ON THE RISK OF STOCKS IN THE LONG RUN: A RESPONSE TO ZVI BODIE

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ABSTRACT

In a paper in 1995 investigating "the risk of stocks in the long run", Zvi Bodie concluded that "for guarantors of money-fixed annuities, the proposition that stocks in their portfolios are a better hedge the longer the maturity of their obligations is unambiguously wrong". This paper investigates this proposition, and finds that Bodie's conclusion is unambiguously wrong, but that everything else is ambiguous. Bodie's model, when properly formulated, in effect required total investment in the risk-free asset for all terms of investment. In this paper different levels and formulations of guarantee are discussed, with the guarantee protected by put options or, equivalently, by dynamic portfolio hedging. The risk of stocks may increase with term, or fall, or rise and fall, or fall and rise, depending on the form of the guarantee and the measure chosen to represent risk.

As an aside, the "value" of options, as opposed to their price, is discussed, and it is shown that, in general, put options are dear and call options cheap, for any but the most risk-averse investor.

KEYWORDS

Risk of stocks, guarantees, value and price of options

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1 Bodie's argument

1.1 In a paper in the *Financial Analysts Journal*, published in 1995, Professor Zvi Bodie investigates the "proposition that investing in common stocks is less risky the longer an investor plans to hold them". He concludes that "for guarantors of money-fixed annuities, the proposition that stocks in their portfolios are a better hedge the longer the maturity of their obligations is unambiguously wrong". It is my contention that Bodie's methodology is flawed, and that his conclusion is far from unambiguous.

1.2 Note that the word "stocks" has quite different meanings in America and Britain, whereas the word "share" is unambiguous; for an international audience I therefore use "share". I also assume, as does Bodie, that all dividends are reinvested, so strictly we invest in a share total return index.

1.3 Bodie rests his argument on "the cost of insurance against a shortfall". He defines the cost of shortfall insurance as "the additional amount of money an investor has to add at the investment starting date to assure that, at the horizon date, the portfolio will have a value at least as great as it would have earning the risk-free interest rate". He assumes $1 invested in shares; he calculates the cost, P, of a (European) put option with an exercise price equal to the rolled-up amount of \$1 invested for t years at the risk free rate. The cost of such an "at the discounted money" put option does not depend on the risk free rate, but only on the standard deviation of the share price process and the term. The Black-Scholes formula simplifies to give a value of P (per unit share price) of $N(\sigma\sqrt{T}/2) - N(-\sigma\sqrt{T}/2)$, where $N(.)$ is the normal distribution function; σ is the standard deviation of the share price process, and T is the term to go. He correctly finds that P increases with T and draws his conclusion.

1.4 A closer look at what is happening shows that Bodie assumes that the amount invested is $1 + P(T)$, where $P(T)$ is the put option price for term T. Thus the amount invested increases with T. This is not very realistic. Let us consider a rather different model, where the amount invested remains constant, and what we pay for the put options comes out of the invested amount. Bodie, in a footnote, does recognise this as a possible model.

1.5 For convenience of exposition, I shall assume that we have \$1,000 to invest, and that we can buy shares at a unit price of \$1. Thus if we invest all our funds in shares we can buy 1,000 shares. Alternatively, we can spend part of our money on put options, and invest less in shares. Assume that we buy n shares, and n corresponding put options, each at a price of p with an exercise price of E. We spend $1 + p$ on each "unit" consisting of a combination of a share and a put option, which costs us $n(1 + p)$. Thus must equal our original investment of \$1,000, so $n(1 + p) = 1{,}000$. If shares perform poorly, and we exercise the options, we receive E for each option instead of the share proceeds, so we get a minimum amount of nE, which we shall call G.

1.6 To jump ahead: we shall find that if we wish to have G equal to the proceeds of risk-free investment (as Bodie would like) then the number of shares we can buy falls to zero, the put option price is infinite, and in effect we have to invest all our funds in a risk-free investment. This is true whatever the term. So Bodie's methodology is saying that, if we wish to guarantee that we shall get at least the proceeds of risk-free investment on all our money, we have to invest wholly in the risk-free investment, and we shall get no more than those proceeds either. (In fact if we could get more than that, we would have an arbitrage opportunity, which is not likely to persist.) Thus, if we are obliged to guarantee the proceeds of risk-free investment, then we cannot afford to invest except in risk-free investments, and investing in shares at all is unsafe. If Bodie had restricted himself to this statement, he would have been on safe and uncontentious ground.

2 A more realistic model

2.1 Now let us investigate my more realistic model further. We have: $n(1 + p) = 1{,}000$ and $nE = G$. The cost of the put option, p, is a function of E. It is also a function of the term, T, the standard deviation, σ, and the risk-free interest rate, which I shall assume is a constant δ, independent of T (I shall also assume for simplicity that σ is constant). Thus we get: $G(1 + p(E)) = 1{,}000E$, an implicit equation in E (assuming G fixed), which has no more than one solution (and may have none). Given G, and also T, δ and σ, and using the standard Black-Scholes formula, we can find E by some recursive method. (I use the secant method, but any other will do). Once we know E, we can get $p(E)$ and also n. This is very similar to what I describe in Wilkie (1987), but there in relation to the guarantees under with profit insurance policies.

2.2 To remind the reader, the Black-Scholes formula for a European put option (exercisable only at the expiry date) in this case (with the share price equal to 1) can be expressed as:

$$W = E.\exp(-\delta T).N(-d_2) - N(-d_1)$$

with

$$d_1 = \log(1/(E.\exp(-\delta T)))/\sigma\sqrt{T} + \tfrac{1}{2}\sigma\sqrt{T}$$
$$d_2 = \log(1/(E.\exp(-\delta T)))/\sigma\sqrt{T} - \tfrac{1}{2}\sigma\sqrt{T}$$

and $N(.)$ is the normal distribution function.

2.3 If G is zero, then $E = 0$, so we spend nothing on put options, and invest all our funds in shares. We find empirically that as G increases so do E and $p(E)$, and n reduces. As G approaches the proceeds of risk-free investment, namely $1{,}000\exp(\delta T)$, so E and also $p(E)$ increase without limit, and n tends to zero, which is the result asserted in ¶1.6. If we choose a value for G greater than $1{,}000\exp(\delta T)$ we find no solution, so we can do no better than to guarantee the proceeds of risk-free investment.

2.4 What should we choose for G? And, having chosen G, how should we compare the results of investing some of our money in shares and some in options over different periods? Clearly, for a fixed term T, the higher we choose G, the higher the lower limit of our final proceeds, but the less we can invest in shares, so the lower the upside potential from share investment. The expected return from our investment reduces, and the riskiness also does, from a top level equal to the return and risk on shares to a bottom level equal to the proceeds on risk-free investment. But the return is highly skewed, so calculating a variance does not seem the natural way forward. Without postulating something about an investor's attitude to risk (or perhaps our own attitude, since I have said that "we invest"), especially to downside risk, or perhaps an explicit utility function, we cannot decide what value of G to choose or to recommend.

2.5 But we can compare results across different terms for comparable values of G. I have investigated three formulations for G. Formulation 1 is a naive one, where G is constant by term. I choose values of $G = L$, where L = 600, 800, 1000, 1,200, etc. If G is greater than the proceeds of risk free investment, then we cannot achieve it, so for values of G greater than 1,000 we need to wait a few years before the guarantee is effective. Formulation 2 is a guarantee that is a constant proportion of the proceeds of risk-free investment, so $G = K.\exp(\delta T)$, where K = 700, 800, 900, 1,000. Formulation 3 is where G falls short of the proceeds of risk-free investment by an absolute amount; thus $G = 1{,}000\exp(\delta T) - H$, where H = 300, 200, 100, 0. When $H = 0$ we get the same result as when K =1,000. When $T = 0$ we get the same value of G for L = 700, K =700 and H=300, but for the same starting level, the second formulation provides a higher guarantee than the first and the third a higher guarantee than the second. Graphs of the three formulations for L = 700 and 1,000, K =700 and 1,000 and H=300 and H=0 (which is the same as K = 1,000) are shown in Figure 2.1.

