

**SMOOTHING IN DEFINED BENEFIT PENSION SCHEMES:  
ASSET VALUATION AND SPREADING GAINS/LOSSES**

---

**STEVEN HABERMAN**  
**and**  
**NEEMA LUTULA**

**Department of Actuarial Science and Statistics**  
**City University, Northampton Square, London EC1V 0HB, UK.**

**Phone: (020) 7040 8470**

**Fax: (020) 7040 8572**

**Email: [s.haberman@city.ac.uk](mailto:s.haberman@city.ac.uk)**

**ABSTRACT**

The paper takes forward the analysis of Owadally and Haberman (2000a) and considers the properties of methods for asset valuation that use exponential smoothing. The dynamics of the pension funding process are investigated for a simple model of a defined benefit pension scheme where gains and losses emerge as a result of random rates of investment return and where the gains and losses are spread indirectly in a proportional manner over a moving term. A simulation-based approach is used, with rates of investment process following a simple autoregressive process, AR(1), a simple moving average process, MA(1) and also the Wilkie model using a static asset allocation strategy. The results reinforce those of Owadally and Haberman (2000a), demonstrating that excessive smoothing of asset values is inefficient, although some smoothing does improve the stability of contributions. The importance of the combined effect of the asset valuation method and the gain and loss adjustment method (for contributions) is emphasised. Efficient combinations of spreading period and exponential smoothing parameter are suggested.

# **1. Pension Scheme Asset Valuation**

## **1.1 Introduction**

Various methods are used by actuaries to value the assets of a defined benefit pension scheme. The market value of these assets can be viewed as the present value of future expected cash flows, with discounting at the market rate of investment return. The choice of the asset valuation method should be consistent with the aim of the valuation. Specific methods are often prescribed for valuations that are performed to establish compliance with regulations for solvency or maximum funding or with accounting standards. In our case we will consider regular funding valuations of benefit pension plans. These ongoing valuations are performed in order to make comparisons between the assets and the liabilities and to recommend contribution rates, from a going concern perspective.

Frequently, actuaries use a market-related method to value the assets. These market-related methods are based approximately based on an economic valuation of asset and liability cash flows. A disadvantage of using the current market value is that it can be affected by short term political and economic fluctuations in the market, which might not affect the long term funding process position of the fund, and this may cause difficulty in finding a consistent basis on which to value the fund. As a result, an average of current and past values may be used to remove short-term volatility.

On the other hand, market values of assets allow for the market expectations of future investment income growth, and this will enable members of the scheme to know if their scheme will continue to meet its pension payments over the long term. This can be taken as an advantage of using market-related methods (Booth et al (1999)).

When valuing assets it is important also to consider the modelling of the pension liabilities in the fund. As argued by Wise (1987), pension liabilities are not traded and so pension liability cash flows must be priced by comparison with similar asset cash flows, and then discounted at a market discount rate, which is risk adjusted.

After finding the values of assets and liabilities, a comparison between them is made to obtain a measure of the funded liability that is consistent and contribution rates may then be recommended to secure the long term funding of pension benefits.

## **1.2 Properties of asset valuation methods**

Asset valuation methods should satisfy certain desirable properties, irrespective of the method employed to value those assets (Owadally and Haberman (2000a)). The most important properties are consistency, objectivity, realism and stability and these are discussed briefly below:

The consistency property refers to the fact that the values placed on assets and liabilities should be comparable since pension scheme valuations involve the comparison of asset and liability cash flows and the subsequent determination of an unfunded liability and contribution rate. For example, historic book (cost) values of assets are not generally relevant to future pension liabilities. Fair pricing of assets and liabilities should be consistent by virtue of the no-arbitrage principle. Smoothing asset prices may arguably distort the comparison of asset and liability cash flows and the measurement of the unfunded liability. Nevertheless, asset values may be smoothed to remove impermanent fluctuations in security prices, driven by speculators or short-horizon investors, if it is believed that such volatility is not reflected in pension liability values and is irrelevant to long-term planning for retirement benefits. Excessive smoothing would not be acceptable, particularly as the scheme sponsor's financial planning tends to be over a shorter horizon and could be influenced by volatile market conditions which cannot be ignored altogether. If any asset valuation method were not consistent with the liability valuation, then systematic gains or losses would emerge, and such a method may not be acceptable according to national regulations (e.g. Canada).

