A GENERAL THEORY OF FINANCIAL RISK

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ABSTRACT

The paper describes the construction and application of a new model of financial risk which recognises that in the perception of risk, whether physical of financial, the human mind acts as an analogue computer at an essentially subconscious level. The new theory is obtained by first of all constructing a theory of physical risk in potentially dangerous sports and then translating this into the financial context. The probability of ruin, variance of return and expected utility are all shown to be restrictive special cases of the new measure of risk obtained. The new theory can explain most of the well documented instances, such as the Allais Paradox, of observed behaviour that are inconsistent with currently accepted financial theories, and in particular the behavioural finance traits of over-confidence, over-reaction bias and myopic loss aversion are predictions of the new theory. It is shown that for long-term investment equities involve far lower risk than long-dated bonds, and some wider investment and risk management implications of the new theory are also discussed.

KEYWORDS

Financial Risk; Physical Risk; Expected Utility; Investment; Behavioural Finance; Stochastic Models

1. INTRODUCTION

1.1 *Objectives*

1.1.1 The primary objective of this paper is to construct from first principles a general theory of financial risk that can not only explain real world behaviour much more accurately than previous theories but can also lead to the attainment of better standards of financial risk management by individuals, financial institutions and governments alike.

1.1.2 The secondary objective is to provide solid foundations on which to build an actuarial theory of finance that might in the not too distant future displace financial economics as the generally accepted scientific framework for prudent financial behaviour.

1.2 *Actuarial approaches to risk*

1.2.1 Actuaries first came to prominence as financial experts through their ability to measure and manage mortality risk in the life assurance and pension fund contexts. The foundation work for this expertise was the empirical investigation set out in Halley (1696), which describes the construction of the first "scientific" life table.

1.2.2 Some actuaries then applied their mathematical and practical skills to general insurance, and in the process developed a new "risk theory" covering loss functions and the probability of ruin.

1.2.3 The actuarial profession responded to the upsurge in interest in financial risk during the 1980s by setting up AFIR (Actuarial Approach for Financial Risk) as the finance section of the International Actuarial Association. However, although AFIR has now been in existence, and running international colloquia, for more than ten years, no dominant single approach to financial risk has emerged. Instead, colloquium papers have typically reflected a wide spectrum of approaches to risk, with papers following the risk methodologies of financial economics tending to become more frequent over recent years.

1.2.4 This absence of a dominant single approach to risk is reflected in influencial actuarial textbooks such as Daykin, Pentikainen & Personen (1994) and Booth, Chadburn, Cooper, Haberman & James (1999). Various different approaches to risk are shown as being appropriate in different application areas, but there is little conceptual cohesion between the different approaches. A very desirable attribute of any new theory of financial risk will accordingly be a sufficiently high degree of generality to allow all useful existing risk methodologies to be regarded as important special cases.

1.3 *Expected utility*

1.3.1 The expected utility approach pioneered by Bernoulli (1738, 1954) and later developed along rigorous mathematical lines by von Neumann & Morgenstern (1944) is one of the cornerstones of present day economic science and is perhaps the most widely used theoretical framework for human choice under conditions of uncertainty and risk.

1.3.2 As will be described in detail below, not only has utility theory been severely criticised by many eminent economists, but its predictions are in many well-documented cases markedly inconsistent with observed real world behaviour. There is accordingly considerable doubt as to whether utility theory can mirror the deliberative thought processes of "reasonable" individuals.

1.4 *Anomaly and paradigm shift*

1.4.1 In the physical sciences, there have been many important instances of revolutionary new theories displacing familiar old theories when, despite considerable effort and ingenuity on the part of their adherents, these old theories cannot be made accurate enough in their explanation of certain features of observed behaviour. The classic example was the Copernican Revolution in astronomy, when the old system - predicated on the wrong causal mechanism of the earth being the centre of the universe - was replaced by the new system in which all planets orbit around the sun.

1.4.2 No less radical a change of world view may be needed in the search for a new and better theory of financial risk. In particular, some of the seemingly innocuous axioms of "rational behaviour" may have to be modified or abandoned completely.

1.5 *Neural mechanics of physical risk*

1.5.1 The starting point in the derivation of the new theory is the recognition that in the assessment of risk of any type the human mind acts as an analogue computer rather than as a digital computer, with the strength of the risk perception neural response being determined almost instantaneously at a subconscious level rather than as a time-consuming quantitative computation. A corollary is that the neural mechanics of financial risk, where the unwanted outcomes are financial distress or financial ruin at the personal or corporate level, will be identical to the neural mechanics of physical risk, where the unwanted outcomes are injury or death.

1.5.2 Since everyday experience provides a vast amount of observational data as to how the human mind assesses and manages physical risk, it should be possible to construct in the first instance a theory of physical risk and then to translate this into a theory of financial risk.

1.6 *Physical risk in sports*

1.6.1 Adam Smith observes in the "division of labour" discussion in his "Wealth of Nations" that "men now do for sport what they once did to earn their livelihood". Given also that our ability to master physical risk has evolved and improved over millions of years, we can deduce that both our perception of physical risk and our ability to avoid unnecessary mistakes in managing it will be far better than for the relatively new challenge of financial risk.

1.6.2 Clearly it will be potentially risky sports such as ski-ing and rapid river canoeing, rather than low risk sports such as tennis and golf, that will provide the necessary insights into human choice under conditions of uncertainty and risk.

1.7 *Translation into financial risk*

1.7.1 The translation of a theory of physical risk, derived from the consideration of human behaviour in potentially risky sports, into a theory of financial risk should be relatively straightforward if the conjecture as to identical neural mechanics is well founded.

1.7.2 It is likely, however, that there will be two crucial areas of divergence. Firstly, in the financial case perceptions of risk will often be inaccurate, possibly wildly so. Secondly, instances of behaviour where avoidable mistakes are made are likely to be far more frequent than for physical risk.

1.8 *Structure of the paper*

1.8.1 If the new theory of financial risk is to have any claim to generality, it must not only be consistent with the broad conceptual guidelines suggested by eminent economists but must also be able, in numerical examples, to account for real world behaviour that is anomalous in the context of "old" theories. Accordingly, Section 2 discusses perceptive observations by various eminent economists and also a number of instances of "anomalous" behaviour.

1.8.2 The construction of the new and more general theory of risk falls into two parts. Section 3 follows the approach first suggested in outline in Clarkson (1989) and describes the construction of a theory of financial risk that represents an extension of the framework described in Clarkson (1989, 1990).

1.8.3 Three levels of application of the new theory are described. Section 5 discusses elementary applications including the resolution of paradoxes relating to behaviour that is clearly anomalous within the paradigm of financial economics. Section 6 applies the new theories to the "equities versus bonds" debate, and Section 7 discusses some wider implications of the new theory as a guide to prudent financial behaviour.

1.8.4 Since the new theory represents, in the terminology of Kuhn (1970), a "new paradigm" rather than a "normal-science" refinement of existing theories, most of the "conclusions" section, Section 8, describes the advantages of the five criteria - accuracy, consistency, scope, simplicity and fruitfulness - suggested by Kuhn (1977) as being of prime importance when the relevant scientific community has to decide whether or not to adopt a new theory.

2. ANOMALIES AND INCONSISTENCIES

2.1 *Observations by Adam Smith*

2.1.1 In his "Wealth of Nations", Smith (1776, 1976) observes how the "absurd presumption in their own good fortune" on the part of most people leads to behaviour that is blatantly inconsistent with the "rational behaviour" cornerstone of present day economics:

> "The chance of gain is by every man more or less over-valued, and the chance of loss is by most men under-valued."

2.1.2 Smith cites the popularity of lotteries, where the expected payout is always well below the price of a ticket, as a classic example of the overvaluation of gains. It is salutory to note that in January 1999 the Chairman of the U.S. Federal Reserve used the phrase "lottery mentality" in connection with his "irrational exuberance" warning that the aggregate market capitalisation of internet-related stocks was vastly in excess of what could be justified by the likely aggregate future profits of the industry. Also, in the first week of January 2000, when major stockmarkets experienced sharp setbacks after the "new millennium" euphoria of December 1999, the Governor of the Bank of England warned that it was very easy for stockmarkets to become seriously overvalued.

2.1.3 As his flagship example of how most people undervalue risk as a result of "thoughtless rashness and presumptuous contempt", Smith cites the failure of many people to insure against serious wealth-destroying hazards such as fire and shipwreck, even although appropriate insurance cover is often readily available at reasonable cost. He also refers to a "nice calculation" on a probabilistic basis whereby risk may be reduced to an acceptable level through what would today be called self-insurance:

> "When a great company, or even a great merchant, has twenty or thirty ships at sea, they may, as it were, insure one another. The premium saved upon them all may more than compensate such losses as they are likely to meet with in the common course of chances."

This provides our first pointer towards a new theory of risk, namely that risk cannot be eliminated completely but must be brought down below some small value that is deemed acceptable in all the circumstances.

2.1.4 Smith begins his discussion of the determinants of wages and profit by observing that the theoretical state of equilibrium that would result from every man's self-interest to "seek the advantageous and to shun the disadvantageous" does not in fact occur, largely because of factors that exist only "in the imaginations of men". This is perhaps the first documented evidence of what might today be called "systematic irrationality".

2.1.5 In the context of commodity prices (of which stockmarket prices are perhaps the most important present day examples), Smith observes that, rather than always being close to what he called their "natural" (or equilibrium) value, the observed market prices often differ markedly from these "natural" prices:

> "The natural price, therefore, is, as it were, the central price, to which the prices of all commodities are continually gravitating. Different accidents may sometimes keep them suspended a good deal above it, and sometimes force them down even somewhat below it. But whatever may be the obstacles which hinder them from settling in this centre of repose and continuance, they are constantly tending towards it."

Smith discusses the annual prices of corn, the most important commodity several hundred years ago, to illustrate the general behaviour of commodity prices. It is instructive to note that his numerical approach of using 10-year moving averages as reference values is identical in principle to the Mean Absolute Deviation analysis of stockmarket prices described in Plymen & Prevett (1972) and Clarkson (1978, 1981).

2.1.6 In his "History of Astronomy", Smith (1795, 1980), when setting out his objectives for the study of philosophy (what we would now call science), comments on an often overlooked "overconfidence" facet of human behaviour, namely the (often wrong) presumption that currently accepted theories in a particular field of human endeavour always represent the best that will ever be available:

> "Let us endeavour to trace philosophy, from its first origin, up to that summit of perfection to which it is at present supposed to have arrived, and to which, indeed, it has equally been supposed to have arrived in almost all former times".