2.6 As one example, Table 2.1 shows the details for T = 20 with L = 0 to 2,600 in 200s, K = 100 to 1,000 in 100s and H = 1,000 to 0 in 100s. The maximum possible guarantee is $1{,}000\exp(\delta T) = 1{,}000\exp(0.05*20)$ = 2,718.28. One can readily confirm that $nE = G$ and $n(1 + p)$ = 1,000 in each case. One can also see that as G increases so do E and p, while n reduces. This is all as one might expect.

Table 2.1. Details for different guarantees for $T = 20$.

Formulation	Amount guaranteed, G	Exercise price, E	Put price, p	Number of units of share plus put, n
$L = 0$	0	0	0	1000
$L = 400$	400	0.401	0.002	998.171
$L = 600$	600	0.605	0.008	992.556
$L = 800$	800	0.815	0.019	981.531
$L = 1,000$	1,000	1.037	0.037	964.171
$L = 1,200$	1,200	1.277	0.064	939.753
$L = 1,400$	1,400	1.543	0.102	907.512
$L = 1,600$	1,600	1.847	0.154	866.438
$L = 1,800$	1,800	2.209	0.227	815.026
$L = 2,000$	2,000	2.664	0.332	750.830
$L = 2,200$	2,200	3.287	0.494	669.414
$L = 2,400$	2,400	4.277	0.782	561.097
$L = 2,600$	2,600	6.606	1.541	393.558
$K = 100$	271.8	0.272	0.000	999.586
$K = 200$	543.7	0.547	0.005	994.634
$K = 300$	815.5	0.832	0.020	980.423
$K = 400$	1,087.3	1.139	0.048	954.421
$K = 500$	1,359.1	1.486	0.093	914.778
$K = 600$	1,631.0	1.898	0.164	859.201
$K = 700$	1,902.8	2.428	0.276	783.840
$K = 800$	2,174.6	3.194	0.469	680.944
$K = 900$	2,446.5	4.617	0.887	529.829
$K = 1,000$	2,718.3	n/a	n/a	0
$H = 1,000$	1,718.3	2.052	0.194	837.409
$H = 900$	1,818.3	2.246	0.235	809.736
$H = 800$	1,918.3	2.463	0.284	778.823
$H = 700$	2,018.3	2.712	0.344	744.195
$H = 600$	2,118.3	3.004	0.418	705.191
$H = 500$	2,218.3	3.357	0.513	660.852
$H = 400$	2,318.3	3.802	0.640	609.688
$H = 300$	2,418.3	4.404	0.821	549.156
$H = 200$	2,518.3	5.311	1.109	474.196
$H = 100$	2,618.3	7.053	1.694	371.249
$H = 0$	2,718.3	n/a	n/a	0

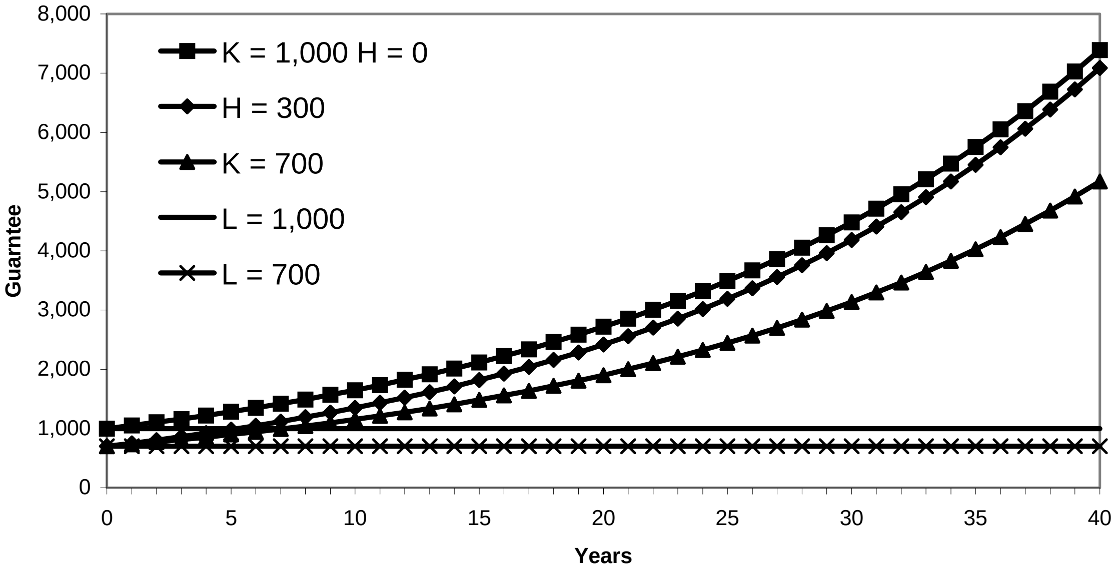

Figure 2.1. Values of guarantees with different formulations; δ=0.5.

3 Comparison of performance

3.1 Now how do we compare the results of these different levels of guarantee? One way is to use a utility function. But what function? I shall assume that the investor's downside risk, whatever it may be, is fully covered by the guarantee amount. Returns below that amount are abhorred, and the guarantee amount (assuming that there is no counterparty risk) fully avoids the investor getting into undesirable territory. Since a return below the guaranteed amount cannot happen, we do not need to worry about what utility function to use in this region. Above the guaranteed amount I suggest that a linear utility function is satisfactory. Since the undesirable downside risk is avoided, we do not need to take a risk-averse view about the upside.

3.2 Other approaches would be possible. In Wilkie (1998) I used a "kinked linear" utility function. In both that case and this I assume a linear (one for one) utility function above some threshold. But in that case the utility function below the threshold was linear with a higher slope than above the threshold, so returns below the threshold were undesirable but admitted. In this present case the utility function below the threshold is assumed to fall vertically, and we cope with this by the protected portfolio strategy.

3.3 Since I assume a linear utility function above the guarantee level, we need to calculate only the expected return on our investment. To do this we need to make an additional assumption about the real world (as opposed to the risk-neutral) return on shares. I assume logarithmic Brownian motion for the share process and I define the risk premium on shares as π, so that the *median* return on shares per year is $\exp(\delta + \pi)$ and the *mean* return is $\exp(\delta + \pi + \frac{1}{2}\sigma^2)$. I then assume initially that $\pi = 0$, so that the only extra return on shares as compared with risk-free investment comes from the skewness introduced by the lognormal distribution. One could well use a higher value of π, as I consider in Section 5; it would be unlikely that one would normally wish to use a lower value for π, though any value greater than $-\frac{1}{2}\sigma^2$ would still produce a small premium on shares as measured by mean returns. The numerical values I have chosen ($\delta = 0.05$ and $\sigma = 0.2$) give an annual rate of return on risk-free investment (which I often call "cash" for convenience) of 5.1271% and a mean rate of return on shares of 7.2508%. The risk premium in log terms is $\frac{1}{2}\sigma^2 = 0.02$, as a difference in means it is 7.2508% – 5.1271% = 2.1237%, and in compound terms it is 100(1.072508/1.051271 –1) = 2.0201%.