Pension scheme assets should also be valued in an objective way. Market values of assets are objective in the sense that, setting aside accounting errors, two actuaries will employ the same value. Unsmoothed market values are clearly understood by regulators, financial managers accountants and the sponsor's shareholders. If averaging techniques are used, their variety and opacity, specially if they are changed frequently, may appear to be somewhat arbitrary. Thus, smooth asset values would certainly not be objective for the determination of solvency.

Asset values must also be realistic. The primary objective of funding valuations is to determine a reasonable rate of contribution rather than to place an absolute value on scheme assets, but the asset value should nevertheless remain close to market values. Market values are relevant because market conditions do affect the sponsor, who ultimately contributes to the pension scheme. Asset values that are off-market and depart markedly from market conditions can lead to artificial values of the unfunded liability and contribution rates.

Asset valuation methods also stabilize and smooth the pension funding process. Actuarial valuations for funding purposes aim at measuring the shortfall (or surplus of assets over liabilities) so that contribution rates may be calculated in order to make good these shortfalls and secure assets to meet pension liabilities as and when they are due (as discussed in section 2). Employers sponsor pension schemes on a voluntary basis as well as through competitive pressures in the employment market. But scheme sponsors are motivated to fund defined benefit retirement benefits in advance when the contributions required are stabilized and spread over time, so that the costs arising from the uncertainty of long-term pension provision are not immediately borne. An unfunded liability is often defrayed over a number of years for that reason. Short-term variations in asset prices conflict with this stability objective. Special methods for valuing assets in the funding valuations of DB schemes may therefore be used both to reduce the volatility in asset values and to generate a stable and smooth pattern of contribution rates (Anderson, 1993, Ezra, 1979, Winklevoss, 1993).

### **1.3 Effect on Dynamics**

As many commentators have noted, the effect of the asset valuation method on the dynamics of the funding process is significant, impacting on asset-liability models (Kingsland 1982; Kemp 1996). The asset valuation method affects the timing of contributions, as the emergence of asset gains and losses depends inherently on the asset values calculated. It is, therefore, sensible to consider together the smoothing caused by the asset valuation method and by the amortization of asset gains and losses. (Trowbridge and Farr, 1976).

## **2. Model of the Pension Funding Process**

We consider smoothed market asset valuation methods employing exponential smoothing. A simple mathematical model of a defined benefit plan, providing a pension based on final salary on retirement at normal retirement age, is set up as follows.

The plan is valued at the beginning of every year and a contribution rate is determined.

Plan assets are directly marketable, and the market value  $f(t)$  and an actuarial asset value  $F(t)$  are calculated. It is assumed from the outset that the actuarial valuation basis is constant and that a fixed individual actuarial cost method is used, generating an actuarial liability  $AL$  and a normal cost  $NC$ . The valuation discount rate  $i$  is chosen such that  $AL$  is a reasonable approximation to the

fair value of the pension liabilities. The actuarial assumption as to the projected long-term rate of investment return on scheme assets is also  $i$ .

