SOLVENCY REQUIREMENTS FOR LIFE ANNUITIES

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Abstract
The traditional approach to reserving focuses on the expected value of future obligations, which is assessed adopting a prudent valuation basis, representing interest rates and mortality from the time of valuation on. Usual assumptions consist in deterministic interest rates and in mortality levels obtained from past data.

In solvency investigations, usual requirements for assessing the capability of the insurer to meet future obligations imply a comparison between the random profile of the portfolio fund and the random profile of the portfolio reserve. However, this approach might be lacking for some insurance covers, in particular when lifetime living benefits are involved. Actually in this case, owing to the uncertainty in mortality trends at adult ages (named longevity risk) and in the future performance of financial markets, it is difficult to judge on the appropriateness of a reserve profile based on a deterministic view of the future scenario. Hence, a comparison between the portfolio fund and the reserve could be meaningless, in particular as far as the capability of the fund to meet future obligations on realistic grounds is concerned.

In this paper, we investigate solvency for a portfolio of life annuities comparing the portfolio fund with the random present value of future obligations, hence without explicit reference to the reserve. Several requirements are considered, which leads to a “required fund” that must be financed both with premiums and with shareholders’ capital. Implications of this approach on the management of the life business are discussed.

Keywords: required solvency margin, risk-based capital, longevity risk, investment risk, pooling and non-pooling risks

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1. Introduction

A rapidly moving scenario is the current framework of life insurance business. In many European countries, tariffs are no longer subject to approval by the supervisory authority, thanks to the introduction of the Third E.U. Life Directive. Moreover, new types of risks affect the management of a life insurance office. A very important example is provided by the so-called longevity risk, arising from long-term mortality trends at adult and old ages. So, efficient tools must be used for monitoring the capability of the insurer to meet future obligations.

Meeting future obligations requires: (i) to set up the portfolio reserve, (ii) to allocate the solvency margin (or risk-based capital). The traditional approach to reserving focuses (at least at individual level) on the expected value of future obligations, which is assessed adopting a prudent valuation basis, representing interest rates and mortality from the time of valuation on. Usual assumptions consist in deterministic interest rates and in mortality levels obtained from past data, possibly via mortality projections.

In solvency investigations, a comparison between the random profile of the portfolio fund (i.e. the assets) and the random profile of the portfolio reserve is usually performed. However, this approach might be lacking for some insurance covers, in particular when lifetime living benefits are involved. Actually, owing to the uncertainty in mortality trends at adult and old ages and in the future performance of financial markets, it is difficult to judge on the appropriateness of a reserve profile simply based on a deterministic view of the future scenario. Hence, comparing the portfolio fund and the reserve could be meaningless, in particular as far as the capability of the fund to meet future obligations on realistic grounds is concerned.

In this paper, we investigate solvency for a portfolio of life annuities comparing the portfolio fund with the random present value of future obligations, hence without explicit reference to a portfolio reserve, whatever its valuation basis might be. Several requirements are considered, leading to a "required fund" that must be financed both with premiums and with shareholders' capital. From a traditional point of view, the actual fund (which should be greater or equal to the required fund) can be split into a portfolio reserve (calculated with a given valuation basis) and a solvency margin (as a residual). Implications of this approach on the management of the life business are discussed.

The paper is organized as follows. In Section 2 risks inherent in a life portfolio and solvency requirements are discussed. A model for risk investigations and solvency assessment is presented in Section 3. With reference to a portfolio of life annuities a procedure, based on stochastic simulation, for assessing the demographic risk, including random fluctuations risk as well as longevity risk, is described in Section 4. Results coming from numerical investigations embedding the financial risk are presented and discussed in Section 5. Finally, Section 6 presents some concluding remarks and suggestions for future research.
Literature on risks and solvency in life insurance is very extensive. Nevertheless, specific references about longevity risk and related solvency requirements are rather scanty. Some bibliographic remarks can help in understanding the genesis of the present paper. First, some references concerning mortality follow. Mortality trends at old-adult ages reveal decreasing annual probabilities of death; the reader can refer to Benjamin and Soliman (1993), also for a list of references. Population mortality trends are investigated in many countries; see Macdonald (1997) and Macdonald et al. (1998). Mortality improvements have obvious effects on pricing and reserving for life annuities; to this purpose see, for example, Marocco and Pitacco (1998) and Olivieri (1999). More generally, mortality trends affect any insurance cover providing some kind of “living benefits”, such as Long Term Care benefits or lifetime sickness benefits; see Olivieri and Pitacco (1999). These three papers focus, in particular, on randomness inherent in mortality projections, from which the longevity risk originates.

Some references concerning solvency requirements in life insurance follow. The approach to solvency evaluation, leading to the definition of a required fund and disregarding in principle the portfolio reserve, adopted in the present article, has been suggested by the paper of the Faculty of Actuaries’ Solvency Working Party (1986); readers can refer to this paper also for an extensive list of references, including the first studies on solvency in the European Community. The book by Webb and Lilly (1995) is specifically devoted to the Risk-based Capital model for life insurance adopted in the U.S. Sources originating risks in life insurance, life office results representing the effects of risks, solvency requirements and the need for convenient short-cut formulae are discussed by Pitacco (1999).

2. Risks and solvency in life insurance
(a) Types of risks. A risk arises from some expectation, usually consisting in an expected scenario and hence represented by an expected result. A risk reveals itself in the possibility that the actual result is better or worse than the expected one. In order to state a terminology to be used in the following sections, a description of the “causes” originating risks in life insurance follows. Causes of risk can be easily found in items that constitute the scenario in which a life office operates.

The mortality risk is originated by the random lifetimes of the insureds; it can be split as follows. The risk of random fluctuations of the actual number of deaths from the expected one arises from purely random variability of mortality. Mortality trends different from the forecasted trend originate the risk of systematic deviation from the expected number of deaths. In particular the longevity risk originates from long-term mortality trends at adult and old ages. Finally, the catastrophe risk arises from the possibility of an exceptionally high mortality due to particular events (natural disasters, wars, etc.).

The investment risk originates from facts concerning the financial market in
which the insurer invests. The investment risk can be split as follows. The performance risk concerns policies with financial guarantees and arises from the possibility that the asset value is lower than the liability value. The basis risk mainly concerns unit-linked products and originates from the possibility that the insurer’s investment portfolio differs from the set of assets whose performance determines the unit value. The risk of default arises from the possibility that the institution issuing the financial instruments purchased by the insurer does not pay the promised amount at maturity.

Expense risks originate from the behaviour of insurer’s expenses compared with the income from expense loadings included in the premiums. Option risks are determined by choices of the policyholders. Actually, many recent products should be regarded as “options packages”, owing to the flexibility allowed by the policy conditions in terms of annual premium amount, annuity options, insurability options, switching options in unit linked contracts, etc.

Finally, economic risk is any risk not classified in the above categories and arising from the evolution of the economic scenario. Inflation, taxes and exchange rates constitute examples of items originating the economic risk.

Correlations among risks clearly emerge from the above considerations. For example, the policyholders’ behaviour also reflects financial and economic issues. So, increasing rates of interest may encourage surrenders should the bonus policy adopted by the insurer be considered not enough profitable.

The “severity” of the various types of risk is influenced by the size of the portfolio and the amount distribution of the assured benefits. As far as the role of the portfolio size is concerned, risks can be classified into (a) pooling risks, whose effect (conveniently assessed) decreases as the size of the portfolio increases, and (b) non-pooling risks, whose effect increases as the size of the portfolio increases. Within the category of mortality risks, random fluctuations constitute a pooling risk, as a larger number of policies helps compensations. On the contrary, longevity risk is a non-pooling risk, since systematic deviations affect all policies in the same direction; see Olivieri and Pitacco (1999) for a discussion of longevity risk in lifetime sickness insurance. Investment risk is, of course, a non-pooling risk.

(b) Portfolio results affected by risks. Several portfolio results can be used as “functions” in order to quantify the effects of risks. From a merely intuitive point of view, portfolio results can be classified as follows.