3.4 Since the mean return on shares is higher than the risk-free rate we can assume that the longer the term of the investment, the more that shares will pull ahead, in spite of the cost of purchasing the put options. So measuring absolute differences would be misleading. I have calculated two figures, the *Ratio*, which is the ratio of the expected amount of our investment to the amount of a risk free investment over the same term; and the *Premium*, which is the *Ratio* converted to an annual rate of return. Thus if the expected return on our investment in shares and puts over time period T (and with some chosen guarantee) is V, and the risk-free return over the same time period is $A = 1{,}000\exp(\delta T)$, then $Ratio = V/A$, and $Premium = 100(Ratio^{1/T} - 1)$.

3.5 The expected return over the period can be calculated by the integral of a normal and a lognormal distribution over limited ranges. Thus the expected return on the puts depends on the probability that the puts are exercised, which is the probability that the return on shares falls below the exercise price, E. The expected return on the shares is the expected return only if the return on shares exceeds the exercise price. Thus we have:

$$V = n[\int_0^E E.f(x)dx + \int_E^\infty x.f(x)dx],$$

where $f(x)$ is the density function of a lognormal distribution with parameters $(\delta + \pi)T$ and $\sigma^2 T$. Thus we get:

$$V = n.[E.N((\log E - (\delta + \pi)T)/\sigma\sqrt{T}) + \exp((\delta + \pi + \tfrac{1}{2}\sigma^2)T).\{1 - N((\log E - (\delta + \pi)T)/\sigma\sqrt{T} - \sigma\sqrt{T})\}]$$

where $N(.)$ is the normal distribution function.

3.6 Table 3.1 shows details of the expected return, V, the *Ratio* and the *Premium*, for $T = 20$, for the same guarantees as shown in Table 2.1. Note that the amount of risk-free investment after 20 years is \$2,718.28, which is the denominator of the *Ratio*. Obviously, as the amount guaranteed, G, increases, and the number of shares purchased falls, the performance reduces, so that V, the *Ratio* and the *Premium* all reduce.

Table 3.1. Details of performance: $T = 20$

Formulation	Amount guaranteed, G	Number of units of share plus puts, n	Expected value of return, V	Ratio = V/2,718.28	Premium %
$L = 0$	0	1,000	4,055.2	1.492	2.020
$L = 200$	200	999.886	4,054.8	1.492	2.020
$L = 400$	400	998.171	4,049.4	1.490	2.013
$L = 600$	600	992.556	4,032.7	1.484	1.992
$L = 800$	800	981.531	4,001.7	1.472	1.953
$L = 1,000$	1,000	964.171	3,955.5	1.455	1.893
$L = 1,200$	1,200	939.753	3,893.6	1.432	1.813
$L = 1,400$	1,400	907.512	3,815.8	1.404	1.710
$L = 1,600$	1,600	866.438	3,721.7	1.369	1.583
$L = 1,800$	1,800	815.026	3,610.3	1.328	1.429
$L = 2,000$	2,000	750.830	3,480.0	1.280	1.243
$L = 2,200$	2,200	669.414	3,327.5	1.224	1.016
$L = 2,400$	2,400	561.097	3,146.2	1.157	0.734
$L = 2,600$	2,600	393.558	2,918.4	1.074	0.356
$K = 100$	271.8	999.586	4,053.8	1.491	2.018
$K = 200$	543.7	994.634	4,038.8	1.486	1.999
$K = 300$	815.5	980.423	3,998.7	1.471	1.949
$K = 400$	1,087.3	954.421	3,930.4	1.446	1.861
$K = 500$	1,359.1	914.778	3,833.0	1.410	1.733
$K = 600$	1,631.0	859.201	3,705.6	1.363	1.561
$K = 700$	1,902.8	783.840	3,545.9	1.305	1.338
$K = 800$	2,174.6	680.944	3,348.2	1.232	1.048
$K = 900$	2,446.5	529.829	3,098.6	1.140	0.657
$K = 1,000$	2,718.3	0	2,718.3	1	0
$H = 1,000$	1,718.3	837.409	3,658.0	1.346	1.496
$H = 900$	1,818.3	809.736	3,599.2	1.324	1.414
$H = 800$	1,918.3	778.823	3,535.7	1.301	1.323
$H = 700$	2,018.3	744.195	3,467.0	1.276	1.224
$H = 600$	2,118.3	705.191	3,392.8	1.248	1.114
$H = 500$	2,218.3	660.852	3,312.2	1.219	0.993
$H = 400$	2,318.3	609.688	3,224.4	1.186	0.857
$H = 300$	2,418.3	549.156	3,127.7	1.151	0.704
$H = 200$	2,518.3	474.196	3,019.5	1.111	0.527
$H = 100$	2,618.3	371.249	2,893.5	1.065	0.313
$H = 0$	2,718.3	0	2,718.3	1	0

3.7 Tables 3.2 to 3.7 show the values of *Ratio* and *Premium* % for different periods (which is what we are really interested in), for selected levels of guarantee: using formulation 1, with $G = L$ for L= 700 to 1,200 in 100s; using formulation 2, with $G = K\exp(\delta T)$, for K = 700, 800, 900 and 1,000; using formulation 3, with $G = \exp(\delta T) - H$, with H= 300, 200, 100 and 0. As noted in ¶2.5 the results for K = 1,000 are the same as for H = 0, and these guarantees require investment wholly in the risk-free investment. Figures 3.2 to 3.7 correspond to Tables 3.2 to 3.7 and show the results graphically.

3.8 From the Tables or the Figures, we can identify a number of features. The *Ratio* is never below 1, so the *Premium* is never negative.

3.9 For formulation 1, with the guarantee constant (relatively the lowest guarantee of the three), the *Ratio* increases steadily with term. For high guarantees the *Premium* also increases with term, approaching its maximum level of just over 2% in all cases. However, for lower levels of guarantee, the *Premium* at first reduces a little as the term increases. This is only just noticeable for L = 700, 800 and 900, where the minimum value is reached within five years but it is more conspicuous in Figure 3.3 where lines for L = 500 and L = 600 are included.

3.10 For formulation 2, with the guarantee a constant proportion of the risk-free return (the intermediate level of the three, and perhaps the most realistic), the *Ratios* also increase with term, except for the limiting case K = 1,000, but the *Premiums* reduce with term for many years, until beyond term 30, after which they increase slightly.

3.11 For formulation 3, with the guarantee a fixed amount below the risk free return (the strongest guarantee of the three) the *Ratios* are much smaller than with the other formulations, but initially rise, and then fall, reaching a peak at terms around 25 to 30 years. The *Premiums* steadily fall with term.

3.12 Thus, depending on whether one considers *Ratios* or *Premiums*, and depending on which type of guarantee one considers, one can find values that rise consistently with term, fall consistently, fall then rise, or rise then fall. However, as already noted, in each case the *Ratio* is greater than 1, so that the mean return from the portfolio of shares plus puts is greater than the return on the risk-free investment. Thus the investor with the utility function I have postulated, namely linear above the guarantee level, and "minus infinity" below it, would still prefer the protected portfolio to the risk-free investment. It seems to me also that when the *Ratio* is rising, the investor would undoubtedly be happier with the protected investment for a marginally longer term, even though the *Premium* may be falling. The *Premium* represents the average return over the years of the term, and so long as the marginal return is greater than zero, which is the case so long as the *Ratio* is rising, the protected portfolio remains preferable as the term increases.

3.13 Another measure that one might look at is the median return. This can take one of two values: if the probability of the put being exercised is less than one half, then the median return is equal to $n.\exp(\delta T)$, which is less than the median (and only) return on the risk-free investment of $1{,}000\exp(\delta T)$; if the probability of the put being exercised is greater than one half, then the median return is equal to the exercise price, but all other returns are greater than that. While the median return is useful measure with a skew distribution that continues in both directions, for a distribution such as we have here, with a probability mass at one end, the median does not seem as useful a measure and I do not consider it further.