It is assumed that the scheme membership remains constant in size and structure over time and evolves exactly according to the survival model used for valuation purposes so that no demographic gain or loss arises. The economic experience of the plan is such that only asset gains and losses emerge owing to a variable rate of return  $r(t)$  in year  $(t-1, t)$ . For simplicity, neither salaries nor benefits are subject to economic inflation. The actuarial liability  $AL$ , normal cost  $NC$  and yearly benefit outgo  $B$  are constant as a result of the stationary nature of pension liabilities and of the fixed liability valuation method. Alternatively, all salaries and benefits (including pensions in payment) could be assumed to increase at the same rate of inflation, and all monetary quantities (including  $i$ ,  $r(t)$ ,  $f(t)$  etc.) are then deflated.  $AL$ ,  $NC$ ,  $B$  are then constant in real terms.

This mathematical model resembles that of Trowbridge (1952), who shows that when the liability structure of the pension plan is in equilibrium,

$$AL = (1 + i)(AL + NC - B). \quad (1)$$

All cash flows are assumed to occur at the start of the scheme year  $(t, t+1)$  so that

$$f(t+1) = (1 + r(t+1))[f(t) + c(t) - B] \quad (2)$$

where  $c(t)$  is the contribution income for the year  $(t, t+1)$ .

The rate of return  $r(t)$  is calculated after simulating the force of interest  $\delta(t)$  as explained in a later section, with  $\delta(t)$  assumed constant over the year  $(t-1, t)$ , so that

$$r(t) = \exp(\delta(t)) - 1. \quad (3)$$

Then the unfunded liability based on the market value of scheme assets for  $t \geq 0$  is

$$\text{defined as: } ul(t) = AL - f(t). \quad (4)$$

The anticipated (or written up) market value of plan assets at time  $t \geq 1$  if the actuarial assumption as to investment returns is borne out during year  $(t-1, t)$  is

$$f^A(t) = (1 + i)[f(t-1) + c(t-1) - B]. \quad (5)$$

The unsmoothed anticipated unfunded liability is

$$ul^A(t) = AL - f^A(t). \quad (6)$$

An unsmoothed asset gain or loss in year (t-1, t) emerges as the actual return on plan assets  $r(t)$  differs from the actuarial assumption “ $i$ ”, is given by

$$\begin{aligned} l(t) &= ul(t) - ul^A(t) \\ &= f^A(t) - f(t) \quad \text{for } t \geq 1 \end{aligned} \quad (7)$$

Therefore  $f(t) + l(t) = u[f(t-1) + c(t-1) - B]$  (8)

where  $u = 1 + i$ .

Upon substitution of equation (2) and (5) into equation (7) we obtain

$$l(t) = (i - r(t))(f(t-1) + c(t-1) - B). \quad (9)$$

The unfunded liability based on the smoothed actuarial value of plan assets is defined as

$$UL(t) = AL - F(t) \quad t \geq 1 \quad (10)$$

The anticipated actuarial value of plan assets is:

$$F^A(t) = (1 + i)[F(t-1) + c(t-1) - B] \quad (11)$$

and therefore, the smoothed loss is

$$\begin{aligned} L(t) &= UL(t) - UL^A(t) \\ &= F^A(t) - F(t) \quad t \geq 1. \end{aligned} \quad (12)$$

Contribution rates are adjusted to pay off the asset gains and losses:

$$c(t) = NC + adj(t) \quad (13)$$

where  $adj(t)$  is a supplementary contribution paid to liquidate gains and losses, as well as any initial unfunded liability. We will analyse the properties of the spread method, which is commonly used in the UK. In the absence of any initial unfunded liability,  $adj(t)$  is defined by

$$adj(t) = k ul(t) \quad (14)$$

where  $k = \ddot{a}_{\overline{m}|}^{-1}$  calculated at the valuation rate of interest  $i$ . Thus, gains and losses are spread by paying a proportion of  $k$  of the unfunded liability in each year.

