain-Ladder method yields the development factors $\widehat{f}_k$ (for the GL run-off) and $\widehat{g}_k$ (for AL run-off) and the parameter estimates $\widehat{\sigma}_k$ (GL run-off) and $\widehat{\tau}_k$ (AL run-off) as given in Table 3. The parameters σ_{14} and τ_{14} which can not be estimated via (7) and (19) since there is only one individual development factor in each run-off for the fourteenth development year, are selected as

$$\widehat{\sigma}_{14}^2 = min(\widehat{\sigma}_{13}^4/\widehat{\sigma}_{12}^2, \widehat{\sigma}_{12}^2)$$

(see Mack (1993)) and $\widehat{\tau}_{14}^2$ analogous. The parameters w_k^2 in the row 5 of Table 3 show that w_k^2 is approximately 1 for all development years in this example i.e. we have $\frac{1}{n-k-1+w^2} \approx \frac{1}{n-k}$. Rows 6 and 7 of Table 3 contain the estimate for ρ_k and for the correlation coefficient $\rho_k/(\sigma_k\tau_k)$. For development years 8 and 12 ρ_k is negative. This should not be overstated since the estimate of the covariance parameter ρ_k is based here only on seven and three observations, respectively and has no substantial contribution to the total errors due to the small ρ_k in the later development periods. ρ_k decays rapidly with respect to k, as it is usually the case for σ_k^2 and τ_k^2 (and also for f_k and g_k). Row 7 shows $\rho_k/(\sigma_k\tau_k)$ which gives the correlation coefficient of the individual development factors. It can be seen, that it is quite stable in the first seven development years. Table 4 shows for each accident year i the estimated reserve $\hat{C}_{in} - C_{i,n+1-i}$ for GL run-off and the estimated reserve $\hat{D}_{in} - D_{i,n+1-i}$ for AL run-off and the sum of these two reserves ("Portfolio"). In the last column of Table 4 the estimated reserve is given when aggregating first both data triangles to one single triangle and then estimating the reserve with the Chain-Ladder method. This (nonsense) calculation is only done for comparison purposes and is denoted "overall calculation" in the following and in the tables. The example shows that the overall calculation leads to another result which can be considered as unusable here since run-offs with different development patterns were added together. The reserve is about 265 Mio. lower than the one by separate calculation of the GL and AL reserves. To evaluate this difference we have to consider the variability in our estimates.

Tables 5-7 show the square roots of the random error, the estimation error and the prediction error, respectively for GL run-off in column 1 and AL run-off in column 2. The column "Portfolio" of these tables shows the corresponding figures for the whole portfolio consisting of the GL and AL subportfolios, computed with our method as described in section 4 taking into account the correlation between the individual development factors. Column 3a gives the implied average coefficent of correlation, i.e. the solution $\rho(X,Y)$ of the equation

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\rho(X,Y)\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)} \qquad (29)$$

where X and Y are the reserves of the GL and AL run-off, and $\mathrm{Var}(X), \mathrm{Var}(Y)$ and $\mathrm{Var}(X+Y)$ are the squares of corresponding errors from columns (1)-(3). Columns 4 to 6 show the results of the calculation (29) but assuming a positive correlation of +1, no correlation and a negative correlation -1 between the corresponding individual development factors of all columns of run-off 1 and 2. In column 7 the roots of the errors are given for the overall calculation. The errors for the reserve of each accident years and all accident years together are between the ones assuming no correlation and a correlation equal to 1. Note that, the overall calculation yields for the accident year 1988 and 1989 errors which are larger than the corresponding error of the portfolio under the assumption of a complete positive correlation between both run-offs. This is a further hint that the overall calculation is not suited for the estimation of portfolio reserves and its range.

Assuming a log-normal distribution for the best estimate of the reserve with mean and variance according to table 4 and 6 a reasonable range for the best estimate for all accident years of our portfolio consisting of the GL and the AL runoff can be calculated. The mean is equal to the estimated reserve and the variance equal to the estimation error. We use the interval containing 50% probability around the mean with 25%-probability on each side as range. This is a fair compromise between a non-informative 99%-range and the straight point estimate which would not contain the true reserve with 100% probability. This 50%-range leads to a lower bound of 8.008.292 and an upper bound of 8.438.171. Within this range, each amount can be taken as best estimate. The reserve estimate of the overall calculation is outside this range, since it is below the lower bound. This shows again that the overall calculation is not reasonable.

6 Final remarks

Correlations between run-off triangles are often attributed to the claims inflation affecting all or most of the run-offs of a portfolio in a similar way. For this reason, it is obvious to derive the correlation between the reserves from the correlation between the estimated inflation rates in the run-offs. But, since the inflation affects the diagonals in the run-offs, the basic Chain Ladder model assumption of independence of the accident years is violated. Therefore, calculating reserve ranges by using calendar year based correlations in conjunction with reserves estimated with the Chain-Ladder method is inadvisable. In principle, all calendar year based dependences should be removed from the run-offs, before the reserves are calculated with the Chain-Ladder method.

Furthermore, the inflation rate of a calendar year does not affect the accident years of a run-off in the same way, since the payments are for different types of claims due to their different development periods. For instance, considering a fixed calendar year in a general liability portfolio, in earlier development years mainly property damages are paid while for later development years payments of bodily injury claims dominate. Moreover, the portions of claims affected by the inflation of a fixed calendar year related to all noticed claims per accident year are different due to the different development periods. To summarize, the inflation of a calendar year does not influence the affected cells of the triangle in the same way.

Our approach comes up with an individual correlation coefficent for each development period. In contrast to this, some other approaches measure the correlation between two run-offs by a single number, e.g. by a correlation coefficient. If one likes to do this with our approach - even though it is not in line with the Chain-Ladder model - one can simply set in the basic assumption for the covariance in section 3 $\rho_k = \psi \sigma_k \tau_k$ with σ_k and τ_k as before and estimate ψ now by using data from all development periods. This simplified model implies a constant correlation coefficient ψ for all development years and ψ can be compared with estimated correlations of other approaches.

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