Table 3.2. Guarantee $G = L$: $T = 1$ to 50: *Ratios*

Term	$L = 700$	$L = 800$	$L = 900$	$L = 1{,}000$	$L = 1{,}100$	$L = 1{,}200$	$L = 1{,}300$
1	1.020	1.019	1.015	1.008			
2	1.039	1.036	1.031	1.021	1.002		
3	1.059	1.054	1.047	1.036	1.019		
4	1.078	1.073	1.065	1.053	1.036	1.010	
5	1.099	1.093	1.084	1.072	1.055	1.032	
6	1.120	1.113	1.104	1.091	1.075	1.053	1.023
7	1.141	1.134	1.125	1.112	1.096	1.076	1.049
8	1.164	1.156	1.147	1.134	1.118	1.099	1.075
9	1.186	1.179	1.169	1.157	1.142	1.123	1.100
10	1.210	1.203	1.193	1.181	1.166	1.148	1.126
15	1.337	1.329	1.320	1.310	1.297	1.282	1.265
20	1.479	1.472	1.464	1.455	1.445	1.432	1.419
25	1.636	1.631	1.624	1.617	1.608	1.598	1.587
30	1.811	1.807	1.801	1.795	1.788	1.779	1.771
35	2.004	2.000	1.996	1.991	1.985	1.978	1.971
40	2.218	2.214	2.211	2.206	2.202	2.196	2.191
45	2.453	2.450	2.447	2.444	2.440	2.436	2.431
50	2.713	2.711	2.708	2.705	2.702	2.699	2.695

Table 3.3. Guarantee $G = L$: $T = 1$ to 50: *Premium* %.

Term	$L = 700$	$L = 800$	$L = 900$	$L = 1{,}000$	$L = 1{,}100$	$L = 1{,}200$	$L = 1{,}300$
1	1.988	1.865	1.533	0.793			
2	1.939	1.794	1.513	1.025	0.098		
3	1.914	1.776	1.544	1.181	0.612		
4	1.903	1.779	1.584	1.298	0.885	0.238	
5	1.900	1.789	1.624	1.391	1.071	0.623	
6	1.901	1.803	1.661	1.468	1.210	0.868	0.386
7	1.905	1.818	1.695	1.532	1.320	1.047	0.689
8	1.911	1.833	1.726	1.586	1.409	1.185	0.903
9	1.917	1.847	1.754	1.633	1.482	1.296	1.066
10	1.923	1.861	1.779	1.674	1.544	1.387	1.196
15	1.953	1.916	1.870	1.815	1.748	1.670	1.581
20	1.975	1.953	1.925	1.893	1.856	1.813	1.764
25	1.990	1.976	1.959	1.940	1.918	1.893	1.864
30	2.000	1.991	1.981	1.969	1.955	1.940	1.923
35	2.006	2.001	1.994	1.987	1.978	1.968	1.958
40	2.011	2.007	2.003	1.998	1.993	1.986	1.980
45	2.014	2.012	2.009	2.006	2.002	1.998	1.994
50	2.016	2.014	2.012	2.010	2.008	2.005	2.003

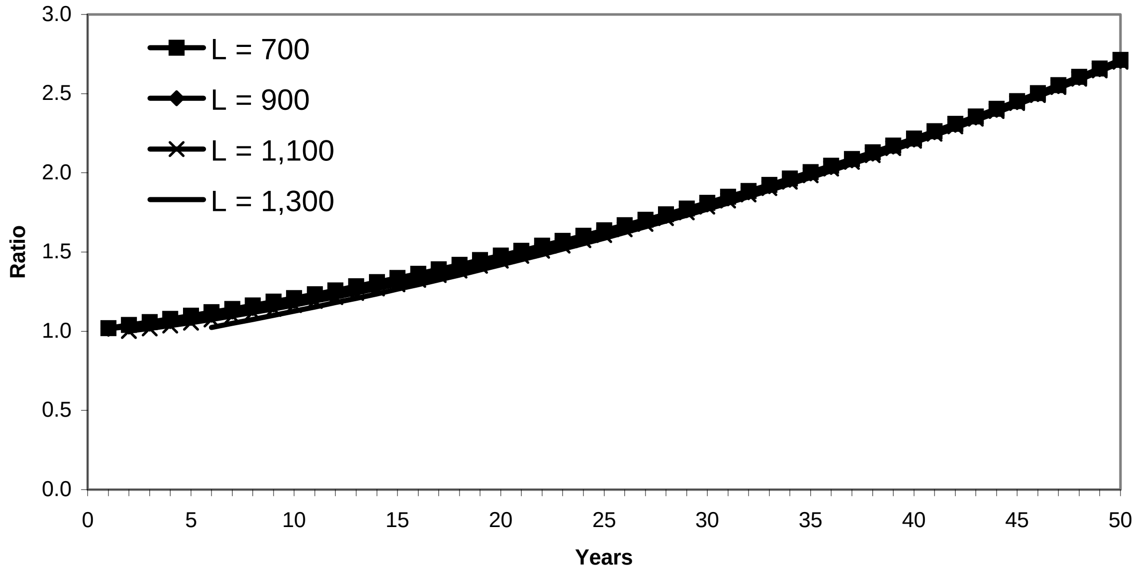

Figure 3.2: Guarantee formulation1: *Ratios*

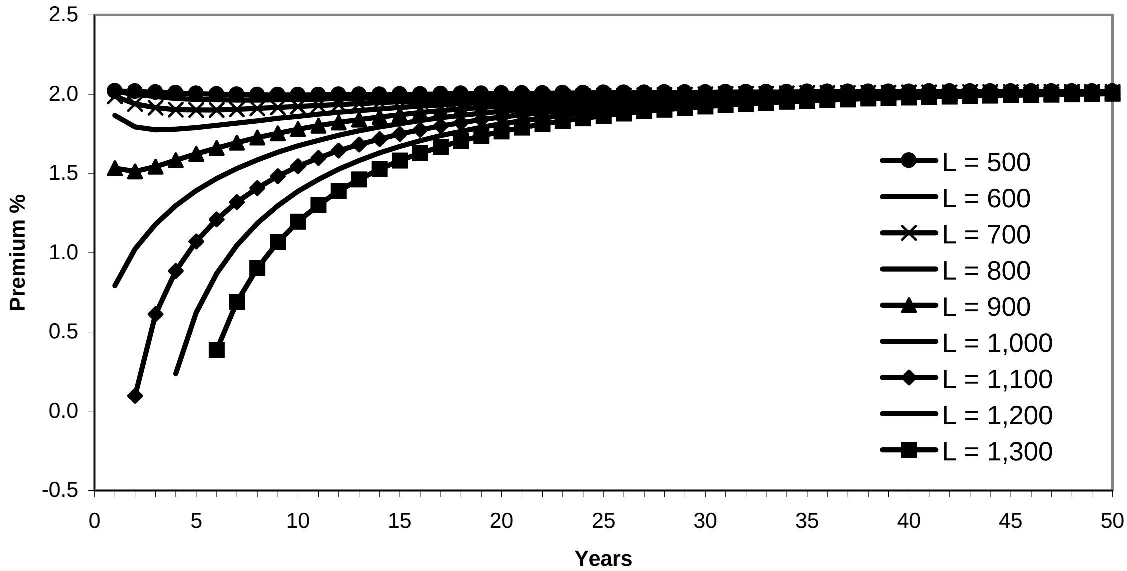

Figure 3.3: Guarantee formulation1: *premiums %*

Table 3.4. Guarantee $G = K\exp(\delta T)$: $T = 1$ to 50: *Ratios*

Term	$K = 700$	$K = 800$	$K = 900$	$K = 1{,}000$
1	1.020	1.018	1.013	1.0
2	1.037	1.032	1.021	1.0
3	1.054	1.044	1.029	1.0
4	1.069	1.056	1.036	1.0
5	1.084	1.067	1.043	1.0
6	1.099	1.079	1.050	1.0
7	1.113	1.090	1.056	1.0
8	1.128	1.100	1.062	1.0
9	1.142	1.111	1.069	1.0
10	1.156	1.122	1.075	1.0
15	1.229	1.176	1.107	1.0
20	1.305	1.232	1.140	1.0
25	1.385	1.291	1.175	1.0
30	1.471	1.354	1.212	1.0
35	1.563	1.422	1.251	1.0
40	1.663	1.495	1.294	1.0
45	1.771	1.575	1.340	1.0
50	1.889	1.660	1.390	1.0