### 3. Asset Valuation Methods

Define the present value of plan assets at time  $t$  written up over  $j \geq 1$  years and allowing for cash flows to be

$$F_j(t) = u^j f(t-j) + \sum_{k=1}^j u^k [c(t-k) - B]. \quad (15)$$

From equation (8), we obtain

$$f(t) = u[f(t-1) + c(t-1) - B] - l(t)$$

and by recursion, it is easily shown that

$$F_j(t) = u[F_j(t-1) + c(t-1) - B] - u^j l(t-j). \quad (16)$$

The smoothed actuarial value of plan assets at time  $t \geq 1$  is then defined as

$$F(t) = (1-\lambda)\{f(t) + \sum_{j=1}^{\infty} \lambda^j F_j(t)\} \quad (17)$$

where  $0 \leq \lambda \leq 1$ .  $F(t)$  is an exponentially weighted average allowing for cash flows and interest. It averages the market values of the scheme assets allowing for cash flows and the time value of

money. We note that the smoothing weights sum to unity since  $\sum_{j=0}^{\infty} \lambda^j (1-\lambda) = 1$ .

Upon substitution of equation (15) into equation (17), we obtain:

$$\begin{aligned} F(t) &= \sum_{j=0}^{\infty} \lambda^j (1-\lambda) u^j f(t-j) + \sum_{j=1}^{\infty} \lambda^j (1-\lambda) \sum_{k=1}^j u^k [c(t-k) - B] \\ &= \sum_{j=0}^{\infty} \lambda^j (1-\lambda) u^j f(t-j) + \sum_{j=1}^{\infty} (\lambda u)^j (c(t-j) - B). \end{aligned} \quad (18)$$

Owadally and Haberman (2000a) prove that (18) can be written as:

$$F(t) = \lambda F^A(t) + (1-\lambda)f(t) \quad (19)$$

which presents  $F(t)$  as a weighted average of the market value of the scheme assets and of the written up anticipated actuarial value.

Similarly for  $t \geq 1$ , we can re-write (19) as:

$$F(t) = f(t) + \lambda[F^A(t) - f(t)]. \quad (20)$$

which presents the asset valuation method as an adjusted market method. The actuarial value of the scheme assets is equal to the market value, smoothed by an adjustment equal to a fraction of the difference between the anticipated actuarial value and the market value.

The asset valuation method can also be defined in terms of the asset losses  $\{l(t)\}$  in a pension plan. Substituting  $F_j(t)$  from equation (16) into equation (17) yields,

$$F(t) = (1-\lambda)f(t) + u \sum_{j=1}^{\infty} \lambda^j (1-\lambda) F_j(t-1) + u\lambda[c(t-1) - B]$$

$$\begin{aligned}
& - \sum_{j=1}^{\infty} \lambda^j (1-\lambda) u^j l(t-j) \\
& = uF(t-1) + (1-\lambda)f(t) - u(1-\lambda)f(t-1) + u\lambda(c(t-1) - B) \\
& - \sum_{j=1}^{\infty} \lambda^j (1-\lambda) u^j l(t-j)
\end{aligned} \tag{21}$$

Upon substituting  $F^A(t)$  from equation (11) and  $l(t)$  from equation (9) into equation (21), Owadally and Haberman (2000a) obtain

$$F(t) = F^A(t) - \sum_{j=0}^{\infty} \lambda^j (1-\lambda) u^j l(t-j), \quad (\text{for } t \geq 1). \tag{22}$$

Then, Owadally and Haberman (2000a) obtain the following results for  $L(t)$

$$L(t) = (1-\lambda)[F^A(t) - f(t)] \tag{23}$$

and

$$L(t) = \sum_{j=0}^{\infty} \lambda^j (1-\lambda) u^j l(t-j). \tag{24}$$

so that  $L(t)$  involves the present value of losses  $\{u^j l(t-j)\}$  being exponentially smoothed.