Firstly (following Colotti and Oliveri (1998)), we can distinguish between: (a1) cash results simply referring to monetary inflows and outflows; (a2) profit results involving mathematical reserves and aiming to measure annual and total profits; (a3) asset/liabilities results which explicitly consider the value of assets and the capital invested into the business.

Portfolio results can be examined at several detail levels as far as time is concerned. In particular it is possible to work with: (b1) annual results, formally
represented by vector-valued functions, for example the sequence of expected annual premium incomes over a given number of years; (b2) single-figure results, aiming to provide appropriate summaries, for example the present value of expected annual premium incomes.

Time intervals assumed in order to define the size of the sequence of annual results or to produce single-figure results lead to the following classification: (c1) short-term results concerning 1 to 2 years, roughly speaking; (c2) medium term results, concerning 3 to 5 years; (c3) cohort results referring to the time interval within which a given policy cohort terminates; (c4) long-term results, concerning more than 5 years (single-figure asymptotic results in principle belong to this category; in practice, however, this latter type of results does not comply with numerical procedures needed in a “corporate” approach to risk analysis).

Working with portfolio results obviously requires a rigorous definition of the concept of “portfolio”. Actually the portfolio can be thought of as a “closed” collection of policies already in force, as well as an “open” collection of policies to which also future business contributes. So, (d1) the run-off approach is based on the assumption that the insurer’s activities are restricted to the in-force portfolio and hence cease when the portfolio comes to its end; (d2) in the going concern approach it is assumed that the insurer’s activities develop with new acquisitions throughout the chosen time span.

Adopting the run-off approach rather than the going concern approach means different effects produced by some types of risks. For example, random mortality fluctuations (i.e. a pooling risk) normally lead to a higher variability in a run-off context, owing to the decreasing portfolio size.

3. A model for risk investigation and solvency assessment
(a) Notation. In this Section we describe the framework within which risk analyses are performed. More precisely, we make reference to a model which allows to assess cash flow, profit and patrimonial results (for more details, see Colotti and Olivieri (1998), in particular as far as portfolio valuations are concerned). What follows applies to a generic portfolio of life insurance covers. The following notation is adopted:

$Z_t$: random portfolio fund at time $t$;
$A_t$: random value of assets at time $t$;
$P_t$: random office premium income at time $t - 1$ (for $t$-th year);
$E_t$: random outcome for expenses paid at time $t - 1$ (for $t$-th year);
$C_t$: random outcome for death benefits paid at time $t$;
$S_t$: random outcome for living benefits paid at time $t$;
$\gamma_t$: random value at time $t$ of future obligations;
$V_t$: random portfolio reserve set up at time $t$;
$I_t$: random financial incomes in year $(t - 1, t)$;
$J_t$: random capital gains or losses on asset value in year $(t - 1, t)$;
random payment (withdrawal) by the insurer to (from) the fund at time $t$

($K_t > 0$ denotes a payment, $K_t < 0$ a withdrawal)

Living benefits paid at the beginning of the year, i.e. at time $t - 1$, can be represented by letting $P_t$ be negative (in absolute value equal to the benefit).

Randomness of the above mentioned quantities comes in general from all the types of risks described in Section 2 (see point (a)). In particular, the portfolio reserve is random because it is the sum of a random number of individual reserves. The individual reserve, usually assessed as the actuarial value of future net payments, can be random in its turn in case of benefits linked to some financial index.

(b) Cash flow analysis. Cash flow analysis can be performed either retrospectively, looking at past incomes and outcomes, or prospectively, looking at future payments. In a retrospective approach, the behaviour of the portfolio fund describes purely cash incomes and outcomes occurred up to the time which the fund is referred to. Let $Z_0$ denote the (known) value of the fund at time 0 (the valuation time). The random path of the fund is recursively described (starting from the given value at time 0) as follows:

$$Z_t = Z_{t-1} + P_t - E_t + I_t - C_t - S_t + K_t \quad (3.1)$$

The random variable $Z_t - V_t$ is usually called the free portfolio fund.

In a prospective approach, we define the random value of future obligations as follows

$$Y_t = \sum_{h=1}^{\infty} [(C_{t+h} + S_{t+h} - K_{t+h}) \nu(h) - (P_{t+h} - E_{t+h}) \nu(h-1)] \quad (3.2)$$

where $\nu(h)$ denotes the (random) discount factor for the period $(t, t + h)$, chosen accordingly to the future performance of investments (to this regard see also point (g) in this Section). Note that a broad sense is attached to the term “obligations”, since $Y_t$ includes payments both to/from policyholders (“industrial” obligations) and to/from shareholders (“corporate” obligations).

(c) Profit analysis. Annual profit emerging from the management of the portfolio can be assessed as follows

$$U_t = P_t - E_t + I_t - C_t - S_t + V_{t-1} - V_t \quad (3.3)$$

The annual profit $U_t$ can be split in several ways, reflecting the profit sources. However, we do not go deeply into this aspect, since it is not directly required in solvency analysis. We just mention the following relationships between profit and the portfolio fund, which can be easily verified

$$U'_t = \Delta Z_t - \Delta V_t - K_t \quad (3.4)$$

$$Z_t - V_t = (Z_0 - V_0) + \sum_{h=1}^{t} (U'_h + K_h) \quad (3.5)$$
where $U'_t = U_t - J_t$ and, for a given function $f_t$, $\Delta f_t = f_t - f_{t-1}$.

**d) Patrimonial analysis.** A patrimonial analysis consists in the investigation of both assets and liabilities. We assume that the value of assets can be recursively described as follows (for a given value of $A_0$)

$$A_t = A_{t-1} + \Delta Z_t + J_t \quad (3.6)$$

where $\Delta Z_t$ expresses new financial resources, and hence new investments, emerging in year $(t-1,t)$, whilst $J_t$ represents increases (capital gains) or decreases (capital losses) on asset value in the same year.

For the “balance condition”, at each time the portfolio reserve (i.e. the value of industrial liabilities) and shareholders’ fund, $M_t$, must be equal to the value of assets. Then we have

$$A_t = M_t + V_t \quad (3.7)$$

whence, trivially, $M_t = A_t - V_t$. We easily obtain

$$U_t = \Delta A_t - \Delta V_t - K_t \quad (3.8)$$

$$M_t = M_0 + \sum_{h=1}^{t} (U_h + K_h) \quad (3.9)$$

Further relationships among the three types of portfolio results obviously hold; readers are referred to Colotti and Olivieri (1998).

**e) Solvency.** Albeit solvency is a fundamental concept in insurance theory and practice, it is not clearly and univocally defined. What is commonly adopted is a stochastic approach to risk analysis. In this framework, solvency is meant as the ability to meet, with an assigned (high) probability, the random liabilities as described by a realistic (experience based) probabilistic structure.

The concept of stochastic solvency requires some specifications. In particular, choices are needed with respect to:

- the portfolio results which can be used to assess the above ability;
- the time span which the above results must be referred to;

in case the time span is longer than one year (as it is common):

- vector-valued results vs single-figure results;
- run-off vs going concern approach;

the meaning of the above choices can be easily understood in terms of the classification scheme presented in Section 2.

In order to make the above mentioned choices, the point of view from which solvency is ascertained must be stated. Policyholders, shareholders and the supervisory authority represent possible viewpoints on the insurance business. However, the policyholders’ and shareholders’ perspectives involve profitability requirements probably higher than those implied by the need of meeting liabilities. Such requirements would lead to a concept of insurer’s solidity, rather
than solvency. So, we will restrict our attention to the supervisory authority’s perspective.

The supervisory authority is charged to protect mainly the interests of present and forthcoming policyholders. From this point of view, cash and patrimonial results are mostly important. A long-term perspective should be considered; shorter horizons are consistent with a careful and frequent monitoring of the portfolio as well as with severe solvency requirements.