Table 3.5. Guarantee $G = K\exp(\delta T)$: $T = 1$ to 50: *Premium* %

Term	$K = 700$	$K = 800$	$K = 900$	$K = 1{,}000$
1	1.960	1.763	1.262	0.0
2	1.843	1.569	1.058	0.0
3	1.752	1.453	0.955	0.0
4	1.681	1.373	0.889	0.0
5	1.626	1.314	0.842	0.0
6	1.581	1.268	0.808	0.0
7	1.544	1.232	0.781	0.0
8	1.513	1.202	0.759	0.0
9	1.486	1.177	0.741	0.0
10	1.463	1.155	0.726	0.0
15	1.383	1.085	0.680	0.0
20	1.338	1.048	0.657	0.0
25	1.310	1.027	0.646	0.0
30	1.294	1.016	0.642	0.0
35	1.284	1.011	0.643	0.0
40	1.279	1.011	0.647	0.0
45	1.278	1.014	0.653	0.0
50	1.280	1.019	0.661	0.0

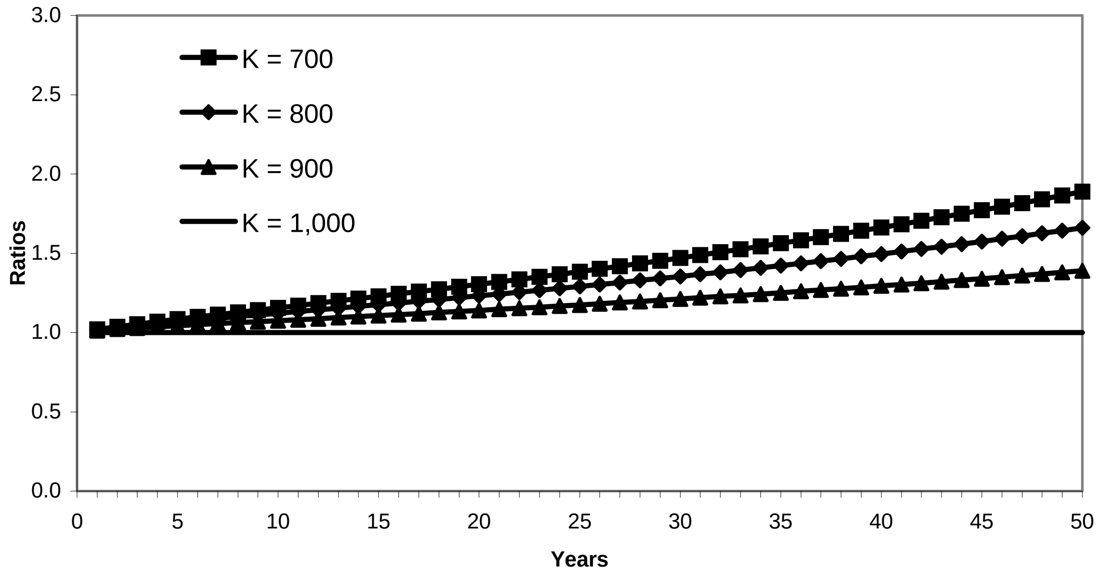

Figure 3.4: Guarantee formulation 2: *Ratios*

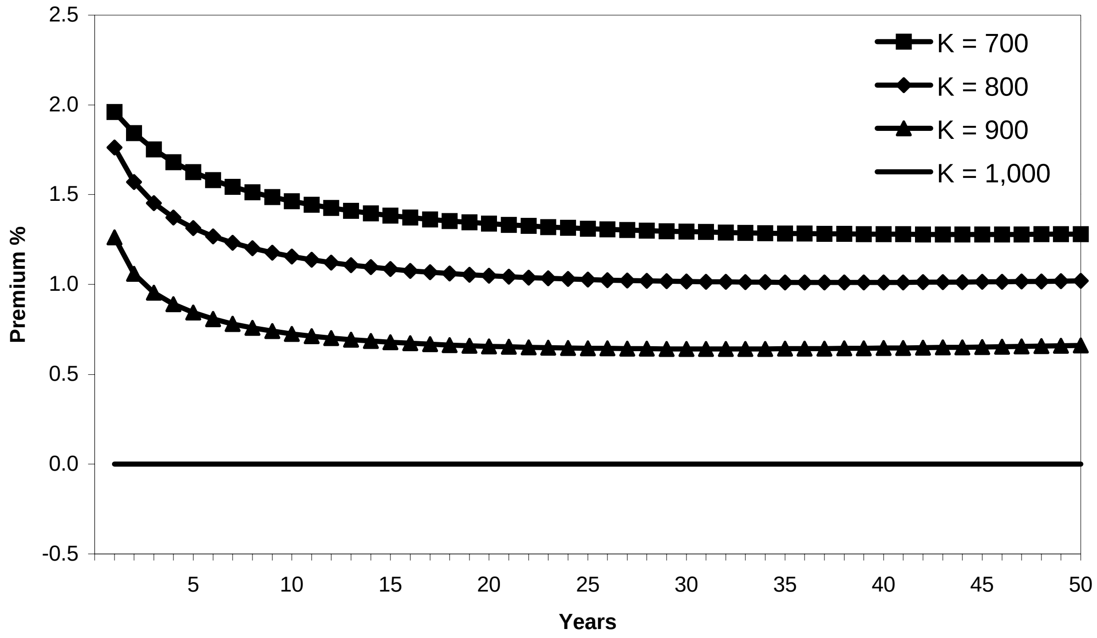

Figure 3.5: Guarantee formulation 2: *Premiums %*

Table 3.6. Guarantee $G = \exp(\delta T) - H$: T = 1 to 50: *Ratios*

Term	$H = 300$	$H = 200$	$H = 100$	$H = 0$
1	1.019	1.017	1.012	1.0
2	1.036	1.030	1.020	1.0
3	1.050	1.041	1.026	1.0
4	1.063	1.050	1.031	1.0
5	1.074	1.058	1.036	1.0
6	1.084	1.065	1.040	1.0
7	1.093	1.071	1.043	1.0
8	1.101	1.077	1.046	1.0
9	1.108	1.082	1.049	1.0
10	1.114	1.086	1.052	1.0
15	1.138	1.103	1.060	1.0
20	1.151	1.111	1.065	1.0
25	1.156	1.114	1.066	1.0
30	1.156	1.113	1.065	1.0
35	1.152	1.110	1.063	1.0
40	1.146	1.106	1.061	1.0
45	1.138	1.100	1.057	1.0
50	1.130	1.094	1.054	1.0

Table 3.7. Guarantee $G = \exp(\delta T) - H$: T = 1 to 50: *Premium* %

Term	$H = 300$	$H = 200$	$H = 100$	$H = 0$
1	1.943	1.730	1.225	0.0
2	1.783	1.494	0.992	0.0
3	1.646	1.338	0.862	0.0
4	1.531	1.222	0.774	0.0
5	1.433	1.129	0.706	0.0
6	1.348	1.053	0.652	0.0
7	1.273	0.987	0.607	0.0
8	1.206	0.929	0.568	0.0
9	1.144	0.878	0.534	0.0
10	1.089	0.831	0.504	0.0
15	0.866	0.653	0.391	0.0
20	0.704	0.527	0.313	0.0
25	0.580	0.432	0.255	0.0
30	0.483	0.358	0.211	0.0
35	0.405	0.299	0.176	0.0
40	0.341	0.251	0.147	0.0
45	0.288	0.212	0.124	0.0
50	0.244	0.179	0.105	0.0