#### 4. Supplementary Contributions

In this paper, we use the spread method for adjusting contributions, as discussed by Trowbridge (1952), Dufresne (1988) and Owadally and Haberman (1999), whereby the gains and losses are spread forward over a moving term ( $m$ ). The initial unfunded liability  $ul_0$  may be implicitly spread forward or explicitly amortized over  $m$  years by payments,

$$\begin{aligned}
P(t) &= \begin{cases} \frac{ul_0}{\ddot{a}_{\overline{m}|}}, & 0 \leq t \leq m-1, \\ 0, & t \geq m. \end{cases}
\end{aligned} \tag{25}$$

The unamortized part of  $ul_0$  is

$$\begin{aligned}
U(t) &= \begin{cases} \frac{ul_0 \ddot{a}_{\overline{m-t}|}}{\ddot{a}_{\overline{m}|}}, & 0 \leq t \leq m-1, \\ 0, & t \geq m. \end{cases}
\end{aligned} \tag{26}$$

Note that,

$$P(t) = U(t) - vU(t+1) \text{ where } v = (1+i)^{-1}. \quad (27)$$

From Owadally and Haberman (1999, 2000a), the supplementary contribution or adjustment is based on the unfunded liability, derived from the smoothed value of plan assets,  $UL(t)$ , and the unamortised part of the initial unfunded liability i.e.,

$$adj(t) = (1-K)[UL(t) - U(t)] + P(t), \text{ where } 1-K = k = \frac{1}{\ddot{a}_{\overline{m}|}}. \quad (28)$$

This method of indirectly spreading gains and losses will be used in preference to the direct amortizations of gains and losses over a fixed term because it is more efficient and leads to less variable contribution rates, as demonstrated by Owadally and Haberman (1999, 2000b).

The relation for the smoothed unfunded liability, for  $t \geq 1$  is

$$UL(t) - uUL(t-1) = L(t) - u.adj(t-1). \quad (29)$$

On substituting for  $adj(t)$  from equation (28) and  $P(t)$  from equation (27) we obtain

$$(UL(t) - U(t)) - uK(UL(t-1) - U(t-1)) = L(t). \quad (30)$$

Hence, for  $t \geq 0$ , the solution to this difference equation is:

$$UL(t) - U(t) = \sum_{j=0}^{\infty} (uK)^j L(t-j) \quad (31)$$

which, on substituting into equation (28), leads to

$$adj(t) - P(t) = \sum_{j=0}^{\infty} K^j (1-K) u^j L(t-j). \quad (32)$$

There is a symmetry between equation (24) and (32). Asset gains and losses (at market) are smoothed twice, first by asset valuation method and then by the gain and loss adjustment method. When the spreading adjustment is used, equation (32) shows that the supplementary contribution

is an exponentially weighted infinite average of smoothed losses  $(\sum_{j=0}^{\infty} K^j (1-K) = 1)$ , since

$(0 \leq K < v < 1 \text{ for } m \geq 1)$ . An identical smoothing mechanism is employed in the asset valuation and the gain/loss adjustment methods that are considered here.

## 5. Moments of the Pension Funding Process: IID Case

We now consider the model for  $r(t)$ . Owadally and Haberman (2000a) consider independent and identically distributed (IID)  $r(t)$  and obtain theoretical results for the limiting moments of  $f(t)$ ,  $F(t)$ ,  $c(t)$ ,  $l(t)$  and  $ul(t)$  which are then compared and analysed. Our approach is to introduce dependency between rates of investment return and use simulation to conduct a corresponding analysis of the moments of these random variables.

For the IID case, Owadally and Haberman (2000a) obtain explicit results for the limiting first and second moments as  $t \rightarrow \infty$ . They note that, for  $f(t)$ ,  $c(t)$ ,  $l(t)$  and  $ul(t)$ , there is symmetry between  $K$  and  $\lambda$ . For  $F(t)$  and  $UL(t)$ , which involve smoothing through the asset valuation mechanism (and not through gain or loss adjustment), this symmetry is absent. When unsmoothed market values of the scheme assets are used (so  $\lambda=0$ ) or when asset gains and losses are not amortized or spread but are paid off immediately (so  $K=0$ ), then the results are similar to those of Dufresne (1988).