2.2 *Observations by John Maynard Keynes*

2.2.1 In Chapter 12 of Keynes (1936) it is observed that there are not, in reality, two separate factors affecting the rate of investment, namely expected return and perceived risk in terms of the "state of confidence"; confidence effectively dominates whether investment will be contemplated or not. In other words, no matter how attractive the expected return, investment will not be contemplated unless the probability of failure (however defined) is acceptably low.

2.2.2 Keynes also observes that a purely quantitative approach is the exception rather than the rule:

> "Most, probably, of our decisions to do something positive, the full consequences of which will be drawn out over many days to come, can only be taken as a result of animal spirits - of a spontaneous urge to action rather than inaction, and not as the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities."

2.2.3 Keynes uses a geometry metaphor to emphasise what he saw as the serious limitations of the then current axioms of economic science:

> "The classical theorists resemble Euclidean geometers in a non-Euclidean world who, discovering that in experience straight lines apparently parallel often meet, rebuke the lines for not keeping straight - as the only remedy for the unfortunate

collisions which are occurring. Yet, in truth, there is no remedy except to throw over the axiom of parallels and to work out a non-Euclidean geometry. Something similar is required today in economics."

This provides a second pointer towards a new and better theory of risk: since the dominant scientific paradigm is the one-dimensional expected utility approach, we should in the first instance attempt to find a two-dimensional approach where expected utility can be regarded as a (possibly inaccurate) special case.

2.2.4 In his much earlier "Treatise on Probability" Keynes (1921) suggests various ways in which to achieve better understanding of how the human mind perceives probability and risk. In particular, he is strongly distrustful of the marginal utility of wealth approach that Daniel Bernoulli (1738, 1954) relied upon to "solve" the famous St Petersburg Paradox, and observes that what might be called tacit knowledge, especially regarding Peter's ability to pay Paul and the enormous risk of Paul incurring a serious loss, leads to considerable "psychological doubt" which makes a purely mathematical approach difficult:

> "We are unwilling to be Paul, partly because we do not believe Peter will pay us if we have good fortune in the tossing, partly because we do not know what we should do with so much money or sand or hydrogen if we won it, and partly because we do not think it would be a rational act to risk an infinitely larger one, whose attainment is infinitely unlikely. When we have made the proper hypotheses and have eliminated these areas of psychological doubt, the theoretical dispersal of what element of paradox remains must be brought about, I think, by a development of the theory of risk."

2.2.5 Keynes also suggests an extension of the second maxim of Jacques Bernoulli (an uncle of Daniel), which states that we must take into account all the information we have:

> "But should this maxim not be reinforced by a further maxim, that we ought to make the weight of our arguments as great as possible by getting all the information we can? .... But there clearly comes a point when it is no longer worth while to spend trouble before acting, and there is no evident principle by which to determine how far we ought to carry our maxim of strengthening the weight of our argument."

2.2.6 Keynes discusses instances of where the human mind appears to ignore the risk when it is below some very small value, and cites an interesting observation by the French philosopher Buffon (1777):

> "I am thinking of such arguments as Buffon's when he names 1/10,000 as the limit, beyond which probability is negligible, on the grounds that, being the chance that a man of 56 taken at random will die within a day, it is practically disregarded by a man of 56 who knows his health to be good."

2.3 *Observations by von Neumann & Morgenstern*

2.3.1 In von Neumann & Morgenstern (1944), the foundation work of modern utility theory, it is assumed that human choice under conditions of uncertainty and risk is based on "rational behaviour" as defined by a set of seemingly innocuous utility axioms. Even the authors themselves had serious doubts about the validity of certain of the axioms, particularly (3:C:b), but they concluded that this axiom was "plausible and legitimate, unless a much more refined system of psychology is used than the one now available for the purposes of economics".

2.3.2 The authors also observed that "the common individual, whose behaviour one wants to describe, does not measure his utilities exactly but rather conducts his economic activities in a sphere of considerable haziness." Despite this admission that "rational behaviour" as defined by their utility axioms was more likely to be the exception rather than the rule in the real world, they expressed their belief that at some future date the benefits of their utility approach might be significant:

> "Once a fuller understanding of economic behaviour has been achieved with the aid of a theory which makes use of this instrument, the life of the individual might be materially affected."

2.3.3. Many eminent economists of the day, such as Friedman, Malinvaud, Samuelson and Savage, were highly critical of the von Neumann & Morgenstern utility axioms. However, mathematicians with no practical experience of economics tended to brush these criticisms aside, as exemplified by the highly favourable review in the Bulletin of the American Mathematical Society:

> "Posterity may regard this book as one of the major scientific achievements of the first half of the twentieth century. This will undoubtedly be the case if the authors have succeeded in establishing a new exact science - the science of economics. The foundation which they have laid is extremely promising".

2.3.4 In the Preface to the Third Edition, von Neumann & Morgenstern brushed aside all criticism of their approach, claiming that they had "applied the axiomatic method in the customary way with the customary precautions".

2.4 *Observations by Maurice Allais*

2.4.1 By far the most powerful attack on the highly mathematical approach of utility theory was from Allais (1953), who argues that there is no single-valued function (such as a value of expected utility) which can provide an accurate guide as to how deliberative choices are made by "reasonable" men. He accordingly concludes that the expected utility maxim cannot be regarded as the criterion of rational behaviour.

2.4.2 The now famous "Allais Paradox" is a counterexample Allais used to support his rejection of the expected utility maxim. When given the choice between receiving £1m with certainty or of receiving nil, £1m and £5m with probabilities of 0.01, 0.89 and 0.1 respectively, most subjects choose the former, but when given the choice between receiving nil or £1m with probabilities of 0.89 and 0.11 respectively or of receiving nil or £5m with probabilities of 0.9 and 0.1 respectively, most of the same subjects choose the latter. Such a combination of choices is inconsistent with any expected utility function. Reference will be made at the end of this section to the following similar but less contrived choice. A businessman has sought your actuarial advice as to whether or not he should put his entire working capital of £1m at risk for a business opportunity which will lead either to a profit of £80,000 with probability 0.95 or to a loss of all £1m of his working capital with probability 0.05, giving an expected profit of £26,000.

2.4.3 In his Nobel Lecture, given on 9 December 1988, Allais (1989) suggests that his paradox demonstrates what he calls "the preference for security in the neighbourhood of certainty". This is a facet of prudential human behaviour that is (to use the phraseology of von Neumann & Morgenstern) "far more refined" than the "rational behaviour" assumptions on which expected utility is based.

2.4.4. Allais (1954) draws attention to the very serious dangers of building an apparently rigorous mathematical theory on simplifying assumptions that have no real world relevance, and accordingly he suggests that only those who have extensive practical experience gained over a period of many years should attempt to formulate economic models.

2.5 *Observations by William Sharpe*

2.5.1 Sharpe (1970) investigates utility theory as a plausible framework for the implementation of the Markowitz (1959) mean-variance approach to portfolio selection, and observes that, of the various possible utility curves that have been proposed, "Only one is completely consistent with choices based solely on expected return and standard deviation of return: the assumption that utility is a quadratic function of wealth". However, on investigating the implications of using a quadratic utility function, he discovers some serious inconsistencies and draws the following conclusions:

> "In some instances, investors will be concerned with more than the expected return and standard deviation of return. In such cases a quadratic utility curve will imperfectly approximate an investor's actual utility curve. If portfolios with radically different prospects are considered by an investor, too much reality may be omitted if his decision is assumed to depend only on expected return and standard deviation of return."

2.5.2 Despite these serious inconsistencies, Sharpe suggests that the use of a utility curve may still be justified if it is assumed that investors choose amongst portfolios of roughly similar risk. However, in many suggested applications of utility theory, such as whether or not to insure against the risk of a serious financial loss, the risk levels of the scenarios being compared differ enormously.

2.6 *Attempted generalisations of expected utility*

2.6.1 Until the late 1970s, most economists believed that agents were rational and that expected utility provided a highly satisfactory framework for human choice under conditions of uncertainty and risk. However, by the early 1980s the voluminous experimental evidence of axiom violations that had been published over the previous decade, particularly by Kahneman & Tversky (1979) and Grether & Plott (1979), forced economic theorists to attempt to build more complex new theories that could give a better explanation of real world behaviour. Anand (1993), Machina (1987) and Quiggin (1993) have been especially prolific in first of all documenting axiom violations (particularly in the areas of "independence" and "transitivity") and then suggesting more and more complex generalised axiomatic approaches.

2.6.2 The situation is reminiscent of the failure of Ptolemaic astronomy and its eventual replacement by the Copernican system. Despite numerous attempted generalisations down through the centuries, and its latterly horrendous complexity, the Ptolemaic system could not be made accurate enough to achieve its primary objective, namely to provide a practical framework for safe navigation around the world. Also, the Ptolemaic system was predicated on the totally erroneous belief that the earth was the centre of the universe; the "rational behaviour" belief of mainstream economic science in general and of expected utility theory in particular may similarly be seen at some future date as having been no less erroneous.

2.7 *Absence of an unambiguous basic measure of risk*

2.7.1 In virtually all branches of science where a mathematical approach is attempted, an unambiguous basic measure of key attributes is a necessary condition for the subsequent successful development of a body of theory which can accurately describe real world behaviour. In particular, we need to be able to make unambiguous statements along the lines of "the value in the case of A is twice the value in the case of B".

2.7.2 No such unambiguous basic measure exists for financial risk. In particular, standard deviation of return and variance of return are both used as a measure. However, if the standard deviation in the case of A is twice that for B, the variance for A is four times that for B. What is the "true" value of risk for A as a multiple of that for B? The probability of ruin is also used as a measure of risk, but there is no obvious link between this "non-parametric" measure and a "parametric" measure such as variance. Furthermore, the Risk Assessment and Management for Projects (RAMP) methodology, which has been put forward jointly by the UK actuarial profession and the Institution of Civil Engineers as a basic framework for practical risk management, does not incorporate an explicit numerical measure of risk. Can such an apparently informal approach be regarded as "scientific"?