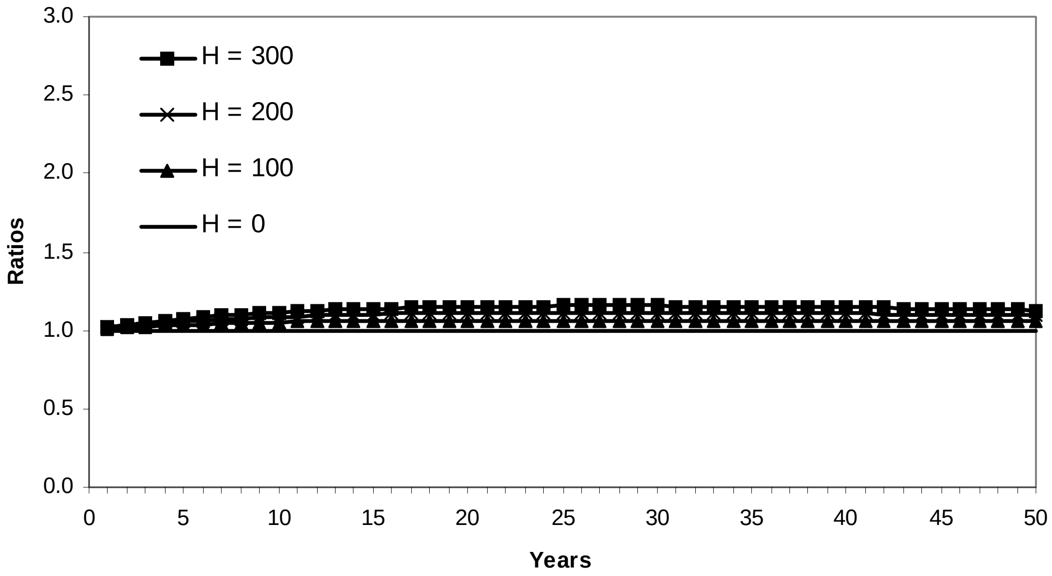

Figure 3.6: Guarantee formulation3: *Ratios*

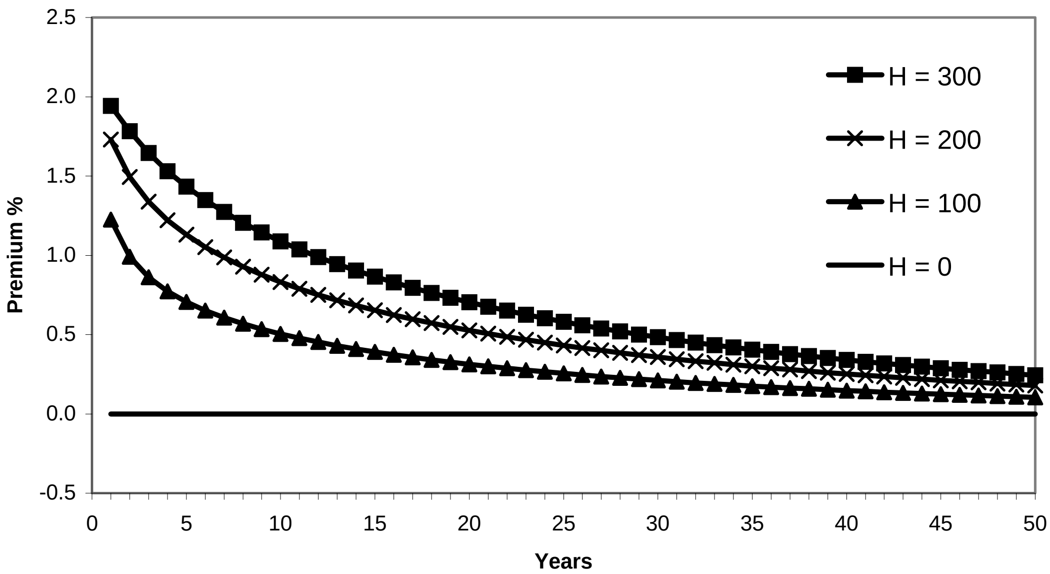

Figure 3.7: Guarantee formulation3: *Premiums %*

4 A portfolio insurance approach

4.1 Another way of looking at the model described in Sections 2 and 3 is through "portfolio insurance". A put option can be hedged by the writer taking the price of the call, selling an appropriate amount of the share short, and investing the combined proceeds in the risk-free investment. The amount of the share to sell is given by the "hedge ratio", h. We find that $h = N(-d_1)$ in the Black-Scholes formula as given in ¶2.2. Instead of buying put options we could arrange the hedge portfolio ourselves. Instead of buying n shares we would need to buy only $n(1 - h)$ shares. The rest of our funds we would invest in the risk-free investment. This is only the initial position. We need to keep on altering the balance of shares and risk-free investment so as to maintain the properly hedged position. In theory this has to be done continuously and costlessly. In practice this is not possible, but it is what is to be assumed by the writer of options in order to find what in insurance terminology would be called the theoretical "net premium", before adding the margins necessary in practice.

4.2 We could equally well have looked at the investment described in Section 2 as consisting of a risk-free investment for the amount of the guarantee, plus a call option that would give us the benefit of the upside performance of shares. If we applied the portfolio insurance strategy to this "cash plus call" investment we would have exactly the same proportions of cash and shares as with our "share plus put" model, since the payoffs from the two strategies are identical.

4.3 The proportion of shares in our initial "insured" or "protected" portfolio gives a very close indication of the overall performance of the portfolio. It is not exact, because different guarantee amounts require different continuous hedging strategies, and so may end up differently. But empirically one can observe that it is close. Thus another way of looking at the relative desirability of shares over different time periods is to look at the different proportions of shares in the initial portfolios.

4.4 Tables 4.1 and 4.2 show details for this way of looking at the model for two examples, with guarantees with L = 1,000 and K = 700. The former is a middling level weak ("money back") guarantee; the latter is a realistic guarantee where there is substantial initial surplus (one can describe it as having \$300 "surplus", with liabilities of \$700). One can see that in the former case the hedge ratio is at quite a high level (0.581) for $T = 1$, and reduces to quite a low level. The number of units purchased increases slowly with term, whereas the equivalent number of shares increases rapidly. The *Ratio* and the *Premium* increase steadily with term. The final column shows the probability of the put being exercised, which is above 0.5 for the first two years, falling off thereafter.

4.5 The second example shows almost the opposite. The hedge ratio starts out low, rises to a peak in years 25 to 30, then falls again. The number of units purchased and the equivalent number of shares both fall from quite high initial levels. The *Ratio* rises steadily, but nearly as much as in the first example. The *Premium* initially falls, bottoms out at about $T = 44$ and then rises slightly. The probability of the put being exercised starts out low and rises, reaching 0.5 by $T = 32$, and remaining above that level thereafter.

Table 4.1. Details for L = 1,000, T =1 to 50.