(f) Reserve-based solvency requirements. In the traditional approach to solvency, the ability to meet random liabilities (see point (e)) is meant as the ability to set up the reserve for each policy in the portfolio and hence the portfolio reserve. In terms of patrimonial results, the insurance company is therefore able to meet its liabilities at a given time \( t \) if the asset value is greater or equal to the portfolio reserve, i.e.

\[ M_t \geq 0 \]  

In this framework, \( M_t \) is usually referred to as the solvency margin assigned to the portfolio.

Let 0 denote the time at which solvency is ascertained and assume \( M_0 \geq 0 \). Given a time horizon of \( T \) years, we say that the insurer has a solvency degree \( 1 - \varepsilon \) if and only if

\[ \Pr \left\{ \bigwedge_{t=1}^{T} M_t \geq 0 \right\} = 1 - \varepsilon \]  

Specific solvency requirements can be found by choosing proper values for the quantities which affect (3.11), i.e.

- the probability \( \varepsilon \) (ruin probability);
- the time span \( T \);
- the capital flows \( K_t \);
- a run-off or a going concern approach.

In particular, if the point of view of the supervisory authority is adopted, it seems natural to choose \( K_t = 0 \) except for the time of valuation, when a proper solvency margin must be financed. On the contrary, when shareholders’ perspective is considered, the choice of the flows \( K_t \) could reflect their targets concerning the timing of shareholders’ capital investment into the portfolio or dividend distribution.

Note that the assumed time span \( T \) implies the consideration of incomes and outcomes over a period of \( n \) years, with \( n \geq T \). Actually, for all \( t \) the portfolio reserve \( V_t \) takes into account premiums and benefits falling due after time \( t \); so, for any fixed \( T \), the time interval actually involved in the solvency ascertainment has a term given by \( T \) plus the largest residual duration of policies virtually in force at time \( T \).

Our aim is to determine the solvency margin required at time 0 such that, for a given choice of the quantities mentioned above, condition (3.11) is satisfied.
Since we adopt the viewpoint of the supervisory authority, we assume $K_t = 0$, $t = 1, 2, \ldots$ and $K_0 \geq 0$. We denote the required solvency margin $M_0^{(R)}$ (the upper letter "R" indicates that a reserve-based requirement has been adopted). Note that

$$A_0^{(R)} = M_0^{(R)} + \nu_0$$

(3.12)

is the required (total) investment at time 0.

Definition (3.11) is based on a vector-valued portfolio result, since the quantities $M_1, M_2, \ldots, M_T$ are taken into account. A different concept of solvency can be stated referring to a single-figure result; typically

$$\Pr \{ M_T \geq 0 \} = 1 - \varepsilon$$

(3.13)

Of course, equation (3.13) leads to a weaker concept of solvency than (3.11). Reasonably, requirements based on (3.13) are disregarded by the supervisory authority.

Solvency requirements can be formulated also with reference to cash results. In this case, the insurance company is solvent at a given time if the free portfolio fund is positive, i.e. if

$$Z_t - \nu_t \geq 0$$

(3.14)

Note that now the solvency margin linked to the portfolio is given by $Z_t - \nu_t$.

Solvency requirements based on the following conditions

$$\Pr \left\{ \bigwedge_{t=1}^{T} Z_t - \nu_t \geq 0 \right\} = 1 - \varepsilon$$

(3.15)

$$\Pr \{ Z_T - \nu_T \geq 0 \} = 1 - \varepsilon$$

(3.16)

lead to a required initial free portfolio fund (or solvency margin), $[Z_0^{(R)} - \nu_0]$, and a required initial fund $Z_0^{(R)}$, for a given choice of the ruin probability $\varepsilon$, the time horizon $T$ and the run-off or going concern approach (we still assume $K_0 \geq 0$ and $K_t = 0$, $t = 1, 2, \ldots$). The main difference between $[Z_0^{(R)} - \nu_0]$ and $M_0^{(R)}$ is due to capital gains or losses. In periods of high investment performance, requirements (3.15) and (3.16) could be more severe than (3.11) and (3.13), since capital gains are disregarded. The reverse obviously holds in periods of investment underperformance. In Section 4 we will make some hypotheses under which cash and patrimonial results are equivalent.

Further requirements can be given considering short-term cash flows, thus involving in particular liquid assets as well as premiums, expenses and sums assured paid in a year. We do not discuss such requirements (see, for example, Kahane, Tapiero and Laurent (1989)).

(g) Obligations-based solvency requirements. The disadvantage of the solvency requirements discussed in point (f) is that they refer to the notion of
reserve, which is usually assessed (at individual level) as an expected value (on a conservative basis) of future industrial obligations (i.e., benefits less premiums). When some types of insurance covers are considered, for example those involving lifetime living benefits, this approach might be lacking. Actually, owing to the uncertainty in mortality trends at adult and old ages and in the future performance of investments, it is difficult to judge on the appropriateness of a reserve profile simply based on a deterministic view of the future scenario. For such covers, solvency requirements based on the random value of future obligations might be more appropriate than those discussed in point (f).

We will therefore consider requirements according to which the insurance company is judged solvent at time $t$ if at that time it is able to meet its future obligations (a similar approach has been adopted in Faculty of Actuaries’ Solvency Working Party (1986)). In patrimonial terms, solvency requirements like the following

\begin{align}
\Pr \left\{ \bigwedge_{t=1}^{T} A_t - Y_t \geq 0 \right\} &= 1 - \varepsilon \\
\Pr \{ A_T - Y_T \geq 0 \} &= 1 - \varepsilon
\end{align}

(can be considered.

Note that, besides what mentioned above, another advantage of definitions (3.17) and (3.18), when compared to (3.11) and (3.13), is that $Y_t$ includes strictly industrial obligations (i.e., with respect to policyholders) as well as corporate obligations (i.e., with respect to shareholders). On the contrary $V_t$, used in (3.11) and (3.13), only considers future payments to policyholders. Hence, definitions (3.17) and (3.18) could be more appropriate when solvency is ascertained from the point of view of shareholders. In the following, we consider the supervisory authority’s perspective. Hence, similarly to point (f), we assume $K_t = 0$ except for the time of valuation.

With regard to the value of future obligations, $Y_t$ should for consistency be calculated with discount factors chosen accordingly to the future behaviour of both financial incomes, $I_{t+h}$, and capital gains/losses, $J_{t+h}$, since they both affect the value of assets (see (3.6)). Further, the summation in (3.2) should be extended to the largest residual duration of policies virtually in force at time $T$. Denoting with $n$ such duration, we have $Y_n = 0$ and we can easily verify that, under the information available at time 0, the following equalities hold (for whatever choice of the flows $K_t$)

\begin{align}
\Pr \left\{ \bigwedge_{t=1}^{T} A_t - Y_t \geq 0 \right\} &= \Pr \{ A_T - Y_T \geq 0 \} = \Pr \{ A_n \geq 0 \}
\end{align}

Relation (3.19) shows in particular that, likely to the reserve-based requirements, a time span greater than $T$ is involved. However, whilst in the reserve-based
approach expected value of future payments are considered (at least at individual level) we now deal with random values only (which allows to make simplifications (3.19)).

For a given choice of the ruin probability $\varepsilon$, under a run-off or going concern approach, conditions (3.17) and (3.18) (which according to (3.19) are equivalent and independent of the time span $T$) leads to a required initial (total) investment $A_0^{(O)}$, to be financed both with premiums and shareholders’ fund (the up-letter “$O$” indicates that an obligations-based requirement has been adopted).

In particular, given the portfolio reserve at time 0 (which should be built with premiums), the required solvency margin according to requirement (3.19) is

$$M_0^{(O)} = A_0^{(O)} - V_0$$

In terms of cash results, future obligations are compared with the portfolio fund. Solvency requirements can be formulated as follows

$$\Pr\left\{ \bigcap_{t=1}^{T} Z_t - \mathcal{Y}_t \geq 0 \right\} = \Pr\{ Z_T - \mathcal{Y}_T \geq 0 \} = \Pr\{ Z_n \geq 0 \} = 1 - \varepsilon$$

(3.21)

Note that in (3.21), for consistency, $\mathcal{Y}_t$ should be based on discount factors chosen accordingly only to the future behaviour of financial incomes, $I_t$ (see (3.1)).