2.8 *Behavioural finance*

2.8.1 Over the past decade or so, numerous economists and psychologists have established a new branch of economic science, namely behavioural finance, which studies behavioural patterns that might be regarded as "systematic irrationality" under mainstream theories of finance where "rational behaviour" is an essential cornerstone.

2.8.2 The four most familiar behavioural finance traits are "framing and coding", where observed behaviour differs depending on how the relevant information is presented to the subject, "over-confidence", as first diagnosed by Adam Smith, "over-reaction bias", which would explain the "excess volatility" documented by Shiller (1989) and others, and "myopic loss aversion", where subjects choose high risk courses of action despite the availability of convincing evidence as to the existence of more profitable courses of action that also involve lower risk.

2.9 *Myopic loss aversion*

2.9.1 Any financial disadvantage resulting from the first three of these four behavioural traits could be mitigated to a considerable extent by the availability of more detailed information, presented in as impartial a manner as possible. Myopic loss aversion, however, is a much more deeply ingained wealth-destroying behaviour trait. A classic physical risk example is a refusal to fly for either business or pleasure purposes, despite the existence of vast amounts of statistical evidence showing that going by car is vastly more risky, in terms of deaths per passenger mile, than flying with a recognised airline.

2.9.2 The classic financial risk example is a preference on the part of many investors for long-term investment in bonds rather than equities, despite very strong evidence that the likelihood of equities outperforming bonds increases to near certainty as the investment horizon increases. This equities versus bond question is discussed in detail in Section 6.

2.10 *The Tversky Paradox*

2.10.1 Tversky (1978) questions the absolutely fundamental assumption that individuals are "risk-averse" in the generally accepted sense of preferring, for a given expected value, the choice which involves the lowest uncertainty of return, as measured, for example, by the standard deviation or variance of return. If investors have the choice between a gain with certainty of £85,000, or an 85% chance of gaining £100,000 and a 15% chance of gaining nothing, most will choose the former, certain, outcome, which is consistent with standard theory.

2.10.2 Suppose now that investors have a choice between losing £85,000 with certainty, or an 85% chance of losing £100,000 and a 15% chance of losing nothing. Most people will "gamble" and choose the latter, which is inconsistent with standard theory.

2.11 *A "risk equals uncertainty" paradox*

2.11.1 An insurance company with assets at present of 100 requires assets at some future date of 110 or more to achieve what it regards as a satisfactory return on capital, and will be insolvent if assets have a value of 105 or less at that future date. The company can invest either in asset class A which will give 105 with certainty, or in asset class B which will give 110 or 115 with equal probability.

2.11.2 The paradox here is that textbooks on stochastic calculus as applied to finance, such as Lamberton & Lapeyre (1996), define asset class A as the "risk-free" asset and asset class B as "risky", in that it involves an uncertain outcome. Common sense shows that it is asset class A that is (pathologically!) risky, whereas asset class B is risk-free in that the criterion of a satisfactory return on capital is achieved with certainty.

2.12 *The other St Petersburg Paradox*

2.12.1 Bernoulli (1738, 1954) applies the logarithmic utility approach that he used to "solve" the St Petersburg Paradox to the case of a merchant shipping goods from St Petersburg to Amsterdam. He can sell his cargo for 10,000 ducats if the ship arrives safely, but there is a probability of 0.05 that the ship and cargo are lost at sea, with the requisite insurance cover being available for a premium of 800 ducats, which the merchant regards as outrageously high. Bernoulli asks what other wealth the merchant should possess for it be rational for him to choose not to insure. Bernoulli obtains the answer of 5,843 ducats, and is very pleased with his approach:

> "Though a person who is fairly judicious by natural instinct might have realized and spontaneously applied much of what I have here explained, hardly anyone believed it possible to define these problems with the precision we have employed in our examples. Since all our propositions harmonize perfectly with experience it would be wrong to neglect them as abstractions resting on precarious hypotheses".

2.12.2 Taking one ducat as being equivalent to £100, this example is identical to that at the end of Section 2.4.2, namely whether or not a businessman should put his entire working capital at risk for a business opportunity which will lead either to a profit of £80,000 with probability 0.95 or to a loss of all £1m of his working capital with probability 0.05, giving an expected profit of £26,000. Even with other wealth of around £600,000, the equivalent of Bernoulli's computation, most actuaries would strongly discourage the businessman from taking up the opportunity. The obvious corollary is that Bernoulli's logarithmic utility approach must after all be based on "precarious hypotheses".

3. PHYSICAL RISK IN SPORTS

3.1 *Severity of consequences of an adverse occurrence*

3.1.1 Consider the consequences of a serious hang-gliding accident, such as equipment failure, as a function of the height above ground at which the accident occurs. From a height of only a few feet, this might result in no injury or nothing worse than a sprained ankle, but from several hundred feet or more death would be almost certain. If we use the same range as for probability to represent the "loss function", we have 0 for no injury, 1 for death, and the general pattern shown in Figure 3.1 with $w(x)$ increasing with the severity of injury in the intermediate zone between no injury and death.

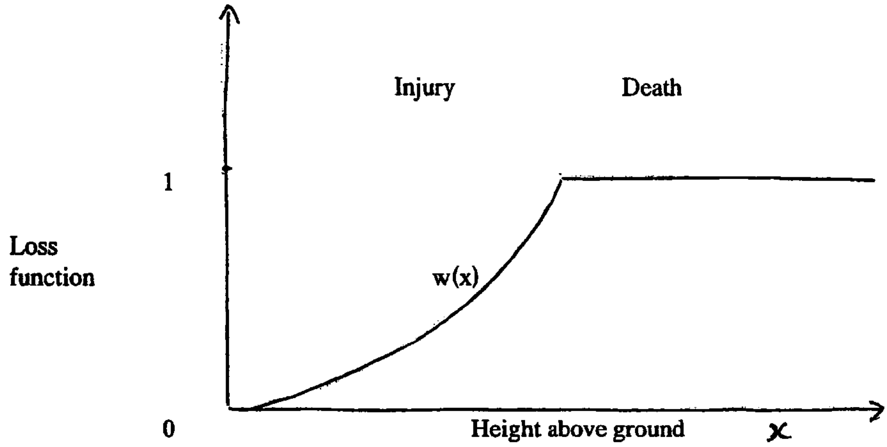

Figure 3.1

3.1.2 Suppose now that we classify the intermediate injuries into five broad categories - minor injury, moderate injury, serious injury, very serious injury and permanent incapacity. The human mind not surprisingly perceives the negative consequences of imminent death as being effectively infinite, with the result that lesser degrees of physical damage are "discounted" at a very high rate. Accordingly, we can, as a first guess, calibrate the severity function w(s) by using a factor of 10 between each reference point from minor injury up to death, giving the values shown below.

Severity	Consequence	w(s)
0	No injury	0
1	Minor injury	0.00001
2	Moderate injury	0.0001
3	Serious injury	0.001
4	Very serious injury	0.01
5	Permanent incapacity	0.1
6	Death	1

3.2 *Equivalent probability of death*

3.2.1 We can now, for example, say that for a given probability of death the perception of risk within the human mind is equivalent to that for serious injury when the associated probability is one thousand times as high. The function w(s) introduces a measure of equivalence between widely differing outcomes, namely death and varying degrees of injury, that could not previously be combined mathematically to produce an overall value for risk.

3.2.2 In ski mountaineering, a very risky sport, there are two dominant adverse occurrences, namely a bad fall and an avalanche. For each of these, we can first of all estimate the probability of an adverse occurrence taking place and then, given that it does in fact take place, estimate the probabilities (which will clearly sum to 1) for each outcome from no injury up to death. In an obvious notation we can now obtain the value of risk as:

$$R = \sum_{n=1}^{2} p(n) \sum_{s=1}^{6} p(n,s)\, w(s)$$

where p(1) and p(2) are the respective probabilities of a bad fall and avalanche and p(2,6) is, for example, the probability that an avalanche, if it occurs, will lead to death. The generalisation to a larger number of adverse occurrences is obvious. This measure of risk R can be called, for obvious reasons, the "equivalent probability of death", with the minimum value of 0 corresponding to no possibility of injury or death and the maximum value of 1 corresponding to imminent death with certainty.

3.3 *Estimation procedures*

3.3.1 The Poisson distribution is ideal for representing the severity function, with the value of 0 corresponding to no injury, 1 corresponding to minor injury, and so on up to 6 or higher corresponding to death. Since the mean of the Poisson distribution is equal to the variance, only one parameter has to be estimated for each adverse occurrence.

3.3.2 The following values of the probability of an adverse occurrence (for typical daily participation), of the associated Poisson parameter, and of the resulting values of risk are set out in Clarkson (1989), and are based not on comprehensive empirical data but on a combination of personal observation and "intelligent guesswork".

Sport	Adverse occurrence	Daily rate	Poisson parameter		Daily risk x10^6
Wind-surfing	Falling off board	5	0.01		0.5
Ski-ing	Bad fall	2	0.1		3.1
Rapid river canoeing	Capsize in rapids	1 in 5	0.5		13.8
Ski-mountaineering	{ Bad fall	2	0.2	}	106.4
	{ Avalanche	1 in 1,000	3	}	
Hang-gliding	Fall to ground	1 in 5,000	7		142.6

3.4 *Royal Society (1992) risk study*

3.4.1 An investigation into fatality rates in sports as part of a wider study of risk by the Royal Society of London covers two of the five sports for which risk values are estimated above. The corresponding values, after appropriate standardisation, are shown below.

Sport	Clarkson (1989) Daily Risk x10^6	Royal Society (1993) Deaths per day x10^6	
Ski-ing	3.1	{ 3.5	USA, 1967-68
		{ 6.5	France, 1974-76
Hang-gliding	142.6	{ 100-325	USA, 1978
		{ 375	UK, 1977-79

3.4.2 The close agreement in the case of the USA experience is quite remarkable, given the totally different methodologies used. The higher values for ski-ing in France and for hang-gliding in the UK are not surprising, but space does not permit a fuller discussion here.

3.5 *Personal threshold of maximum acceptable risk*

3.5.1 It is obvious from everyday experience that different individuals have vastly different tolerances of the level of physical risk in sports. The present author is prepared to participate in ski-mountaineering, which has a daily risk value of the order of 0.0001, but regards hang-gliding, where the risk may be around 50% higher, as "too risky". We can first of all observe that for each individual there is a personal threshold of acceptable risk and then, given that ski-mountaineering is towards the top end of the risk spectrum, take 0.0001 per day as an "upper guideline" value, possibly around the 90th percentile, for the maximum acceptable limit.