Term	Hedge ratio	No. units	No. shares	Ratio	Premium	Prob put
1	0.581	895.0	374.8	1.008	0.793	0.620
2	0.463	892.8	479.8	1.021	1.025	0.519
3	0.386	896.1	550.1	1.036	1.181	0.454
4	0.330	901.0	603.5	1.053	1.298	0.405
5	0.287	906.4	646.4	1.072	1.391	0.367
6	0.252	911.8	682.2	1.091	1.468	0.336
7	0.223	917.1	712.6	1.112	1.532	0.309
8	0.199	922.1	739.0	1.134	1.586	0.286
9	0.178	926.9	762.1	1.157	1.633	0.267
10	0.160	931.5	782.5	1.181	1.674	0.249
15	0.099	950.5	856.8	1.310	1.815	0.183
20	0.064	964.2	902.8	1.455	1.893	0.141
25	0.042	974.1	932.8	1.617	1.940	0.111
30	0.029	981.2	953.0	1.795	1.969	0.088
35	0.020	986.4	966.9	1.991	1.987	0.071
40	0.014	990.1	976.5	2.206	1.998	0.058
45	0.010	992.8	983.3	2.444	2.006	0.047
50	0.007	994.8	988.1	2.705	2.010	0.039

Table 4.2. Details for K = 700, T =1 to 50.

Term	Hedge ratio	No. units	No. shares	Ratio	Premium	Prob put
1	0.031	997.4	966.8	1.020	1.960	0.038
2	0.087	987.3	901.1	1.037	1.843	0.112
3	0.130	974.3	847.9	1.054	1.752	0.170
4	0.161	960.4	805.8	1.069	1.681	0.215
5	0.185	946.5	771.8	1.084	1.626	0.250
6	0.203	932.8	743.4	1.099	1.581	0.279
7	0.218	919.5	719.3	1.113	1.544	0.303
8	0.230	906.6	698.5	1.128	1.513	0.324
9	0.239	894.2	680.2	1.142	1.486	0.342
10	0.248	882.3	663.9	1.156	1.463	0.357
15	0.273	828.8	603.0	1.229	1.383	0.414
20	0.283	783.8	561.9	1.305	1.338	0.450
25	0.287	745.5	531.7	1.385	1.310	0.475
30	0.287	712.3	508.2	1.471	1.294	0.494
35	0.284	683.2	489.1	1.563	1.284	0.508
40	0.280	657.4	473.3	1.663	1.279	0.520
45	0.275	634.4	459.9	1.771	1.278	0.529
50	0.270	613.7	448.3	1.889	1.280	0.537

5 Alternative parameters

5.1 All the calculations so far have been done on the same "basis", i.e. the same set of parameters, δ, π and σ. It is interesting to see what difference changes in these make. Obviously the numerical results are different, but are the qualitative results also different?

5.2 First I change the value of the risk premium, π. I had set it at zero, as being the lowest plausible value. But it is worth trying a smaller value, such as –0.01 or –0.02. The latter would imply that the *mean* return on shares was the same as the risk-free rate, so the real-world probabilities were the same as the risk-neutral ones. One must note that the value of π does not affect the initial portfolio; the exercise price, the number of units, the hedge ratio and the equivalent number of shares remain unchanged. All that differs is the expected performance, and hence the *Ratio* and the *Premium*.

5.3 If the value of π is set to –0.02, then all investments, cash, shares and options, have the same expected return, all the *Ratios* are 1.0, and all the *Premiums* are 0.0%. (This, incidentally, is a useful check on the calculations.) If we put $\pi = -0.0199$ then the *Ratios* for any level guarantee that is not up to the maximum is greater than 1, though not by much. Thus purchasing the put options still allows the portfolio to outperform on average; this is not obvious, because the price of a put option is (normally) greater than its "value", as is discussed in Section 6.

5.4 If the value of π is set to +0.02 then the *Ratios* and *Premiums* are of course larger than when $\pi = 0$, but the overall shape of the results is much the same, with falling, rising, humped and hollowed patterns as before. Where the pattern is not monotonic, the maxima and minima are shifted, sometimes to earlier years, sometimes to later.

5.5 Now consider changes in the value of σ. Increasing it to 0.25 or reducing it to 0.15 have the expected results. Consider the increased value first: the options are dearer, the hedge ratio is increased, so the equivalent number of shares is reduced. However, the expected return on shares is increased, so the *Ratios* and *Premiums* everywhere increase, with the overall pattern remaining the same. The reverse happens when the value of σ is reduced. If one reduces the value of σ to nearly zero (reducing it to zero introduces lots of divisions by zero in the calculations!), then shares have almost the same expected return as cash.

5.6 Now consider changes in the value of δ. Increasing δ changes the relative strengths of the three formulations for the guarantees. Formulation 2 is relatively unchanged, and gives identical values for the *Ratios* and *Premiums*. Formulation 1 is relatively weaker, and gives higher values for the *Ratios* and *Premiums*. Formulation 3 is relatively stronger, and gives lower values for the *Ratios* and *Premiums*. If δ is reduced the reverse occurs. If δ is set to zero, all three formulations for the guarantee give the same results.

5.7 One can therefore conclude that the general results obtained are not dependent on the specific values of the parameters, δ, π and σ, except when these are taken to certain extreme values and degenerate results are obtained.

6 The value of options

6.1 I observed in ¶5.3 that the "value" of a put option was less than its price. This requires some explanation. Consider the put/call parity statement. "Cash plus call" is equivalent to "share plus put" (both options being European and for the same exercise price and term), in the sense that the payoff at the exercise date is the same for both bundles, and is equal to the greater of the payoff from the share and cash.

6.2 To be specific: let the share price now be P, the exercise price be E, the risk free rate be δ, the standard deviation of the share price process be σ, and the time period be T. Then the price of a call option, using the standard Black-Scholes formula for a European option, is:

$$WC = P.N(d_1) - E.\exp(-\delta T).N(d_2)$$

with

$$d_1 = \log(P/(E.\exp(-\delta T)))/\sigma\sqrt{T} + \tfrac{1}{2}\sigma\sqrt{T}$$
$$d_2 = \log(P/(E.\exp(-\delta T)))/\sigma\sqrt{T} - \tfrac{1}{2}\sigma\sqrt{T}$$

and the price of a put option is

$$WP = E.\exp(-\delta T).N(-d_2) - P.N(-d_1)$$

6.3 Then the cost of "cash plus call" is given by:

$$\begin{aligned} E.\exp(-\delta T) + WC &= E.\exp(-\delta T) + P.N(d_1) - E.\exp(-\delta T).N(d_2) \\ &= E.\exp(-\delta T)(1 - N(d_2)) + P.N(d_1) \\ &= E.\exp(-\delta T)(N(-d_2)) + P.N(d_1) \end{aligned}$$

and the cost of "share plus put" is given by:

$$\begin{aligned} P + WP &= P + E.\exp(-\delta T).N(-d_2) - P.N(-d_1) \\ &= P(1 - N(-d_1)) + E.\exp(-\delta T).N(-d_2) \\ &= P(N(d_1)) + E.\exp(-\delta T).N(-d_2) \end{aligned}$$

which is the same as the cost of "cash plus call".

6.4 Now imagine that one does not need to pay the prices for all these immediately, but is able to defer payment till time T, with interest at the risk-free rate. (One's credit is assumed to be good enough for this! The technical name for the interest on late payment for shares is contango, and I suppose it could be the same for options.). The amounts to be paid become:

for cash E, instead of $E\exp(-\delta T)$;
for the share $P\exp(\delta T)$, instead of P;
for the call option $WC\exp(\delta T). = P.\exp(\delta T).N(d_1) - E.\ N(d_2)$, instead of WC;
for the put option $WP\exp(\delta T). = E.N(-d_2) - P.\exp(\delta T).N(-d_1)$, instead of WP.

Put/call parity is still preserved.