For a given choice of the ruin probability $\varepsilon$ and the approach (run-off or going concern), condition (3.21) leads to a required initial fund $Z_0^{(O)}$ to be assigned to the portfolio. Given the initial reserve $V_0$, the required initial free portfolio fund (or solvency margin) is then $[Z_0^{(O)} - V_0]$.

4. Demographic risk in a portfolio of life annuities
(a) Hypotheses. The meaning and implications, in numerical terms! of the solvency requirements discussed in the previous Section are investigated with reference to life annuities. Since these covers are in particular affected by longevity and investment risk, we concentrate on demographic and financial risk only, disregarding expense, option and economic risks.

More precisely, we consider a portfolio of immediate (individual) life annuities, homogeneous in terms of entry time, age, annual amount, etc. For simplicity, we disregard expenses and consider constant benefits. Let $N_t$ be the random number of annuitants at time $t$, $t = 0, 1, \ldots$ (in particular, $N_0$ is a known quantity), $R$ the annual amount for each policy and $y$ the age at time 0.

In this Section, we adopt a deterministic hypothesis with reference to investment performance, denoting with $i_t$ the rate of interest for the $t$-th year (investment risk is dealt with in Section 5). Note that in a deterministic framework capital gains/losses are meaningless; hence $A_t = Z_t$ at each time $t$. 

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Under our hypotheses, the portfolio fund, future obligations and the portfolio reserve are now described (letting $R_t = -R_t$) as follows

$$Z_t = Z_{t-1} - R_t + I_t = (Z_{t-1} - N_{t-1} R)(1 + i_t^*)$$  \hspace{1cm} (4.1)

$$\gamma_t = \sum_{h=1}^{\omega - y - t} R_{t+h} v(h - 1) = \sum_{h=1}^{\omega - y - t} R N_{t+h-1} v(h - 1)$$  \hspace{1cm} (4.2)

$$V_t = N_t V_t = N_t R \tilde{a}_{y+t}$$  \hspace{1cm} (4.3)

where $\omega$ is the maximum age (whence $\omega - y - t$ is the maximum duration of a policy at time $t$), $v(h - 1)$ is calculated according to the interest rates $i_{t+1}^*, i_{t+2}^*, \ldots, i_{t+h-1}^*$ and $\tilde{a}_{y+t}$ denotes the actuarial value of a unitary immediate annuity for a person aged $y + t$, calculated according to a proper valuation basis. We remind that shareholders’ capital flows $K_t$ are disregarded because the viewpoint of the supervisory authority is considered.

Requirements (3.15) and (3.21) are adopted. In order to perform the valuation, we still need to assume a mortality model.

(b) Mortality assumptions. With reference to a newborn, we assume that the distribution of his/her future lifetime, $T_0$, can be represented with the Heligman-Pollard law, which is often used in the analysis of mortality because of its capability to represent mortality satisfactorily over the whole life span (see, for example, Benjamin and Soliman (1993)). Denoting with $q_z$ and $p_z$ the mortality and survival rates at age $x$, under the law of Heligman-Pollard it is assumed

$$q_z = A(z+B)^C + D e^{-E(\ln x - \ln F)^2} + G H^z$$  \hspace{1cm} (4.4)

where the $q_z$’s are the so-called “odds”. As far as the meaning of the law is concerned, the first term in (4.4), $A(z+B)^C$, describes infant mortality, the second term, $D e^{-E(\ln x - \ln F)^2}$, mortality at young adult ages (mainly accidental) and the third term, $G H^z$, mortality at old ages. When ages over 50 are considered, the contribution to the odds from the first two terms is negligible. Hence, in the representation of life annuitants’ mortality we disregard such quantities, assuming the following simplification

$$q_z = G H^z$$  \hspace{1cm} (4.4')

whence the expression of the mortality rate $q_z$ and the survival function $S(x)$ ($S(x) = Pr\{T_0 > x\}$) can be easily obtained. Note that the parameter $G$ expresses the level of senescent mortality and $H$ the rate of increase of senescent mortality itself.

In order to represent mortality trends, we will adopt projected survival functions. Mortality trends at adult ages reveal decreasing annual probabilities of death and in particular:
(i) the concentration of deaths around the mode of the density function \( f_0(x) \) 
\( (f_0(x) = -\frac{dS(x)}{dx}) \), also called "curve of deaths"), increases with time; in terms 
of the graph of the survival function, this implies the so-called "rectangularization";
(ii) the mode of the curve of deaths moves towards very old ages, originating the 
so-called "expansion".

Commonly, mortality projections are extrapolations of (recent) trends as far 
as these can be perceived from mortality statistics. A different approach leads 
to models expressing the basic characteristics of the evolving scenario in which 
mortality improvements take place. In this case the projection model should 
catch the aspects (i) and (ii). To this purpose, analytical mortality laws should 
be used as, for example, the Heligman-Pollard law. According to this approach, 
mortality trends are represented assuming that the parameters of the mortality 
law are functions of the calendar year. Hence, also the mode and the variance 
of the curve of deaths depend on the calendar year. Then, the adequacy of the 
projection model can be checked comparing the behaviour of the curve of deaths 
with the scenario characteristics above described (see (i) and (ii)).

It should be stressed that the increasing concentration of deaths around the 
mode of the curve of deaths reduces the risk affecting annuity benefits, whatever 
the location of the mode may be. Hence, any given mortality projection im-
plying this feature reveals a reduction of the demographic risk originating from 
random fluctuations, with respect to the results produced by a non-projected 
mortality model. However, the mortality projection itself is affected by uncertain-
ty. Evaluating the degree of uncertainty and incorporating this evaluation 
in the actuarial model, a higher demographic risk emerges. The additional risk, 
called the longevity risk, is attributable to systematic deviations of the mortality 
from the projected mortality assumed in the calculation basis (used in pricing or 
reserving).

As far as the demographic risk is concerned, we adopt the following approaches:
(1) a "deterministic" approach, implemented by using a given projected survival 
function; this approach only allows for the random fluctuation risk;
(2) a "stochastic" approach, implemented by using a set of projected survival 
functions, representing the uncertainty inherent in the projection; a "degree 
of belief" will be assigned to each function; this approach allows both for the 
random fluctuation risk and for the longevity risk.

More precisely, we will define three projected survival functions, denoted by 
\( S^{[\text{min}]}(x) \), \( S^{[\text{mod}]}(x) \) and \( S^{[\text{max}]}(x) \), expressing, respectively, a little, a medium 
and a high reduction in mortality. In the deterministic approach only \( S^{[\text{mod}]}(x) \) 
is used, whilst the stochastic approach involves the three functions.

As stated at the beginning of this Section, the Heligman-Pollard model is 
adopted. The parameters of the three survival functions are shown in Table 4.1 
(the maximum age, \( \omega \), is set equal to 110). As it emerges from such parameters,
the projected functions have been obtained so that they perform the trends of expansion and rectangularization. Figures 4.1 and 4.2 show, respectively, the three survival functions and the related density functions.

### Table 4.1 - Heligman-Pollard law for annuities

<table>
<thead>
<tr>
<th></th>
<th>min</th>
<th>med</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>G</td>
<td>0.000042</td>
<td>0.000002</td>
<td>0.000001</td>
</tr>
<tr>
<td>H</td>
<td>1.09803</td>
<td>1.13451</td>
<td>1.17215</td>
</tr>
</tbody>
</table>

Table 4.1 - Heligman-Pollard law for annuities

(c) **Solvency in a deterministic approach to mortality.** The portfolio we are investigating consists of identical annuities, paid to persons of initial age $y = 65$, with annual amount $R = 100$. We assume that the future lifetimes of the annuitants have a common distribution and are independent of each other (conditional on any given survival function). The single premium (to be paid at entry) is calculated, for each policy, according to the survival function $S^{[\text{med}]}(x)$ and with a constant annual interest rate $i = 0.03$. Further we assume that for each policy in force at time $t$, $t = 0, 1, \ldots$, a reserve must be set up, which is calculated according to such hypotheses.