3.5.2 Many individuals regard ski-ing, where avalanches and collisions with trees or other skiers have given rise to many highly publicised fatalities in recent years, as "far too risky". In the light of the estimated risk values in Sections 3.3 and 3.4, the seemingly very low value of 0.000001 per day, equivalent to a chance of one in a million of being killed, can be taken as a plausible "lower guideline" value, corresponding perhaps to the 10th percentile.

3.6 *Indifference curves and axioms of choice*

3.6.1 There is clearly a risk, however small, in all day-to-day activities. But most people are prepared to fly on normal passenger aircraft for business or holiday purposes, and virtually everyone is prepared to travel by car, without giving any thought to the physical risk involved. Such travel decisions are thus made solely on the basis of other criteria such as cost, convenience and reliability. This accords with the observation by Buffon (1777) that the human mind ignores risk completely when its perceived value is very small. In potentially risky sports it is therefore reasonable to assume that very low levels of perceived risk will be ignored completely.

3.6.2 Suppose now that for every available sporting activity we can make a quantitative estimate E of "expected enjoyment" in addition to our "equivalent probability of risk" estimate R. Then the above discussion suggests that the situation can be portrayed by patterns of indifference curves in the E-R diagram of the type shown in Figure 3.2

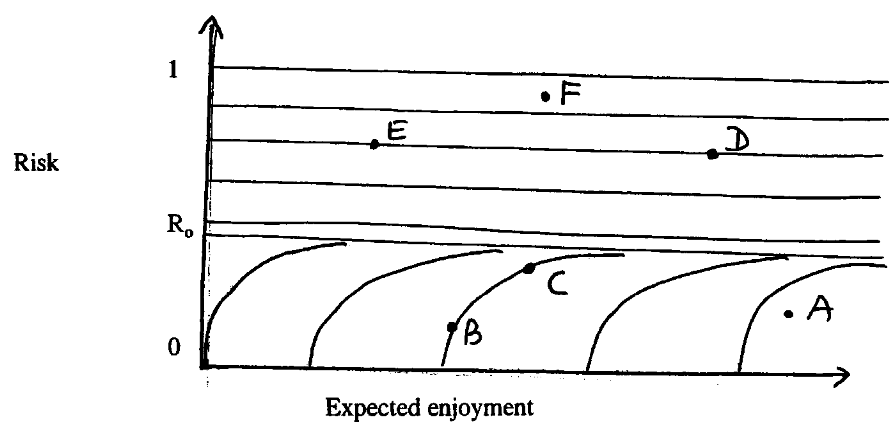

Figure 3.2

Above R_o, the indifference "curves" are essentially horizontal lines, while below R_o they are asymptotic to vertical lines for very low risk, have continuous and decreasing gradient as risk increases, and are asymptotic to the horizontal line $R = R_o$ as risk increases towards R_o.

3.6.3 The four simple rules required to determine preference and optimality can be illustrated by the points A, B, C, D, E and F in Figure 3.2. If at least one point has a risk value below R_o, the preferred point is A, the one to the right of the most indifference curves, whereas B and C, lying on the same indifference curve, are equally preferable. Otherwise, both D and E, lying below more indifference lines, are preferred to F, whereas - regarding the "horizontal" indifference lines as having infinitesimally small negative gradient - D is preferred to E.

3.6.4 It is obvious that transitivity is preserved under these preference and optimality rules, even although the situation is two-dimensional rather than one-dimensional as in the case of expected utility.

3.7 *The quantitative theory of physical risk*

The theory developed above can be summarised as follows:

(i) Identify all possible adverse occurrences than can lead to injury or death.

(ii) For each adverse occurrence, identify the probabilities p(n) and p(n,s) as defined in Section 3.2.

(iii) Identify the appropriate severity weighting function w(s) as defined in Section 3.

(iv) Calculate the values of risk as defined in Section 3.2.2.

(v) Calculate the values of expected enjoyment.

(vi) Identify the value of R_o, the maximum acceptable value of risk, and appropriate indifference curves for values of risk below R_o.

(vii) Use the preference rules in Section 3.6.3 to identify the activity that is optimal on risk/expected enjoyment considerations.

3.8 *The reality of physical risk*

3.8.1 Explicit quantitative evaluation along the lines summarised in Section 3.7 is of course totally impracticable; the required probability distributions cannot be determined with any precision, and the effective values of w(s) and R_o are determined at a subconscious level in response to innate self-protection mechanisms (such as a fear of heights) or to previous practical experience. Accordingly, the quantitative theory is put forward as a counsel of perfection as to how the human mind *should* perceive risk and determine optimality.

3.8.2 In line with the conjecture in Section 1.6.1 that both our perception of physical risk and our ability to manage it have evolved and improved over millions of years, it will be illuminating to explore some ways in which observed behaviour departs from what is predicted by the quantitative theory. Most of the examples below are very brief summaries of some of those discussed in Clarkson (1999).

3.9 *Over-confidence*

3.9.1 Although nearly all participants in potentially risky sports accept their "beginner" status when they first take part and only tackle more difficult challenges after they have acquired - generally under expert tuition and supervision - an understanding of the basic skills, it is only human nature that some individuals will be over-confident about their abilities and accordingly underestimate the true level of risk.

3.9.2 An unfortunate example of this has been the alarmingly high number of fatalities on Scottish mountains in recent years as a result of what has been called the "Surrey syndrome". Whereas those living not far distant will simply stay at home if the weather conditions are likely, expecially in winter, to make climbing unduly dangerous, those who have come a considerable distance will, on account of the time and expense they have devoted to their planned climbs, be much less likely to abandon their plans in the face of adverse weather forecasts.

3.10 *Innate failsafe behaviour*

3.10.1 Many people, when encountering a sport for the first time, will say to themselves "this looks great fun, but it is too risky for me!" However, after observation over a sufficiently long period and assuming that no accidents have occurred, they will often adopt the quite different mental attitude that the inherent level of risk is sufficiently low for them to be able to participate without undue worry.

3.10.2 This in many ways answers the question discussed in Section 2.2.5, namely how much further information should we seek out before pursuing a particular course of action. The above answer is to participate only after we have obtained sufficient information for us to be confident that the inherent level of risk is acceptably small.

3.11 *Over-reaction bias*

3.11.1 An actuary whom the author met at the 1994 AFIR International Colloquium in the United States had booked, and paid for, an introductory hang-gliding flight. But when he returned for his lesson a few days later he found that, of the three instructors at the company, one had a leg in plaster and another had an arm in a sling, both as a result of hang-gliding accidents in the previous two days. He immediately asked for, and was given, a full refund for the lesson that he now perceived as being foolhardily dangerous.

3.11.2 Such a change of mind is entirely reasonable, even although the accident experience might not - on purely statistical arguments - be inconsistent at a given confidence level with previous average rates. Just as with Einstein's General Theory of Relativity for physical behaviour, human behaviour is not "absolute" but is very much dependent on "locality", in this case recent personal experience.

3.12 *Temporary insanity*

3.12.1 For most people, the joys of sailing before long outweigh the serious discomfort of occasional bouts of seasickness. However, on an overnight yacht race in which the author participated one crew member became so ill and mentally disturbed through seasickness that he attempted to jump overboard to bring an end to his suffering and had to be tied down in the cockpit for his own safety until the gale force winds abated. The following morning he had returned to a normal frame of mind, and he later bought his own yacht.

3.12.2 Seasickness is the result of the human mind shutting down certain physical functions in response to being overwhelmed by contradictory signals from the delicate fluid canal balance system within the inner ear. A useful palliative measure is to help the mind to restore an effective sense of perspective by focussing on the horizon. In acute cases, however, all power of rational thought is lost and temporary insanity results.

4. TRANSLATION INTO FINANCIAL RISK

4.1 *Severity of consequences of an adverse occurrence*

4.1.1 For an insurance company, the "loss function" for hang-gliding portrayed in Figure 3.1 translates into the loss function in Figure 4.1, with the horizontal axis x being "assets less liabilities", and the vertical axis again being risk on a scale of 0 to 1.

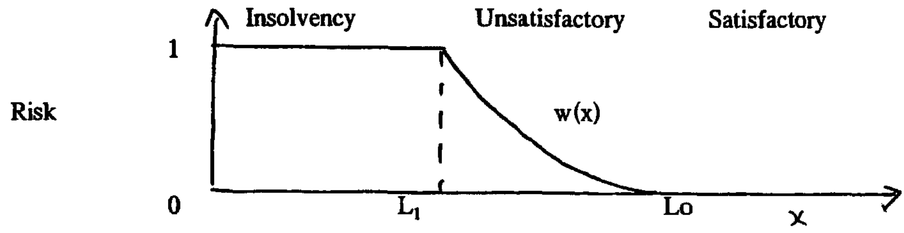

Figure 4.1

Below L_1, the outcome is "insolvency" and the risk value is 1. From L_0 to L_1, the outcome is "unsatisfactory" and risk follows a smooth curve w(x). Above L_0, the outcome is "satisfactory" and risk is zero.

4.1.2 For portfolio investment, assuming short positions are not permissible, L_1 corresponds to a portfolio value of nil, while L_0 corresponds to the point above which the outcome is deemed to be "satisfactory".

4.2 *Equivalent probability of financial ruin*

4.2.1 Assuming that we know p(x), the probability density function of outcome x, the financial risk R(x) translates into:

$$R(x) = \int_{-\infty}^{L_1} p(x)dx + \int_{L_1}^{L_0} w(x)\, p(x)\, dx,$$

where the first and second terms correspond respectively to "non-parametric risk" (the probability of financial ruin) and "parametric risk". On this formulation, the risk measure R(x) can, for obvious reasons, be interpreted as "the equivalent probability of financial ruin".

4.2.2 This framework unifies two existing approaches to risk that are useful in differing application areas. In "catastrophe" general insurance the "risk of ruin" component dominates and the parametric component can for most practical purposes be ignored. In portfolio investment, on the other hand, the parametric component dominates. In very high risk investment areas, where short positions may arise, both components may be important.