6.5 Now consider the expected values of the proceeds at time T. For this we need also the risk premium on shares, π, as used previously. The expected returns are:

for cash: E (with certainty);
for the share: $P \exp((\delta + \pi + \frac{1}{2}\sigma^2)T)$;
for the call option: $\int_{E/P}^{\infty}(Px - E)\, f(x)dx$;
for the put option: $\int_0^{E/P}(E - Px)\, f(x)dx$,

where $f(x)$ is the density function of a lognormal distribution with parameters $(\delta + \pi)T$ and $\sigma^2 T$. Simplifying we get:

for the call option: $P.\exp((\delta + \pi + \frac{1}{2}\sigma^2)T).N(f_1)) - E.N(f_2))$

where

$f_1 = (\log(P/E) + (\delta + \pi)T))/\sigma\sqrt{T} + \sigma\sqrt{T}$
$f_2 = (\log(P/E) + (\delta + \pi)T))/\sigma\sqrt{T} \qquad = \Pr\{x > E/P\}$

and for the put option: $E.N(-f_2) - P.\exp((\delta + \pi + \frac{1}{2}\sigma^2)T).N(-f_1)$.

Put/call parity is still preserved.

6.6 Now let us put some numerical values into the formulae. I assume $T = 1$, $\delta = 0.05$, $\pi = 0$, $\sigma = 0.2$, all is in the main assumptions above, and also $P = 100$, $E = 110$. We then get

	Cost at time 0	Cost at time T	Expected value at T	Excess = ex val – cost	Percentage excess
Cash	104.64	110.00	110.00	0.00	0.00
Share	100.00	105.13	107.25	2.12	2.02
Call	6.04	6.35	7.35	1.00	15.71
Put	10.68	11.22	10.10	–1.13	–10.04
Cash + Call = Share + Put	110.68	116.35	117.35	1.00	0.86

6.7 Thus we can see that, on these assumptions, the expected value of the share at $T = 1$ is 2.02% more than the (contangoed) cost of it; that the expected value of the call option is also higher than the cost, by a smaller absolute amount (1.00), but a larger percentage amount (15.71%); and that the expected value of the put option is less than the cost, by 1.13 or 10.04%. The numbers differ for other exercise prices and on other assumptions, but the pattern is preserved, unless we reduce π to equal $-\frac{1}{2}\sigma^2 = -0.2$, so that the real-world probabilities are the same as the risk-neutral probabilities, and the expected values equal the (contangoed) cost.

6.8 Table 6.1 shows the results for different terms, and Table 6.2 shows the results for different exercise prices for term $T = 20$. One can observe that for a fixed exercise price, longer terms the amount by which the contangoed price of a put exceeds the expected value rises with term up to T = 39 and then falls slightly, whereas the percentage excess rises throughout. By contrast, for fixed term, the "excess" rises with the exercise price, but the percentage excess falls.

6.9 Whether shares or options are "good value" or "poor value" to an investor may be considered to depend on the investor's utility function, but for an investor with a linear utility function who expects that on average shares outperform the risk-free investment, then the theoretical prices for put options are in general dear and those for call options are cheap. On occasions the investor may take the view that share prices are "too high", so that the expected return on shares is below that of cash, and then put options may seem to be good value, and both shares and call options poor value. These are the not unexpected consequences for an investor who "takes a view" on the market.

Table 6.1. The values of options for E =110, for T = 1 to 50.

Term	Share Excess	Call Excess	Put Excess	Share % excess	Call % excess	Put % excess
1	2.12	1.00	-1.13	2.02	15.71	-10.04
2	4.51	2.66	-1.85	4.08	21.04	-15.21
3	7.18	4.76	-2.43	6.18	25.25	-19.19
4	10.17	7.25	-2.92	8.33	28.91	-22.55
5	13.50	10.16	-3.34	10.52	32.23	-25.50
6	17.21	13.49	-3.72	12.75	35.33	-28.16
7	21.32	17.28	-4.05	15.03	38.28	-30.60
8	25.88	21.54	-4.34	17.35	41.12	-32.85
9	30.93	26.33	-4.60	19.72	43.88	-34.95
10	36.50	31.67	-4.84	22.14	46.59	-36.93
15	74.07	68.38	-5.69	34.99	59.86	-45.37
20	133.69	127.52	-6.17	49.18	73.43	-52.17
25	226.43	220.01	-6.42	64.87	87.96	-57.83
30	368.45	361.95	-6.50	82.21	103.85	-62.66
35	583.37	576.90	-6.47	101.38	121.42	-66.83
40	905.56	899.20	-6.36	122.55	140.95	-70.46
45	1384.83	1378.63	-6.20	145.96	162.73	-73.65
50	2093.30	2087.30	-6.00	171.83	187.02	-76.45

Table 6.2. The values of options for E = 50 to 150, for T = 20

Exercise price	Share Excess	Call Excess	Put Excess	Share % excess	Call % excess	Put % excess
50	133.69	132.99	-0.70	49.18	59.66	-64.87
60	133.69	132.45	-1.24	49.18	61.95	-62.16
70	133.69	131.76	-1.93	49.18	64.25	-59.76
80	133.69	130.90	-2.79	49.18	66.56	-57.60
90	133.69	129.90	-3.79	49.18	68.86	-55.63
100	133.69	128.77	-4.92	49.18	71.15	-53.83
110	133.69	127.52	-6.17	49.18	73.43	-52.17
120	133.69	126.17	-7.53	49.18	75.69	-50.62
130	133.69	124.73	-8.97	49.18	77.92	-49.18
140	133.69	123.21	-10.48	49.18	80.14	-47.83
150	133.69	121.63	-12.06	49.18	82.33	-46.56

7 Conclusions

7.1 Bodie showed, in effect, that, if one wishes to guarantee the return from risk-free investment on one's portfolio, then the only possibility is to invest in the risk-free investment. This fuller investigation shows that, with lower levels of desired guarantee different levels of investment in shares are possible, ranging from quite low to very high (strictly from nothing to all) depending on the level of guarantee, the term and one's other assumptions. The "protected portfolio" is constructed from investment in shares and put options (or cash and call options) or by synthetic "portfolio insurance" with the proportion of shares being changed continuously to replicate the option.

7.2 Such a protected portfolio will deliver as a minimum the guaranteed amount, so it can be considered a suitable portfolio for an investor who has fixed money liabilities (or liabilities fixed in real terms if we treat the risk free investment as being fixed in real terms) which can be met exactly by the chosen guarantee at the end of the specified term. One can look at the present value of the liabilities as the initial value of a guarantee using formulation 2, and the balance of the present funds as the "surplus" or "free estate".

7.3 Assuming that the investor does not take such a pessimistic view about shares that would assume a lower expected return on shares than on cash, then the initial portfolio will consist of shares to some extent. For a fixed term, the lower the guarantee, the more can be invested in shares, and the higher the expected return. As the term increases the proportion of shares may rise, or may fall, depending on how the guarantee changes with term. In the circumstances assumed in the last sentence of ¶7.2 the proportion of shares that should be held *falls* with duration, which is contrary to popular preconceptions (and contrary to mine, before carrying out this investigation).

7.4 The *Ratio*, that is the ratio of the expected return from the protected portfolio and the return on cash, may rise, fall, or rise and then fall for different formulations of guarantee. The *Premium*, that is the annual compound equivalent of the *Ratio*, may rise, fall, or fall and then rise again, also for different formulations of guarantee. All the numerical details depend on the parameters chosen, though the overall patterns remain the same. Nothing is unambiguous.

7.5 If we compare the theoretical Black-Scholes price of options (contangoed for late payment) with the expected value of the option at exercise, we find that, in normal circumstances, shares and call options are "good value", whereas put options are "poor value". Thus, unless the investors utility function really requires a guarantee that certain liabilities can be met from the existing funds (and not, for example from the funds of a sponsoring employer, or from "slack" such as a with-profits system, or discretionary pension increases), investing so as meet a fixed guarantee with absolute certainty may not be the most effective investment policy.

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