In the deterministic approach to mortality, the distribution of the future lifetime of each insured is known, the only cause of uncertainty consisting in the time of death. The assessment of the solvency margin required according either to a reserve-based or an obligations-based condition is performed through simulation. In order to obtain results more easily to interpret, we disregard profit; the actual life duration of the annuitants is thus simulated with the survival function $S^{[\text{med}]}(x)$. Further, we assume $i_t = i = 0.03$. A run-off approach is adopted (i.e. only the cohort entering at time $0$ is considered).

In table 4.2 the solvency margin required at time 0 is quoted according to conditions (3.15) and (3.21). In the former case, the time spans $T = n = \omega - y$ and $T = 5$ years are alternatively chosen. When the largest time span (i.e. $T = n$) is chosen, $[Z_0^{(R)} - V_0]$ and $[Z_0^{(O)} - V_0]$ are quite similar. However, the reserve-based approach seems to be more severe than the obligations-based one. In order
to understand why consider that, since \( V_n = 0 \), \( Z_n - V_n = Z_n \). According to requirement (3.21), solvency is in practice ascertained only at time \( n \) (checking the positivity of \( Z_n \)), whilst under (3.15) it is ascertained not only at time \( n \) (checking the positivity of \( Z_n - V_n = Z_n \)), but also at time \( t, t < n \), when a negative free portfolio fund can occur. When \( T = 5 \), \( [Z_0^{(R)} - V_0] \) seems to be insufficient when compared to \([Z_0^{(O)} - V_0]\). This is due to the fact that the risk of random fluctuations of the portfolio fund from the reserve (roughly speaking, of the number of survivors from their expected value) is heavy in the long run, but not within a short horizon. As far as the size of the portfolio is concerned, table 4.2 shows that the required margin decreases as \( N \) increases. This is due to the fact that a deterministic approach to mortality only catches the demographic risk of random fluctuations which, as it is well-known, is a pooling risk (i.e. its effect decreases when larger numbers of similar policies are dealt with). Finally, note that since by definition the initial reserve \( V_0 \) is equal to the single premium, the solvency margin at time 0 must be financed with shareholders’ funds.

<table>
<thead>
<tr>
<th>( N_0 )</th>
<th>( T=n )</th>
<th>( [Z_0^{(R)} - V_0] )</th>
<th>( [Z_0^{(R)} - V_0] ) %</th>
<th>( T=5 )</th>
<th>( [Z_0^{(R)} - V_0] )</th>
<th>( [Z_0^{(R)} - V_0] ) %</th>
<th>( [Z_0^{(O)} - V_0] )</th>
<th>( [Z_0^{(O)} - V_0] ) %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>32,059</td>
<td>2.180%</td>
<td>17.242</td>
<td>1.140%</td>
<td>29.744</td>
<td>1.967%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>44,037</td>
<td>1.486%</td>
<td>25.253</td>
<td>0.835%</td>
<td>40.653</td>
<td>1.344%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3000</td>
<td>54,398</td>
<td>1.199%</td>
<td>27.771</td>
<td>0.612%</td>
<td>48.462</td>
<td>1.068%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4000</td>
<td>65,460</td>
<td>1.082%</td>
<td>33,340</td>
<td>0.551%</td>
<td>58.894</td>
<td>0.974%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5000</td>
<td>73,242</td>
<td>0.969%</td>
<td>36,621</td>
<td>0.484%</td>
<td>64.319</td>
<td>0.851%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6000</td>
<td>79,956</td>
<td>0.881%</td>
<td>39,597</td>
<td>0.437%</td>
<td>68.790</td>
<td>0.758%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7000</td>
<td>87,585</td>
<td>0.828%</td>
<td>44,861</td>
<td>0.424%</td>
<td>80.891</td>
<td>0.764%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8000</td>
<td>90,332</td>
<td>0.747%</td>
<td>47,302</td>
<td>0.391%</td>
<td>81.071</td>
<td>0.670%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9000</td>
<td>100,555</td>
<td>0.739%</td>
<td>49,744</td>
<td>0.366%</td>
<td>90.192</td>
<td>0.663%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10000</td>
<td>103,149</td>
<td>0.682%</td>
<td>52,632</td>
<td>0.344%</td>
<td>92.398</td>
<td>0.611%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Table 4.2 - Required solvency margin; \( \varepsilon=0.025 \) |

Results in tables 4.3 and 4.4 have been obtained by performing the valuation at time \( h, h = 0, 5, 10, \ldots, \) in order to inspect the behaviour of the required margin through time (amendments to formulae in Section 3 are straightforward). At each valuation time \( h \) we have considered, for a given initial size \( N_0 \), a portfolio of \( E(N_h) \) insureds, each aged \( y + h \), and a portfolio reserve \( V_h = E(N_h) V_h \) (the expected value \( E(N_h) \) is calculated according to the assumed mortality distribution, i.e. \( S^{[med]}(x) \)). The increase in the (relative) solvency margin, whatever requirement is adopted, is due to the fact that the size of the portfolio reduces and the age of the annuitants increases with time. Note, however, that the increase is stronger when a reserve-based approach with horizon \( T = 5 \) is chosen. The example shows that, as it is quite intuitive, such approach requires proper adjustments once the given horizon has been reached. Moreover, it implies a different capital allocation than the other two conditions examined numerically.
(i.e. \(T = n\) and obligations-based approach), in the sense that the lower initial required fund must be followed, at the subsequent times, by a greater increase of the fund itself. As far as financing of the required fund \(Z_h^{(R)} (Z_h^{(O)})\) is concerned, in case the reserve actually cumulated is not equal to its expected value, shareholders’ fund greater than \([Z_h^{(R)} - V_h] ([Z_h^{(O)} - V_h])\) could be necessary.

\[
\begin{array}{cccccc}
  h & T=n & N_0=1000 & N_0=5000 & N_0=10000 \\
  0 & 2.150 & 0.969 & 0.682 & 1.140 & 0.484 & 0.344 \\
  5 & 2.613 & 1.966 & 0.980 & 1.501 & 0.663 & 0.450 \\
 10 & 3.124 & 1.430 & 0.981 & 2.079 & 0.953 & 0.649 \\
 15 & 4.146 & 1.761 & 1.289 & 2.922 & 1.321 & 0.937 \\
 20 & 5.442 & 2.270 & 1.675 & 4.372 & 1.823 & 1.371 \\
 25 & 7.489 & 3.357 & 2.351 & 6.222 & 2.984 & 2.046 \\
\end{array}
\]

Table 4.3 - Required solvency margin \([Z_h^{(R)} - V_h]/V_h \times 100; \varepsilon=0.025\)

\[
\begin{array}{cccccc}
  h & N_0=1000 & N_0=5000 & N_0=10000 \\
  0 & 1.967 & 0.851 & 0.611 \\
  5 & 2.345 & 1.024 & 0.741 \\
 10 & 2.972 & 1.243 & 0.910 \\
 15 & 3.809 & 1.634 & 1.154 \\
 20 & 5.022 & 2.136 & 1.511 \\
 25 & 7.148 & 3.103 & 2.242 \\
 30 & 13.996 & 5.329 & 3.807 \\
 35 & 49.934 & 14.407 & 9.928 \\
\end{array}
\]

Table 4.4 - Required solvency margin \([Z_h^{(O)} - V_h]/V_h \times 100; \varepsilon=0.025\)

Finally, tables 4.5 to 4.8 shows dependence on the ruin probability.