4.3 *Formulating the equivalence function*

For portfolio investment, the consequences of losing a given proportion of investment value are far more than twice as serious as losing half that proportion. We can therefore infer that the "parametric" function w(x) is concave upwards. A highly convenient formulation for w(x) is a power (greater than one) of the proportionate shortfall below L_0 towards L_1. Taking this power as two is not only eminently plausible in that the severity of the consequences quadruples as the shortfall doubles but is also broadly consistent with vast tranches of the financial economics literature in that variance of return and semi-variance can then be regarded as special cases when L_0 is equal to the mean return.

4.4 *Thresholds of maximum and minimum risk*

4.4.1 In the case of a financial company the threshold of maximum risk for which the value of risk attains its maximum value of 1 can normally be taken as the value of financial outcome below which insolvency results. For an individual, the threshold below which bankruptcy results will normally be appropriate.

4.4.2 For the threshold L_0 above which risk is zero it will normally be appropriate to use either the current value of wealth or net assets, or the minimum future value that is deemed to be satisfactory. Where, however, a higher value is available with certainty under one of the available choices, then this higher value should be used. For instance, if the choice for an individual with negligible current wealth is between receiving £1m with certainty and receiving either nil or £3m with equal probability, £1m is the appropriate value for L_0. This means that L_0 and hence the value of risk is not absolute but may depend on the available choices.

4.5 *Maximum level of acceptable risk*

4.5.1 We can use the uncertain outcome under the first half of the Allais Paradox to begin the calibration. The probability of 0.01 of receiving nothing as against being able to receive £1m with certainty involves a risk of 0.01 which can, on the basis of the experiments carried out by Allais, be taken as being in excess of the risk that nearly all "reasonable" individuals would accept. This suggests 0.005 as a plausible guess as to an "upper guideline" value that might correspond to the 90th percentile.

4.5.2 A probability of ruin of 0.001, one in a thousand, is generally regarded as a very prudent risk of ruin criterion. This suggest 0.001 as a plausible guess as to a "lower guideline" value that might correspond to around the 10th percentile.

4.6 *Indifference curves and axioms of choice*

Subject only to replacing "expected enjoyment" by "expected return", Section 3.6 translates word for word into the indifference curves and axioms of choice framework in the context of financial risk.

4.7 *The quantitative theory of financial risk*

Again subject only to replacing "expected enjoyment" by "expected return", Section 3.7 translates word for word into the corresponding summary of the new quantitative theory of financial risk.

4.8 *The reality of financial risk*

Again explicit quantitative evaluation, except in the simplest of cases, will be totally impracticable. Whereas the learning process in risky sports is very short term in nature, in that reasonable proficiency in, for instance, skiing or canoeing can be achieved under expert supervision within a few weeks, the relatively very slow evolution of economic and stockmarket scenarios means that the learning process will, at best, normally involve many years.

4.9 *Over-confidence*

The essentially short term neural response mechanism of the perception of risk, combined with the relatively long timescales of episodes of economic and stockmarket behaviour, suggests that in the face of apparently attractive investment opportunities far too little weight will be paid to historical evidence of similar scenarios that led to serious losses. Accordingly, over-confidence through an underestimation of the true level of risk is likely to be a not uncommon feature of stockmarket behaviour at a collective level. Classic examples are the "nifty fifty" boom in US growth stocks during the 1960s, the Poseidon nickel boom of 1969, and the meteoric rise in Japanese equities in the late 1980s.

4.10 *"Lack of confidence" and "myopic loss aversion"*
As with unfamiliar risky sports, most people have an innate awareness that certain financial or investment opportunities may be "too risky" for them until they have investigated the risks in sufficient detail. To begin with such a refusal to participate might be described as understandable and laudable "lack of confidence". However, once more background information is available, the relatively short term neural response mechanism for the perception of risk can be expected to attach too little weight to relevant long term data. This leads to "myopic loss aversion", the classic example of which is a refusal to contemplate long term investment in equities even although all the UK and US data show that the probability of equities outperforming bonds increases to near certainty as the investment horizon increases.

4.11 *Over-reaction bias*
The laudable human trait of trying to find causes for events, particularly those involving adverse consequences, means that financial reporting tends to give undue prominence to negative background factors when stockmarket prices have been falling. Accordingly, the neural response nature of the perception of risk will tend to amplify the recent trend, particularly when it is downwards. The equilibrium position implied by the cornerstone "rational behaviour" axioms of modern finance theory will never be achieved.

4.12 *Temporary financial insanity*
The obvious financial risk parallel to the sea-sick yacht crew member who tried to jump overboard is the "flight to liquidity" that occurs at the bottom of a bear market, when some investors try to sell at any price, no matter how far below any realistic long term value, to escape from the mental anguish of a seemingly endless series of price falls.

5. ELEMENTARY APPLICATIONS

5.1 *The Allais Paradox*

5.1.1 Assume, as is implicit, that any other wealth of the subjects can be ignored. Then for the first pair of choices, A and B, say, £1m is available with certainty if A is chosen and thus L_0 is £1m, and the obvious choice for L_1 is nil. Clearly the value of risk is nil for choice A. For choice B, the only shortfall below £1m is the outcome of nil, which corresponds to L_1 with probability 0.01; the value of risk is thus 0.01 x 1, i.e. 0.01. Since this value is greater than the higher guideline value for maximum acceptable risk, namely 0.005, A in chosen in preference to B, agreeing with the choice of most subjects.

5.1.2 For the second pair of choices, C and D, say, no value higher than nil is available with certainty, and hence L_0 is nil, and the value of risk is zero for both C and D. Since D has by far the higher expected value, namely £0.445m as against £0.1m, D is chosen is preference to C, again agreeing with the choice of most subjects.

5.1.3 A subject with enormous existing wealth, such as £1bn, would perceive the risk in B to be negligible and would accordingly choose B on account of the higher expected value, namely £1.39m as against £1m. It is interesting to calculate what other wealth a subject should possess for the risk under choice B to be equal to either the lower or the upper subject should possess for the risk under choice B to be equal to either the lower or the upper guideline value of maximum acceptable risk. For other wealth W (in millions of pounds), L_0 is now 1+W, L_1 is still nil, and the proportionate shortfall is $\frac{1}{1+W}$ if nil occurs under choice B. Using the square of the proportionate shortfall as the parametric risk function, we have:

$$0.01\,(\frac{1}{1+W})^2 = R_0,$$

which gives W = 2.16 when R_0 = 0.001 and W = 0.41 when R_0 = 0.005. These values of £2.16m and £0.41m for subjects with low and high risk tolerances respectively seem eminently sensible. Accordingly, a subject with existing wealth comfortably in excess of the appropriate minimum value would be expected to choose B in preference to A.

5.1.4 This extension of the Allais Paradox corresponds exactly to the thought experiment at the beginning of Bernoulli (1738, 1954), namely the deduction that, if a very poor man had somehow obtained a lottery ticket that would pay either 20,000 ducats or nil with equal probability, he would be unwise not to sell it for 9,000 ducats, whereas a very rich man would be unwise not to buy it for 9,000 ducats. The corresponding values of existing wealth for maximum acceptable risk are 195,000 ducats when R_0 = 0.001 and 81,000 ducats when R_0 = 0.005; again these values seem eminently reasonable.

5.2 *The Tversky Paradox*

5.2.1 Assume, as is implicit, that the level of existing wealth is less than £85,000, so that a loss of either £85,000 or £100,000 leads to financial ruin. Then a loss of £85,000 with certainty has 1 as the risk value, whereas the alternative, namely either a loss of £100,000 with probability 0.85 or a loss of nil with probability 0.15 has 0.85 as the value of risk, taking L_0 as the value of existing wealth. Since both values of risk are vastly in excess of any acceptable level, the latter will be chosen as having the lower, though still dangerously high, value of risk. This is in accord with Tversky's experiments.

5.2.2 It can easily be shown that, no matter how high the level of existing wealth, the value of risk is always lower on the latter, uncertain, scenario. Since the expected value is the same in both cases, the uncertain scenario should always be chosen under the new theory.

5.3 *A "risk equals uncertainty" paradox*

5.3.1 In the paradox described in Section 2.11, L_0 = 110 and L_1 = 106. Investment in asset class A, which gives 105 with certainty, has a risk value of 1, whereas investment in asset class B has a risk value of nil and also a higher expected value, namely 112.5. Investment in asset class B, which involves the uncertain scenario, is the blindingly obvious choice under the new theory.

5.3.2 An immediate corollary is that any so-called "theorem" relating to financial risk that has been derived using stochastic calculus is likely to be dangerously unsound, since in the abstract world of stochastic calculus asset class A would in the case of the present example be the "risk-free" asset.

5.4 *The other St Petersburg Paradox*

5.4.1 With W as the level of other wealth over and above the 10,000 ducats expected from the safe arrival of the ship at Amsterdam, the choice is between W + 10,000 with certainty if insurance is taken out, or W + 10,800 with probability 0.95 and W + 800 with probability 0.05 if insurance is not taken out. Taking L_0 = W + 10,000 and L_1 as nil, and using the square of the proportionate shortfall as the parametric risk measure, we obtain W =

55,054 for $R_0 = 0.001$ and W= 19,093 for $R_0 = 0.005$. Hence the value of other assets must be in excess of 55,054 ducats or 19,093 ducats for a merchant with low or high risk tolerance respectively before the option of not insuring the ship can be contemplated.

5.4.2 Since the corresponding minimum wealth value of 5,843 ducats that Bernoulli derives using his logarithmic utility function is very significantly lower than even the value of 19,093 ducats using the higher guideline value of acceptable risk under the new theory, it is difficult to avoid the conclusion that the use of a logarithmic utility function in financial risk investigations may often lead to dangerously unsound conclusions, in this case an equivalent probability of financial ruin of 0.020, four times the suggested upper guideline of maximum acceptable risk.

5.5 *Comparing profiles of financial outcomes*

5.5.1 On the present management strategy, strategy A, the financial outcome of an insurance company follows a normal distribution with mean 5 and variance 2. The value of 2, around two standard deviations below the mean, is seen as the minimum satisfactory outturn and hence is equivalent to L_0. L_1 is zero, since insolvency will occur for an outcome lower than this. The management of the company wish to investigate, subject to an over-riding requirement of the risk of insolvency not exceeding 0.001, the merits of adopting an alternative management strategy, strategy B, where the financial outcome corresponds to a normal distribution with mean 6 and variance 2.5.