Providing that certain stability conditions enable so that the limiting first and second moments exist as  $t \rightarrow \infty$ , Owadally and Haberman (2000a) establish the following three propositions, inter alia:

### Proposition 1

Provided that the stability conditions hold,  $\lim \text{Var}f(t)$  increases monotonically with both  $m$  and  $\lambda$ .

### Proposition 2

Suppose  $m > 1$  and  $\lambda > 0$ . Providing that the stability conditions hold,

1. as  $m$  increases,
  - $\lim \text{Var}c(t)$  has at least one minimum at some  $m \geq m^*$ , provided  $0 < \lambda < \lambda^*$ ;
  - $\lim \text{Var}c(t)$  increases monotonically, provided either  $\lambda \geq \lambda^*$  or  $m \geq m^*$ ;
2. as  $\lambda$  increases,
  - $\lim \text{Var}c(t)$  has at least one minimum at some  $\lambda < \lambda^*$ , provided  $1 < m < m^*$ ;
  - $\lim \text{Var}c(t)$  increases monotonically, provided either  $m \geq m^*$  or  $\lambda \geq \lambda^*$ .

### Proposition 3

Under the objectives of minimizing  $\lim \text{Var}f(t)$  and  $\lim \text{Var}c(t)$ ,

1. it is not efficient to smooth asset values by weighting current market value by less than  $1 - \lambda^*$ ;
2. it is not efficient to adjust gains/losses by spreading them over periods exceeding  $m^*$ .

We will investigate the applicability of these Propositions when  $r(t)$  is modelled by a dependent process.

## 6. Dependent Rates of Investment Return

### 6.1 Introduction

The assets of the pension fund are likely to have rates of return which are statistically dependent because of the correlation that can exist between the assets. Markets may not be efficient over time, as demonstrated by the statistical analysis of Wilkie (1995). Whether or not markets are efficient, not all the securities held by the fund will typically be traded every year and some dependence in the returns from individual securities will occur. Hence, we model the rates of return on the pension scheme assets allowing for dependence.

The two most common time series mathematical actuarial models will be considered, these are the autoregressive model applied to defined benefit pension scheme by Haberman (1994) and the moving average model introduced by Dufresne (1990) and explored further by Haberman and Wong (1997). Apart from those, the Wilkie (1995) model will also be used to model rates of investment return. Simulation will be used to analyse the behaviour of the different models under consideration.

### 6.2 Moving Average Rates of Investment Return

In this paper, we will take the case where logarithmic rates of return process (net of wage inflation) are modelled as a stationary Gaussian moving average process of order 1 (MA(1)):

$$\delta(t) = \delta + e(t) - \phi e(t-1) \quad (33)$$

where  $\{e(t)\}$  is a sequence of zero mean independent and identically normally distributed variables for  $t = 1, 2, 3, \dots$  and  $|\phi| < 1$  so that the process is invertible and may be expressed as an

autoregressive process. The arithmetic rate of return  $r(t)$  (net of wage inflation) is  $\exp(\delta(t)) - 1$  and in the simulations its mean is specified as 5%, that is equal to the valuation discount rate net of salary inflation, and its standard deviation is specified as 20%.  $\phi$  is the moving average parameter and different values of this parameter will be used in the analysis to show how the variance of the fund level and contribution rate change as a result.

Thus, the values of  $\delta$  and  $\sigma$  which are to be used in the simulations are determined by using the following relations:

$$\log 1.05 = \delta + \frac{1}{2}\sigma^2(1 + \phi^2) \quad (34)$$

$$\log(1.05^2 + 0.20^2) = 2\delta + 2\sigma^2 + 2\phi^2\sigma^2. \quad (35)$$

### 6.3 Autoregressive rates of investment return

In this case, the logarithmic rate of return process (net of wage inflation) is modelled as a stationary Gaussian autoregressive process of order 1 (AR(1)):

$$\delta(t+1) = \delta + \phi(\delta(t) - \delta) + e(t+1) \quad (36)$$

where  $|\phi| < 1$  and  $\{e(t)\}$  is a sequence of zero-mean independent and identically normally distributed variables. The process is assumed to be stationary from the start.