\[
\begin{array}{cccccc}
  N_0 & T=n & \varepsilon=0.01 & \varepsilon=0.025 & \varepsilon=0.05 & T=5 & \varepsilon=0.01 & \varepsilon=0.025 & \varepsilon=0.05 \\
  1000 & 2.503 & 2.180 & 1.887 & 1.254 & 1.140 & 1.024 \\
  2000 & 1.736 & 1.486 & 1.272 & 0.949 & 0.835 & 0.696 \\
  3000 & 1.475 & 1.199 & 1.080 & 0.688 & 0.612 & 0.543 \\
  4000 & 1.226 & 1.082 & 1.082 & 0.636 & 0.551 & 0.488 \\
  5000 & 1.147 & 0.969 & 0.844 & 0.581 & 0.484 & 0.420 \\
  6000 & 0.969 & 0.881 & 0.773 & 0.511 & 0.437 & 0.390 \\
  7000 & 0.846 & 0.728 & 0.715 & 0.473 & 0.424 & 0.376 \\
  8000 & 0.868 & 0.747 & 0.676 & 0.454 & 0.391 & 0.324 \\
  9000 & 0.861 & 0.739 & 0.642 & 0.471 & 0.366 & 0.308 \\
 10000 & 0.775 & 0.682 & 0.588 & 0.380 & 0.344 & 0.293 \\
\end{array}
\]

Table 4.5 - Required solvency margin \([Z_h^{(R)} - V_h]/V_h \times 100\)

\[
\begin{array}{cccccc}
  N_0 & \varepsilon=0.01 & \varepsilon=0.025 & \varepsilon=0.05 \\
  1000 & 2.314 & 1.967 & 1.644 \\
  2000 & 1.389 & 1.344 & 1.107 \\
  3000 & 1.347 & 1.068 & 0.927 \\
  4000 & 1.116 & 0.974 & 0.974 \\
  5000 & 1.053 & 0.851 & 0.725 \\
  6000 & 0.927 & 0.758 & 0.645 \\
  7000 & 0.845 & 0.764 & 0.647 \\
  8000 & 0.808 & 0.670 & 0.589 \\
  9000 & 0.762 & 0.663 & 0.576 \\
 10000 & 0.764 & 0.611 & 0.523 \\
\end{array}
\]

Table 4.6 - Required solvency margin \([Z_h^{(O)} - V_h]/V_h \times 100\)
To conclude the examples relating to the deterministic approach, we just mention that investigations performed with either the survival function $S_{\text{med}}(z)$ or $S_{\text{max}}(z)$ suggest comments similar to those above discussed. However, due to the phenomenon of rectangularization, the solvency margin required according to $S_{\text{med}}(z)$ is lower (higher) than that assessed using $S_{\text{med}}(z)$. Finally, it must be mentioned that the relatively low levels of $\mu_{\text{ih}}$ in all the examples presented is due to the fact that, given the deterministic approach to mortality (and to investment as well), only the risk of random fluctuations has been accounted for.

(d) Solvency in a stochastic approach to mortality. The assessment of the solvency margin is now obtained considering explicitly uncertainty in mortality projections. To this aim, we consider the three survival function $S_{\text{min}}(x)$, $S_{\text{med}}(x)$ and $S_{\text{max}}(x)$, assuming that each of them is meant as a possible distribution of the future lifetime. The weights $\rho_{\text{min}}$, $\rho_{\text{med}}$ and $\rho_{\text{max}}$ represent, respectively, the "degree of belief" of such functions.

The single premium for each policy and the individual reserve are still calculated with the survival function $S_{\text{med}}(x)$ and the interest rate $i = 0.03$; we let $i^* = i = 0.03$. Unless otherwise stated, we assume $\rho_{\text{min}} = 0.2$, $\rho_{\text{med}} = 0.6$, $\rho_{\text{max}} = 0.2$ (reflecting the fact that $S_{\text{med}}(x)$, which is used for pricing and reserving, is assumed to give the most reliable mortality description).

Obviously, the investigation is carried out through simulation. We now deal with two causes of uncertainty: the actual distribution of the future lifetimes and the time of death of each insured. Firstly, the survival function must be chosen (through simulation) and then, assuming that under a given lifetime distribution the annuitants are independent risks, the actual duration of life of each person is simulated.
Tables 4.9 to 4.15 show the results of valuations of the same type of those performed in the deterministic framework. The following aspects must be stressed.

For a given choice of parameters, the comparison between the deterministic and the stochastic case shows a heavy increase of the required solvency margin in the latter. This is due to the fact that a stochastic framework allows to analyse not only the risk of random fluctuations in the number of survivors, but also that of systematic deviations (i.e. the longevity risk), which is a non-pooling risk. Considering for example table 4.9, the decreasing behaviour of the relative required solvency margin with respect to $N_0$ is due to the pooling effect of random fluctuations; however, its magnitude is rather stable and its value seems to tend to a large positive amount (as could be checked considering larger portfolios); hence, the non-pooling effect of longevity risk is witnessed.

Requirement (3.15) with time span $T = 5$ leads to a solvency margin significantly lower than either the case $T = n$ or requirement (3.21). This shows first of all that the longevity risk reveal itself in the long run. Secondly, the choice of assessing the required solvency margin according to (3.15) with $T = 5$ implies a strong postponement of the solvency margin building up (as witnessed by tables 4.10, 4.11, 4.14 and 4.15) and the need to monitoring carefully the portfolio in order to adjust the solvency margin in case it is insufficient.

Finally, table 4.16 shows that dependence of the required solvency margin on the parameters $p$'s is not very significant, unless they bring back to the deterministic approach (i.e. in the case $p^\text{[min]} = 0$, $p^\text{[med]} = 1$, $p^\text{[max]} = 0$).

<table>
<thead>
<tr>
<th>$N_0$</th>
<th>$T=n$</th>
<th>$T=5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{[Z_0^\text{(R)} - V_0]}{V_0}$</td>
<td>$\frac{[Z_0^\text{(R)} - V_0]}{V_0}$</td>
</tr>
<tr>
<td>1000</td>
<td>218,172 , 14.431 %</td>
<td>43,869 , 2.902 %</td>
</tr>
<tr>
<td>3000</td>
<td>632,324 , 13.941 %</td>
<td>123,672 , 2.727 %</td>
</tr>
<tr>
<td>4000</td>
<td>840,797 , 13.903 %</td>
<td>162,592 , 2.689 %</td>
</tr>
<tr>
<td>5000</td>
<td>1,046,066 , 13.838 %</td>
<td>202,827 , 2.683 %</td>
</tr>
<tr>
<td>6000</td>
<td>1,253,963 , 13.824 %</td>
<td>241,871 , 2.666 %</td>
</tr>
<tr>
<td>7000</td>
<td>1,457,367 , 13.771 %</td>
<td>289,904 , 2.643 %</td>
</tr>
<tr>
<td>8000</td>
<td>1,660,461 , 13.729 %</td>
<td>318,527 , 2.634 %</td>
</tr>
<tr>
<td>9000</td>
<td>1,867,065 , 13.722 %</td>
<td>357,704 , 2.629 %</td>
</tr>
<tr>
<td>10000</td>
<td>2,072,830 , 13.710 %</td>
<td>395,813 , 2.618 %</td>
</tr>
</tbody>
</table>

Table 4.9 - Required solvency margin, $\epsilon=0.025$

5. Investigations embedding financial risk

(a) Investment hypotheses. The examples in the previous Section show the importance of the demographic risk, namely of the longevity risk, in life annuity portfolios. Another important source of risk comes from investment. In this
Table 4.10 - Required solvency margin \([Z^{(R)}_h - V_h]/V_h \times 100; \varepsilon = 0.025\)