5.5.2 Using the square of the proportionate shortfall as the measure of parametric risk, the values of risk are as below:

Strategy	Probability of ruin	Parametric risk	Total risk
A	0.0002	0.0018	0.0020
B	0.0001	0.0005	0.0006

Strategy B is accordingly chosen in preference to strategy A, since the expected value is higher, the value of total risk is very significantly lower, and the insolvency risk constraint is satisfied very comfortably.

5.5.3 This example has been chosen to correspond to Example 1.1 of the actuarial risk textbook Bowers *et al* (1986), where an exponential utility function is used to "prove" that the distribution N (5,2) has a higher value of expected utility than the distribution N (6,2.5), and hence is preferable, the heuristic justification being that the more diffuse nature of the distribution N (6,2.5) is a highly adverse factor.

5.5.4 To investigate this further example of an expected utility approach giving the opposite result to that obtained using the new "equivalent probability of ruin" theory, consider what might be called a "pseudo-utility" function u(x) defined as follows:

$$u(x) = \begin{cases} x & x \geqslant L_0 \\ x - \lambda\, w(x) & L_1 < x < L_0 \\ x - \lambda & x \leqslant L_1 \end{cases}$$

with w (x) having the properties of a parametric risk weighting function as described in Section 4, L_0 and L_1 being the thresholds of zero and maximum risk respectively, and λ being a constant. Then if p(x) is the probability density function of the financial outcome x, we can express the integral U(x) of "expected pseudo-utility" as:

$$U(x) = \int_{-\infty}^{\infty} u(x)p(x)dx = \int_{-\infty}^{\infty} xp(x)dx - \lambda\left\{\int_{-\infty}^{L_1} p(x)dx + \int_{L_1}^{L_0} w(x)\,p(x)dx\right\}$$
$$= E(x) - \lambda R(x)$$

where $E(x)$ is the expected value of x, and $R(x)$ is the value of risk under the new theory.

5.5.4 Since this expression appears to correspond, with λ interpreted as a Lagrange multiplier, to the utility theory solution where the indifference curves in the E - R diagram are straight lines with constant positive gradient, it might appear at first sight that expected utility could give the same results as the new theory, at least as a good first approximation. This is not the case, for two reasons. First, under the new theory the indifference curves in the E - R diagram do not have constant positive gradient. Below the threshold of maximum acceptable risk they are asymptotic to vertical straight lines for very low values of risk, with the gradient decreasing to zero as risk increases towards the threshold of maximum acceptable risk. Above this threshold they are essentially horizontal straight lines. Second, the shape of the "pseudo-utility" function is rectilinear, with the same positive gradient, both below L_1 and above L_o, as shown in Figure 5.1, whereas all standard utility curves have continuous gradient and convexity and hence, outside the range L_1 to L_o, follow the general shapes shown by the dotted curved lines. The new theory can thus predict that standard utility functions will often lead to unsound conclusions, with in particular an exponential utility function - which tends to minus infinity very rapidly as x decreases - attaching a pathologically high weighting to minuscule levels of downside risk that are of no real world significance.

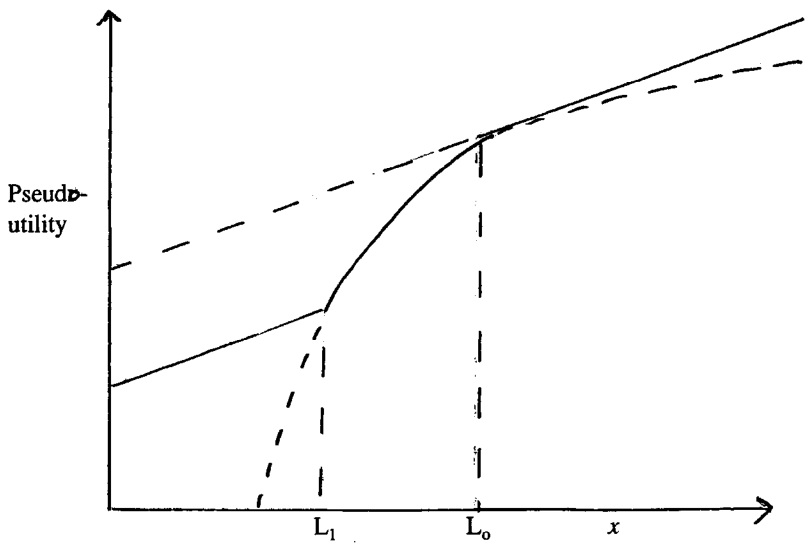

Figure 5.1

5.6 *Risk Assessment and Management for Projects (RAMP)*

5.6.1 The RAMP initiative, a joint undertaking of the Faculty and Institute of Actuaries and of the Institution of Civil Engineers, is a formal framework for risk management in which all reasonably practicable scenarios by which risk can be eliminated or reduced or transferred are investigated before coming to a reasoned choice as to which scenario best meets the risk and return requirements of the sponsor.

5.6.2 The new theory not only encapsulates this seeking out of additional information (in accordance with the extension of Bernoulli's second maxim as discussed in

Section 2.2.5) but can also be used as a structured framework for the identification of the optimal solution once all this further necessary information has been elucidated.

5.7 *Application of RAMP to the Allais Paradox*

5.7.1 Having been offered the choice between A, namely receiving £1m with certainty, and B, namely receiving nil, £1m or £5m with probabilities 0.01, 0.89 and 0.1 respectively, we can investigate the possibility of transferring the risk inherent in B through insurance. Suppose that for £15,000, 50% above the "pure" premium, we can obtain insurance of £1m for the 0.01 probability of receiving nil under B. Then B, with the associated insurance, gives £985,000 with probability 0.9 and £4,985,000 with probability 0.1. Taking, as before, L_0 as £1m and L_1 as nil, the value of risk under B as modified by insurance is $0.015^2 \times 0.9$, ie 0.0002, assuming that the measure of parametric risk is the square of the proportionate shortfall.

5.7.2 Since this risk value is very significantly below the lower guideline value of maximum risk, namely 0.001, and the expected value is much higher than for A, namely £1.385m as against £1m, most subjects might now prefer B (as modified by the addition of insurance) to A.

5.8 *A new resolution of the St Petersburg Paradox*

5.8.1 An inherent feature of the new theory is that individuals will take into account "tacit knowledge", i.e. knowledge acquired either first hand through practical experience or second hand through education or reading, and held at an essentially subconscious level, when assessing choices amongst different scenarios. In the case of the St Petersburg Paradox, the legend of the wise man who asked his king for one grain of rice for the first square of a chess board, two grains for the second square, four grains for the third square, and so on, comes readily to mind. The king thought that such a reward for services well rendered was trivial in the extreme, whereas the weight of rice for all 64 squares of the chess board was of the order of many thousands of millions of tons.

5.8.2 Suppose that Paul knows that Peter has certified wealth of at least 16 ducats. Then if Peter has to pay Paul up to 16 ducats, we can expect him to pay this with certainty. But if he has to pay 32 ducats, 64 ducats, 128 ducats, and so on, plausible probabilities of Paul receiving payment might be 0.5, 0.25, 0.125, and so on, reducing by a factor of two each time, in which case Paul's expectation is $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + 1/8 + 1/16 + \ldots$, which is 3 ducats rather than being infinite. Similarly if Peter has certified wealth of at least 128 ducats or 1,024 ducats, realistic appraisals of Paul's expectation would be 4.5 ducats and 6 ducats respectively. If, however, the role of Peter is played by a soundly financed company or by a government agency (e.g. a national lottery) payment can be assumed to be made with certainty, but for obvious prudential reasons there will be a limit, possibly very high, to the amount that can be paid out. It is interesting to note that a highly popular television game show in the UK offers cash prizes that increase by around a factor of two up to £1m for correct answers to general knowledge questions. If we take one ducat as being equivalent to £100 and assume a maximum payout of £100 x 10^{15}, i.e. £3.28m, then the expected value of the payout (ignoring diminishing marginal utility of wealth) is £800. Since houses, yachts, classic cars and other expensive items can readily be bought by a winner of the maximum prize regardless of his or her previous level of wealth, diminishing marginal utility of wealth is unlikely to be of any significant importance until perhaps £1m is reached, so that allowance for this effect will reduce the expectation only slightly, possibly to an equivalent of around £750.

5.8.3 Consider now the maximum price that Paul should pay, on risk considerations alone, to enter this constrained version of the St Petersburg Paradox, where the possible payouts are £100, £200, £400, and so on up to £3.28m. For values of £1,000, £10,000 and £100,000 as Paul's existing wealth (these correspond to the values of 10, 100 and 1,000 ducats used by Bernoulli) the maximum amount is shown in the table below for both the lower and the higher guideline value of maximum acceptable risk. The amounts in brackets are those in excess of the upper limit of £750 obtained in Section 5.8.2. Also shown are the values of Paul's expectation (if he already owns the entitlement to the gamble) and of the purchase price he should pay (if he does not already own it) using the logarithmic utility approach pioneered by Bernoulli.

	Maximum under new theory		Logarithmic utility	
Paul's wealth	Low risk	High risk	Expectation	Purchase price
£	£	£	£	£
1,000	145	200	304	285
10,000	495	(920)	438	430
100,000	(3,450)	(7,420)	588	551

5.8.4 Two important conclusions can be drawn. First, the acute counterparty risk when Peter is an individual and the limited liability "cap" that will apply otherwise ensure that in real life the expectation is so modest that declining marginal utility of wealth is of little significance. Second, when Paul's wealth is at the lowest of the three levels investigated, Bernoulli's logarithmic utility "solution" leads to a dangerously high level of risk on the new "equivalent probability of financial ruin" basis, namely 0.019, which is almost four times the suggested higher guideline value of 0.005 for the maximum acceptable level of risk.

6. THE "EQUITIES VERSUS GILTS" DEBATE

6.1 *Testing the conventional wisdom*

6.1.1 It is generally accepted, by many actuaries as well as by nearly all financial economists, that equity investment, while having a higher expected return than fixed-interest investment, involves a higher level of risk. The new theory, within which risk to an appropriate investment horizon can be measured in an unambiguous manner, allows this conventional wisdom to be tested more scientifically than has previously been possible.

6.1.2 Since the principal aim of long-term investment is to maintain or exceed the purchasing power of money, we can take for UK investments the threshold of zero risk as the change in the Retail Price Index (RPI) over the period. Taking the threshold of maximum risk as an end-period investment value of nil, the risk value is then the square of the proportionate shortfall (if any) of the total return below the return on the RPI. For instance, if investment of 100 gave an end period value of 114 whereas the RPI increased 20% over the period, the proportionate shortfall is 0.05 and the risk value is the square of this, i.e. 0.0025.

6.2 *The UK experience from 1918 to 1998*

6.2.1 The most comprehensive data source for UK investment returns is the annual Barclays Capital (formerly BZW) Equity-Gilt Study. Using these data for total returns after reinvestment of gross dividends, average risk values were calculated for UK equities and conventional long-dated gilts (British Government securities) for all available durations (from 1918 to 1998) of 1 to 10 years and 15 and 20 years. In the case of equities, a very prominent feature is the contribution to the total risk that arises from the period ending December 1974, which represented the bottom of a very severe bear market. The average risk values and the 1974 percentage contributions are shown below.

UK equities and gilts from 1918 to 1998

Duration in Years	Risk		1974 contribution	
	Equities	Gilts	Equities %	Gilts %
1	0.0104	0.0052	40.6	19.9
2	0.0126	0.0106	53.0	20.4
3	0.0143	0.0139	44.6	20.9
4	0.0145	0.0162	32.7	12.2
5	0.0155	0.0189	35.7	11.9
6	0.0142	0.0231	46.5	11.0
7	0.0125	0.0278	37.1	11.4
8	0.0095	0.0313	29.8	10.2
9	0.0071	0.0350	48.0	9.1
10	0.0076	0.0391	39.6	8.2
15	0.0032	0.0571	100.0	7.0
20	0	0.0767	-	7.7

The (geometric) average real (i.e. inflation-adjusted) rates of total return from 1918 to 1998 are 8.0% pa for equities and 2.4% for gilts.

6.2.2 For equities, risk increases up to a maximum of 5 years and then decreases to around 50% of this maximum at 10 years and to around 20% of this maximum at 15 years. The risk at 20 years is zero, indicating that at this duration the real total return was never negative. For gilts, risk increases steadily with duration, being below the equity value at 1 and 2 years, virtually identical to it at 3 years, and higher than it at all longer durations. At 10 years the risk on gilts is more than five times that on equities, and at 15 years it is almost twenty times that on equities. A remarkable feature of the risk on gilts is that the values at 10, 15 and 20 years are almost exactly two, three and four times the value at 5 years, mirroring the linear increase with time exhibited by the variance of a pure diffusion process.

6.3 *Unconventional wisdom*

6.3.1 These results lead to the unconventional conclusion that equities are less risky than gilts for investment horizons of four years or longer. Given also the higher average returns, equities are accordingly the vastly superior asset class to investment horizons of four years or longer.

6.3.2 It is often recommended that, if an individual invests mainly in equities while in paid employment, he or she should switch into the supposedly "less risky" asset class of fixed interest bonds on retirement. Unless there were significant fixed liabilities (such as having to repay a mortgage) within the first few years of retirement, any such recommendation would - on the basis of the above results - represent exceptionally bad advice.

6.4 *The academic literature*

6.4.1 As a result of a prevalence of the simplistic "risk equals uncertainty of return" teaching of financial economics and the indisputably higher variability of equity returns, the academic literature is pervaded with articles which foster the notion that equities are riskier than bonds, even when the investment horizon is fairly long. An excellent article setting out the opposite viewpoint is Thaler and Williamson (1994) in the twentieth anniversary issue of "The Journal of Portfolio Management". Using data from the Ibbotson Associates Yearbook, it is shown that the likelihood of US equities outperforming bonds increases to near certainty as the investment horizon increases. Such empirical articles, however, are the exception rather than the rule.

6.4.2 The article by Nobel Laureate Paul Samuelson in the same issue of "The Journal of Portfolio Management" attempts to dismiss as "unscientific" articles such as those of Thaler & Williamson (1994) which use empirical data to justify the message of "buy and hold equities for sure-thing long term performance". In an end-note, Samuelson makes the comment:

> "No-one can prove to me that I am too risk-averse. Long-run risk (in equities) is not ignorable".

In the context of the risk results set out above for UK equities and bonds and the virtually identical risk results that could be derived for US equities and bonds, continued adherence by academics to the notion that long-term risk on equity investment is significant would amount to a classic instance of "myopic loss aversion" in a scientific context.

7. WIDER CAPITAL MARKET IMPLICATIONS

7.1 *The Hurst exponent and Mean Absolute Deviation analysis*

7.1.1 Harold Edwin Hurst (1880-1978) spent almost his entire working career as a hydrologist in Egypt struggling with the problem of reservoir control. As Peters (1991) observes:

> "An ideal reservoir would never overflow; a policy would be put in place to discharge a certain amount of water each year. However, if the influx from the river were too low, then the reservoir level would become dangerously low. The problem was: What policy of discharges could be set, such that the reservoir never overflowed or emptied?"

Hurst (1955) describes how the range of the reservoir level fluctuated around its average level; if successive influxes were random (i.e. statistically independent) this range - as with standard deviation in the Black-Scholes option pricing model - would increase over time in line with the square root of time. Hurst obtained a dimensionless statistical exponent by dividing the range by the standard deviation of the observations, and hence his approach is generally referred to as rescaled range (R/S) analysis. By taking logarithms, we obtain the Hurst exponent H from the equation:

$$H \log (N) = \log (R/S) + \text{constant},$$

where N is the number of observations and R/S is the rescaled range. In practice the best way to obtain an estimate of H is to find the gradient of the log/log plot of R/S against N. In strict contrast to the "statistical mechanics" independence value of 0.5 for H, Hurst found not only that for almost all rivers the exponent was well in excess of 0.5 (0.9 for the Nile!) but also that for a vast range of other quite distinct natural phenomena, from temperatures to sunspots, the estimates of H clustered very closely around the value of 0.71, indicating the existence of a powerful "long-term memory" causal dependence.

7.1.2 The "over-reaction bias" effect predicted in Section 4.11 from the corresponding behaviour in potentially risky sports can be expected to generate a powerful "long-term memory" effect. Share prices which have been driven by "over-reaction bias" to extreme values either above or below what Adam Smith would call their "natural values" will tend to revert towards these "natural values" rather than follow an essentially random progression from the extreme values, thereby giving rise to a Hurst exponent significantly in excess of 0.5.

7.1.3 Peters (1991) and others have shown that equity market indices and the price series of individual equity shares do indeed exhibit Hurst exponents well in excess of 0.5, with typical values of around 0.7. Such behaviour is anomalous in the context of the mainstream "rational behaviour" teachings of financial economics but is a prediction of the new general theory of risk and the associated underlying patterns of real world investor behaviour.

7.1.4 In the Mean Absolute Deviation analysis approach described in Plymen & Prevett (1972) and Clarkson & Plymen (1988), the multiplier value of 1.6 that was found to work best for equity price series in practice is significantly lower than the theoretical value of 2 implied by statistical independence. There is a strong (and obviously numerically inverse) correspondence to the Hurst exponent being significantly in excess of 0.5. Furthermore, if we regard "over-reaction bias" as a "momentum" effect superimposed on a mean-reverting "value" effect, the resultant equity price dynamics would approximate to a sine wave, the optimal Mean Absolute Deviation multiplier value of which is 1.57, in very close agreement with the empirically obtained value of 1.6.

7.2 *Pension fund investment strategy*

7.2.1 There are very strong parallels between reservoir control and pension scheme funding, where the investment return corresponds to the influx of water from unpredictable levels of rainfall within the catchment area, while the difference between payments to beneficiaries and contributions from employer and employees corresponds to the controlled level of discharge of water from the dam. The pension funding problem is to find a reasonably stable strategy that does not lead either to excess surplus (the dam overflowing) or to financial or regulatory insolvency (the reservoir emptying). Assuming that there are no significant short term liabilities, and ignoring for the moment any relevant solvency regulations, the investment horizon is very long and it is blindingly obvious from the risk and return results set out in Section 6.2 that, if equities and long-dated fixed interest stocks are the only two available asset classes, then the "natural" investment strategy is 100% equities.

7.2.2 A Hurst exponent of around 0.7 for equities means that, just as Hurst found for reservoir control, financial control policies which use only means and standard deviations estimated from past data will seriously understate the extremes of investment value that will occur. Accordingly, mean-variance analysis could give dangerously unsound results.

7.2.3 Another formalised mathematical approach which could give rise to highly unsatisfactory results is utility theory. Well intentioned attempts to encapsulate a balance between funding level and insolvency risk using a standard type of utility function may, for the reasons set out in Section 5.5.4, magnify the insolvency risk out of all proportion and force the sponsor into an unnecessarily low equity exposure and, accordingly, an unnecessarily high funding rate.

7.2.4 The crucial difference between the new theory, which suggests 100% equities as the "natural" investment strategy, and the financial economics approach, which suggests 100% fixed interest as the "matched" position, corresponds very closely to the two different world views in the "risk equals uncertainty of return" paradox described in Sections 2.11 and 5.3. In the pension fund investment strategy context, this paradox can be translated into the following question that the scheme actuary might ask the chief executive of the sponsoring company:

> "Which would you prefer: 100% investment in fixed interest, which means that in theory the actuarial liability and the asset value are highly correlated over the short term, or - for a significantly lower funding rate which will translate into immediate and continuing higher profits and earnings per share for your company - 100% investment in equities, which involves higher asset value volatility that does not in any way threaten the long-term solvency of your scheme?"

7.3 *Solvency regulations*

It is natural for governments and regulatory bodies to try to put in place rules intended to reduce the financial risk to which members of the public are exposed. However, the new theory of risk developed in this paper recognises that, on occasions such as during 1974, investors will act "irrationally" and drive equity share prices to unrealistically low values that do not in any way affect the satisfactory long term returns that will be achieved. If the basic solvency test for an insurance company or pension fund relates purely to market values with equities being classed as "more risky" than bonds or cash, a strong decline in equity prices, once established, could become self-feeding in the same way as resulted from margin calls in the Wall Street Crash of 1929. The well-intentional rules would be potentially destabilising and could vastly increase the inherent level of financial risk.