<table>
<thead>
<tr>
<th>(T = n)</th>
<th>(h)</th>
<th>(N_0 = 1000)</th>
<th>(N_0 = 5000)</th>
<th>(N_0 = 10000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0</td>
<td>14.431</td>
<td>13.838</td>
<td>13.710</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>16.525</td>
<td>15.830</td>
<td>15.631</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>18.236</td>
<td>17.320</td>
<td>17.073</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>18.979</td>
<td>17.585</td>
<td>17.272</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>17.766</td>
<td>16.608</td>
<td>15.602</td>
</tr>
<tr>
<td></td>
<td>25</td>
<td>15.647</td>
<td>13.960</td>
<td>11.498</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>16.758</td>
<td>11.035</td>
<td>10.072</td>
</tr>
<tr>
<td></td>
<td>35</td>
<td>45.498</td>
<td>24.129</td>
<td>21.122</td>
</tr>
</tbody>
</table>

Table 4.11 - Required solvency margin \([Z^{(O)}_h - V_h]/V_h \times 100; \varepsilon = 0.025\)

<table>
<thead>
<tr>
<th>(T = n)</th>
<th>(h)</th>
<th>(N_0 = 1000)</th>
<th>(N_0 = 5000)</th>
<th>(N_0 = 10000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0</td>
<td>14.389</td>
<td>13.826</td>
<td>13.700</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>16.479</td>
<td>15.815</td>
<td>15.640</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>18.158</td>
<td>17.202</td>
<td>17.036</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>18.845</td>
<td>17.536</td>
<td>17.218</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>17.538</td>
<td>15.880</td>
<td>15.481</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>16.250</td>
<td>10.968</td>
<td>10.061</td>
</tr>
<tr>
<td></td>
<td>35</td>
<td>45.258</td>
<td>23.992</td>
<td>21.098</td>
</tr>
</tbody>
</table>

Table 4.12 - Required solvency margin \([Z^{(R)}_h - V_h]/V_h \times 100\)

<table>
<thead>
<tr>
<th>(T = n)</th>
<th>(N_0)</th>
<th>(h)</th>
<th>(\varepsilon = 0.01)</th>
<th>(\varepsilon = 0.025)</th>
<th>(\varepsilon = 0.05)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>14.897</td>
<td>14.431</td>
<td>13.999</td>
<td>3.033</td>
<td>2.902</td>
</tr>
<tr>
<td>5000</td>
<td>13.958</td>
<td>13.838</td>
<td>13.658</td>
<td>2.768</td>
<td>2.683</td>
</tr>
<tr>
<td>8000</td>
<td>13.879</td>
<td>13.729</td>
<td>13.590</td>
<td>2.700</td>
<td>2.634</td>
</tr>
<tr>
<td>10000</td>
<td>13.843</td>
<td>13.710</td>
<td>13.584</td>
<td>2.685</td>
<td>2.618</td>
</tr>
</tbody>
</table>

Table 4.13 - Required solvency margin \([Z^{(O)}_h - V_h]/V_h \times 100\)

<table>
<thead>
<tr>
<th>(T = n)</th>
<th>(N_0)</th>
<th>(h)</th>
<th>(\varepsilon = 0.01)</th>
<th>(\varepsilon = 0.025)</th>
<th>(\varepsilon = 0.05)</th>
</tr>
</thead>
</table>

Table 4.14 - Required solvency margin \([Z^{(R)}_h - V_h]/V_h \times 100; N_0 = 5000\)

<table>
<thead>
<tr>
<th>(T = n)</th>
<th>(h)</th>
<th>(\varepsilon = 0.01)</th>
<th>(\varepsilon = 0.025)</th>
<th>(\varepsilon = 0.05)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3000</td>
<td>17.600</td>
<td>17.320</td>
<td>17.037</td>
<td>6.829</td>
</tr>
<tr>
<td>5000</td>
<td>18.609</td>
<td>18.008</td>
<td>17.485</td>
<td>11.753</td>
</tr>
<tr>
<td>8000</td>
<td>27.794</td>
<td>24.129</td>
<td>20.302</td>
<td>25.656</td>
</tr>
</tbody>
</table>

Table 4.15 - Required solvency margin \([Z^{(O)}_h - V_h]/V_h \times 100; N_0 = 5000\)

565
As far as financial modelling is concerned, we do not deal with the problem of assessing the market value of the different types of financial instruments in which the insurance company can invest, nor with the problem of choosing a proper investment strategy. We simply adopt a model for the short interest rate, whose (random) behaviour should reflect interest incomes as well as capital gains/losses arising from the performance of the assets linked to the portfolio.

Let \( r_t \) be the short interest rate, expressing the instantaneous total rate of return on assets. We assume that its behaviour can be described with the Vasicek model (i.e. with an Ornstein-Uhlenbeck process). Thus

\[
    dr_t = \alpha (\gamma - r_t) \, dt + \sigma \, dW_t
\]

where \( \{W_t\} \) is a standard Wiener process and \( \alpha, \gamma \) and \( \sigma \) are positive constants. Note that \( \gamma \) represents the long-term mean of the short rate, \( \alpha \) a friction force bringing the process back towards \( \gamma \) and \( \sigma \) the diffusion coefficient. We do not discuss further the features of model (5.1) (see Vasicek (1977)). We just mention that the Vasicek model is often used in actuarial applications since it represents quite satisfactorily the long-term development of the rate of return (see, for example, Parker (1997) also for a list of references). However, (5.1) could lead to a
negative value for the short rate; under our hypotheses this is anyway acceptable, since capital losses are admitted.

Having assumed that the behaviour of $r_t$ includes both financial incomes and capital gains/losses, solvency is investigated in patrimonial terms. Assets are now described as follows

$$A_t = A_{t-1} - N_{t-1} R + I_t + J_t$$

whilst the value of future obligations and the portfolio reserve are still given by (4.2) and (4.3), respectively (considering the perspective of the supervisory authority, the flows $K_t$ are disregarded). Requirements (3.11) and (3.19) are considered.

(b) Numerical investigations. We refer to the same portfolio considered in Section 4; hence, in particular, we adopt a run-off approach. As far as mortality is concerned, both a deterministic and a stochastic framework are dealt with. In any case, we still assume that the future lifetimes of the annuitants have a common distribution and that the single premium and the individual reserve are calculated according to $S^{\text{med}}(x)$ and $i = 0.03$.

Within a deterministic approach to mortality, the causes of uncertainty are two: the time of death of each annuitant and the level of the short interest rate at any time $t$. We assume, as it is quite common, independence between demographic and financial variables and we proceed to the assessment of the solvency margin through simulation (assuming independence among the insured risks, as far as mortality is concerned, conditional on a given survival function). Parameters for the financial simulation are given in table 5.1. They are (arbitrarily) chosen so that they reflect the overall performance of assets, resulting from both market behaviour and the investment strategy of the insurance company. Once the short rate has been simulated, values for $I_t + J_t$ and $v(t)$ can be easily obtained. For example

$$v(t) = \exp \left( - \int_0^t r_u \, du \right)$$

<table>
<thead>
<tr>
<th>$r_0$</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.1</td>
<td>0.04</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Table 5.1 – Parameters for the short rate

Note that under the parameters of table 5.1, an expected investment profit derives (the average simulated annual interest rate is slightly greater than 0.04). Hence, results quoted in this Section are not comparable with those of Section 4.