7.4 *Stochastic investment models*

7.4.1 The risk values described in Section 6.2, as corroborated by a Hurst exponent significantly in excess of 0.5, show that the statistical behaviour of equity prices over periods of up to five years cannot be used to extrapolate equity price behaviour into the indefinite future. In particular, the value of risk as measured against an appropriate benchmark (such as an increase in line with either price or salary inflation) will, as a result of the "long-term memory" effect that is consistent with behavioural finance conclusions, increase to a maximum at around four or five years and then decrease steadily thereafter. However, the most widely used stochastic investment models within the actuarial profession, and in particular the Wilkie (1984, 1986, 1995) model, follow an autoregressive approach that takes no account whatsoever of this crucial "long-term memory" effect. Within such models, the risk on equity investment, as measured on the "equivalent probability of financial ruin" basis of the new theory, will increase steadily with duration rather than reflecting the real world behaviour of decreasing from around five years onwards. This distortion will lead to a very significant overestimation of the long-term risk of equity investment, and accordingly to the recommendation of unnecessarily low levels of equity investment.

7.4.2 The absence of statistical independence between successive annual equity returns will, mainly as a result of "over-reaction bias", often cause estimates of expected returns on the basis of recent history to be either far too high or far too low. The oil industry analogy just over two decades ago comes readily to mind: in the aftermath of the two very traumatic "oil shocks", the projection models used by some major oil companies did not allow for the possibility of future oil prices falling below the then current value of 40 dollars per barrel.

8. CONCLUSIONS

8.1 *The Kuhn criteria*

The general theory of financial risk developed in this paper represents, in the terminology of Kuhn (1970), a "new paradigm" rather than a "normal science" refinement or extension of existing theories. Having been widely misrepresented by others as suggesting that the process of scientific thought is random or chaotic in nature, Kuhn (1977) wrote a further book in which he set out five criteria - accuracy, consistency, scope, simplicity and fruitfulness - which are of prime importance when the relevant scientific community has to choose between currently accepted theory and an "upstart competitor". The advantages of the new theory in terms of each of these five criteria are discussed below.

8.2 *Accuracy*

The new theory scores well under accuracy. In particular, observed areas of what was previously regarded as anomalous behaviour, such as the Allais Paradox and the Tversky Paradox, can be explained very easily, and it provides for the first time a plausible answer to the important practical question of "what is the maximum acceptable level of financial risk?"

8.3 *Consistency*

The new theory is consistent with important risk methodologies, such as the probability of ruin, variance of return, semi-variance of return and expected utility, each of which can be regarded as a useful, but less general and hence less accurate, interpretation that could be used in certain specific instances.

8.4 *Scope*

There are two areas where the scope of the new theory is far wider than what had been envisaged when the present approach was still in the embryonic form as set out in Clarkson (1989, 1990). First, it predicts when the one-dimensional expected utility approach is likely to lead to unsound conclusions. Second, important behavioural finance traits such as over-confidence, over-reaction bias and myopic loss aversion are predictions of the new theory rather than anomalous behaviour, as is the case within the mainstream theories of modern finance.

8.5 *Simplicity*

The new theory of financial risk can be described in terms of a few simple principles that are much more transparent than the highly mathematical axioms of utility theory. Furthermore, the theory of physical risk from which it has been translated can be verified by most individuals as being a simple but nevertheless accurate portrayal of everyday experience and observation.

8.6 *Fruitfulness*

This criterion relates to the potential for others to work on a "normal science" basis within the framework of a new paradigm and thereby arrive at new research findings which not only explain observed behaviour better than previous theories but also extend and embellish the new theoretical framework. The opportunities in both these areas are considerable, particularly as regards behavioural finance and more effective generalisations of utility theory.

8.7 *Primary objective*
In terms of the primary objective of constructing a general theory of financial risk that can explain real world behaviour better than previous theories and also enhance standards of risk management, these advantages under the Kuhn criteria suggest that much useful progress has been made. Clearly, a vast amount of further work requires to be carried out to develop the new financial risk framework to its full potential.

8.8 *Secondary objective*
In Clarkson (1996) it is suggested that a new and essentially actuarial theory of finance is well within our grasp and might before long offer a better scientific framework for prudent financial management than the general teachings and methodologies of what has become known as financial economics. Since the management of risk is absolutely central to prudent financial behaviour, the generalised financial risk framework set out in this paper would appear to offer a solid foundation on which to build a far more satisfactory theoretical structure than the essentially axiomatic and highly mathematical approach of currently accepted economic science.

REFERENCES

ALLAIS, M. (1953). Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école Americaine. *Econometrica* 21, 503-46.

ALLAIS, M. (1954). L'outil mathématique en economique. *Econometrica.* 22, 58-71.

ALLAIS, M. (1989). An outline of my main contributions to economic science. Nobel Lecture, December 9,1988. *Les Prix Nobel,* Stockholm.

ANAND, P. (1993). *Foundations of rational choice under risk.* Clarendon Press, Oxford.

BERNOULLI, D. (1738, 1954). Specimen theoriae novae de mensura sortis. *Commentarii Academiae Scientiarum Imperialis Petropolitanea,* 5, 175-192. English translation in *Econometrica,* 22 (1954), 22-36.

BOOTH, P. CHADBURN, R. COOPER, D. HABERMAN, S. & JAMES, D. (1999). *Modern actuarial theory.* Chapman & Hall/CRC, London.

BOWERS, L.N., GERBER, H.U., HICKMAN, J.C. , JONES, A.J. & NESBITT, C.J. (1986). *Actuarial mathematics.* The Society of Actuaries.

BUFFON, J. (1777). *Essai d'arithmétique morale.* Supplément a l'Histoire Naturelle, **4.**

CLARKSON, R.S. (1978). A mathematical model for the gilt-edged market. *Transactions of the Faculty of Actuaries*, **36**, 85-160 and *Journal of the Institute of Actuaries*, **106**, 85-148.

CLARKSON, R.S. (1981). A market equilibrium model for the management of ordinary share portfolios. *Transactions of the Faculty of Actuaries*, **37**, 349-607 and *Journal of the Institute of Actuaries*, **110**, 17-134.

CLARKSON, R.S. (1989). The measurement of investment risk. *Transactions of the Faculty of Actuaries*, **41**, 677-750 and *Journal of the Institute of Actuaries*, **116**, 127-178.

CLARKSON, R.S. (1990). The assessment of financial risk. *Transactions of the 1st AFIR International Colloquium*, Paris, **2**, 171-194.

CLARKSON, R.S. (1991). A non-linear stochastic model for inflation. *Transactions of the 2nd AFIR International Colloquium*, Brighton, 3, 233-253.

CLARKSON, R.S. (1996). Financial economics - an investment actuary's viewpoint. *British Actuarial Journal*, **2**, 809-973.

CLARKSON, R.S. (1999). Artificial intelligence and stockmarket success. The Staple Inn Actuarial Society, May 1999.

CLARKSON, R.S. & PLYMEN, J. (1988). Improving the performance of equity portfolios. *Journal of the Institute of Actuaries*, **115**, 631-691 and *Transactions of the Faculty of Actuaries*, 41, 631-675.

DAYKIN, C.D., PENTIKAINEN, T. & PERSONEN, M. (1994). *Practical risk theory for actuaries.* Chapman & Hall, London.

GRETHER, D.M. & PLOTT, C.R. (1979). *Economic theory of choice and the preference reversal phenomenon.* American Economic Review, **69**, 623-638.

HALLEY, E. (1693). An estimate of the degrees of mortality, *Philosophical Transactions of the Royal Society of London*, **17**, 596-610.

HURST, H.E. (1955). Methods of using long-term storage in reservoirs. *Proceedings of the Institution of Civil Engineers*, Part 1, 5, 519-577.

KAHNEMA, D. & TVERSKY, A. (1979). Prospect theory. *Econometrica*, **47**, 263-291.

KEYNES, J.M. (1921). *A Treatise on Probability*. Macmillan, London.

KEYNES, J.M. (1936). *A General Theory of Employment, Interest and Money*. Macmillan, London.

KUHN, T.S. (1970). *The structure of scientific revolutions*. The University of Chicago Press.

KUHN, T.S. (1977). *The essential tension: selected studies in scientific tradition and change.* The University of Chicago Press.

LAMBERTON, D. & LAPEYRE, B. (1996). *Introduction to stachastic calculus applied to finance.* Chapman & Hall, London.

MACHINA, M.J. (1987). Choice under uncertainty: problems solved and unsolved. *Journal of Economic Perspectives*, **1**, 121-54.

MARKOWITZ, H.M. (1959, 1991). *Portfolio selection*. Second edition (1991), Blackwell, Oxford.

von NEUMANN, J.R. & MORGENSTERN, O. (1944). *Theory of games and economic behavior*. Princeton Univerity Press.

PETERS, E.E (1991). *Chaos and order in capital markets*. Wiley, New York.

PLYMEN, J. & PREVETT, R.M. (1972). The computer for investment research. *Transactions of the Faculty of Actuaries*, 33 143-186.

ROYAL SOCIETY OF LONDON (1992). Risk: analysis, perception and management. Report of a Royal Society Study Group.

QUIGGIN, J. (1993). *Generalised expected utility theory.* Kluwer Academic Publishers, London.

SAMUELSON, P.A. (1994). The long-term case for equities. *Journal of Portfolio Management,* 21, 15-24

SHARPE, W.F. (1970). *Portfolio selection and capital markets.* McGraw-Hill, New York.

SHILLER, R.J. (1989). *Market volatility.* Massechusetts Institute of Technology.

SMITH, A. (1776, 1976). *An enquiry into the nature and causes of the wealth of nations. Volume II of the Glasgow Edition of the Works and Correspondence of Adam Smith*, Oxford University Press.

SMITH, A. (1795, 1980). *Essays on Philosophical Subjects. Volume III of the Glasgow Edition of the Works and Correspondence of Adam Smith.* Oxford University Press.

THALER, R.H. & WILLIAMSON, J.P. (1994). College and university endowment funds: why not 100% equities? *Journal of Portfolio Management*, **21**, 15-24.

TVERSKY, A. (1990). *The psychology of risk. In quantifying the market risk premium phenomena for investment decision making.* Institute of Chartered Financial Analysts, Charlottesville, VA.

WILKIE, A.D. (1984). Steps towards to a comprehensive stochastic investment model. O.A.R.D.P. 36.

WILKIE, A.D. (1986). A stochastic investment model for actuarial use. T.F.A. **39**, 341-403.

WILKIE, A.D. (1995). More on a stochastic investment model for actuarial use. *B.A.J.* **1**, 777-977.