Table 5.2 shows the solvency margin required at time 0 according to condition (3.11) ($M_0^{(R)}$), with both $T = n$ and $T = 5$, as well as to (3.19) ($M_0^{(O)}$). The greater severity of requirement (3.11) (both in the case $T = n$ and $T = 5$) is due to the fact that assets, cumulating at a random rate, are compared with the reserve.
calculated according to a financial hypotheses which could be significantly different from the actual investment performance. On the contrary, under condition (3.19) assets are compared with the present value of future obligations, calculated under the same investment scenario (actually, (3.19) reduces to the inspection of $A_n$); hence, a sort of offsetting effect arises for the investment risk when the obligations-based approach is considered, i.e. when the ability to meet future liabilities is ascertained on realistic grounds. It must be stressed, in particular, that contrarily to the deterministic financial setting (Section 4), $M_0^{(O)} < M_0^{(R)}$ also in the case $T = 5$, witnessing the fact that investment risk reveals itself already in the short run. As far as the size of the portfolio is concerned, in any case the entity of the relative required solvency margin ($\frac{M_0}{V_0}$) is almost constant; actually, the investment risk is non-pooling and its effect, in relative terms, is independent of the size of the portfolio. Finally, it should be kept in mind that results in table 5.2 (and 5.3 as well) have been obtained in presence of an expected (positive) investment profit. In order to understand its effect, we mention that adopting a deterministic financial scenario and assuming $i^*_t = 0.04$ (hence including an investment profit whose magnitude is comparable to that obtained in the stochastic financial framework) it turns out that no solvency margin is required at time 0, the demographic risk being completely covered by the expected financial profit.

\[
\begin{array}{cccccc}
N_0 & T=n & & & T=5 & \\
 & M_0^{(R)} & \frac{M_0}{V_0} & M_0^{(O)} & \frac{M_0}{V_0} & M_0^{(O)} & \frac{M_0}{V_0} \\
1000 & 89,226 & 5.092 \% & 69,633 & 4.606 \% & 27,737 & 1.835 \% \\
2000 & 177,550 & 5.872 \% & 140,135 & 4.634 \% & 55,639 & 1.840 \% \\
3000 & 265,559 & 5.855 \% & 211,177 & 4.656 \% & 82,428 & 1.817 \% \\
4000 & 352,875 & 5.835 \% & 281,198 & 4.650 \% & 108,223 & 1.790 \% \\
5000 & 441,260 & 5.837 \% & 351,888 & 4.655 \% & 135,558 & 1.793 \% \\
6000 & 529,288 & 5.835 \% & 422,707 & 4.660 \% & 162,775 & 1.794 \% \\
7000 & 616,138 & 5.822 \% & 492,382 & 4.653 \% & 188,919 & 1.785 \% \\
8000 & 703,182 & 5.814 \% & 563,850 & 4.662 \% & 216,284 & 1.788 \% \\
9000 & 790,749 & 5.811 \% & 634,518 & 4.663 \% & 243,852 & 1.792 \% \\
10000 & 879,061 & 5.814 \% & 705,166 & 4.664 \% & 271,403 & 1.795 \% \\
\end{array}
\]

Table 5.2 - Required solvency margin; demographic deterministic approach, $\varepsilon = 0.025$

In a stochastic approach to mortality, the causes of uncertainty are three: the distribution of future lifetimes, the time of death of each insured and investment performances. As before, we assume independence between demographic and financial variables; further, we assume that, once the mortality distribution has been assigned, annuitants have independent durations of life. Results in table 5.3 have been obtained through simulation (assuming $\rho_{[\text{min}]} = 0.2$, $\rho_{[\text{med}]} = 0.6$ and $\rho_{[\text{max}]} = 0.2$), with valuation time 0. Note the sharp increase in the margins $M_0^{(R)}$ under $T = n$, and $M_0^{(O)}$ with respect to table 5.2, obviously due to longevity risk. In particular, since longevity risk emerges in the long run, $M_0^{(O)}$ is now greater than $M_0^{(R)}$ when $T = 5$. Note also that comparing table 5.3 with 5.2 it turns out
that $M_0^{(O)}$ is more seriously affected by longevity than investment risk. Actually it must be stressed that, contrarily to table 5.2, $M_0^{(R)}$ under $T = n$ and $M_0^{(O)}$ have the same magnitude. This shows that for longevity risk neither a pooling effect (as for the random fluctuation risk) nor an offsetting effect (as for the investment risk) can be obtained, albeit the investigation is performed on realistic grounds. In any case, as in table 5.2, the relative required solvency margin $\frac{M_0}{V_0}$ is almost constant, due to the non-pooling effect of both longevity and investment risk. In order to catch the effect of investment profit, we point out that numerical evaluations performed within a deterministic financial setting lead to a required solvency margin of nearly 3% of the initial reserve in the reserve-based approach with $T = n$ and in the obligations-based approach and to no margin required in the reserve-based approach with $T = 5$.

<table>
<thead>
<tr>
<th>$N_0$</th>
<th>$T=n$</th>
<th>$\frac{M_0^{(R)}}{V_0}$</th>
<th>$\frac{M_0^{(O)}}{V_0}$</th>
<th>$T=5$</th>
<th>$\frac{M_0^{(R)}}{V_0}$</th>
<th>$\frac{M_0^{(O)}}{V_0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>234.809</td>
<td>15.531 %</td>
<td>105,820</td>
<td>6.999 %</td>
<td>239.627</td>
<td>15.453 %</td>
</tr>
<tr>
<td>2000</td>
<td>465.021</td>
<td>15.379 %</td>
<td>211,859</td>
<td>7.007 %</td>
<td>463.350</td>
<td>15.324 %</td>
</tr>
<tr>
<td>3000</td>
<td>696.735</td>
<td>15.361 %</td>
<td>318,642</td>
<td>7.095 %</td>
<td>694.796</td>
<td>15.319 %</td>
</tr>
<tr>
<td>4000</td>
<td>927.784</td>
<td>15.358 %</td>
<td>424,604</td>
<td>7.021 %</td>
<td>926.516</td>
<td>15.321 %</td>
</tr>
<tr>
<td>5000</td>
<td>1,160.254</td>
<td>15.349 %</td>
<td>531,348</td>
<td>7.029 %</td>
<td>1,157,321</td>
<td>15.310 %</td>
</tr>
<tr>
<td>6000</td>
<td>1,394.717</td>
<td>15.375 %</td>
<td>638,309</td>
<td>7.037 %</td>
<td>1,391,178</td>
<td>15.336 %</td>
</tr>
<tr>
<td>7000</td>
<td>1,622.656</td>
<td>15.333 %</td>
<td>743,647</td>
<td>7.027 %</td>
<td>1,618,809</td>
<td>15.296 %</td>
</tr>
<tr>
<td>8000</td>
<td>1,857.986</td>
<td>15.362 %</td>
<td>851,135</td>
<td>7.037 %</td>
<td>1,853,379</td>
<td>15.324 %</td>
</tr>
<tr>
<td>9000</td>
<td>2,090.759</td>
<td>15.366 %</td>
<td>957,496</td>
<td>7.037 %</td>
<td>2,086,065</td>
<td>15.331 %</td>
</tr>
<tr>
<td>10000</td>
<td>2,317.505</td>
<td>15.329 %</td>
<td>1,063,397</td>
<td>7.034 %</td>
<td>2,313,219</td>
<td>15.300 %</td>
</tr>
</tbody>
</table>

Table 5.3 - Required solvency margin; demographic stochastic approach, $\varepsilon = 0.025$

Fig. 5.1 and 5.2 offer an overall comparison among some of the results discussed in Sections 4 and 5.

6. Concluding remarks
Solvency requirements for a portfolio of life annuities have been analyzed. Particular emphasis has been given to the longevity risk, i.e. the demographic risk originated by possible systematic deviations from the projected mortality assumed in the pricing basis.

Solvency is traditionally defined in terms of comparisons between assets and portfolio reserve. In order to avoid problems arising from the choice of the valuation basis used in reserving, solvency has been defined also in terms of random values of future obligations.

Several numerical examples illustrate solvency requirements produced by the two different approaches. In particular, the results obtained taking into account the demographic as well as the financial risk provide an interesting description of the riskiness inherent in a portfolio of life annuities.

Numerical results only provide an illustration, since they obviously depend on

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the specific assumptions concerning the randomness in mortality as well as on the model used to describe the investment performance. Nevertheless we think that the results underline the dramatic importance of a sound evaluation of the solvency requirements for life annuities (and, more generally, for insurance covers.
providing lifetime living benefits).

Further work should concern the construction of appropriate short-cut formulae, expressing the required fund in terms of quantities which properly reflect the portfolio characteristics (number of policies, average age of the annuitants, etc.); in particular, the required fund should be related to some "objective" quantity as, typically, the total amount yearly paid to the annuitants, rather than the portfolio reserve which is usually considered in traditional short-cut formulae, but whose amount heavily depends on the chosen valuation basis.

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