The arithmetic rate of return (net of wage inflation) in year  $(t-1, t)$  is  $\exp(\delta(t)) - 1$  and in the simulations its limiting mean is specified to be 5% and its limiting variance is specified to be 20%.  $\phi$  is the autoregressive process parameter and different values of this parameter will be used to analyse the properties of the model.

Thus, the values of  $\delta$  and  $\sigma$  to be used in the simulations are calculated by using the following relations

$$\ln(1.05) = \delta + \frac{\sigma^2}{2(1-\phi^2)} \quad (37)$$

$$\ln(1.05^2 + 0.20^2) = 2\delta + \frac{2\sigma^2}{1-\phi^2} \quad (38)$$

#### 6.4 How the above models will be used in the simulations

2000 scenarios are generated, with a time horizon of 300 years in each, for the forces of interest arising from the MA(1) and the AR(1) models. Hence, the rate of return each year will be calculated and will be used in the simulation of the experience of the pension scheme model. Finally, the pension scheme model will be used to calculate the standard deviations of the funding level and contribution rate which will be used to investigate the behaviour of the calculated standard deviations when the length of the spread period (m) and the value of the smoothing parameter ( $\lambda$ ) in the asset valuation method change. The pension scheme model to be used in the analysis is described in the next section.

### 7. Moments of the Pension Funding Process Using Dependent Rates of Return

#### 7.1 Calculating the Actuarial Liability (AL) and Normal Cost (NC)

The Actuarial Liability and Normal Cost of the pension scheme are calculated by using Unit Credit Funding Method. The following are the assumptions considered in calculations:

Population structure is based on A1967/70 life table.

- There are no withdrawals, no early retirements and no salary scale.
- Age of entry to the scheme is 25 years.
- Retirement age is 65 years.
- Earned salary equals unity ( $S = 1$ ).
- Valuation rate of interest is 5%.
- Rate of wage inflation is assumed to be zero.

Defining  $AL_x$  as the expected present value of pension fund's Actuarial Liability for each employee who is currently working aged  $x$ , we have:

$$AL_x = \frac{40}{60} \frac{l_{65}}{l_x} v^{65-x} \ddot{a}_{65}$$

The total Active Liability ( $AL_1$ ) is the summation over all working ages that is 25 – 64 years, this is then:

$$AL_1 = \frac{2}{3} \sum_{x=25}^{64} n_x \frac{l_{65}}{l_x} v^{65-x} \ddot{a}_{65}$$

where  $n_x$  is the number of persons aged  $x$  between 25 and 64 years.

The total liability for pensioners ( $AL_2$ ) is:

$$AL_2 = \frac{2}{3} \sum_{x=65}^{\infty} m_x \ddot{a}_x$$

where  $m_x$  is the number of persons aged 65 years and above.

Pension scheme Actuarial Liability ( $AL$ ) is the summation of  $AL_1$  and  $AL_2$ , and it is given by:

$$AL = \frac{2}{3} \left( \sum_{x=25}^{64} \frac{l_{65}}{l_x} v^{65-x} \ddot{a}_{65} n_x + \sum_{x=65}^{\infty} m_x \ddot{a}_x \right) \quad (39)$$

The expected present value of Normal Cost ( $NC_x$ ) for a member aged  $x$  is:

$$\begin{aligned} NC_x &= \frac{1}{60} {}_{65-x}p_x \ddot{a}_x \\ &= \frac{1}{60} v^{65-x} \frac{l_{65}}{l_x} \ddot{a}_{65} \end{aligned}$$

Then the expected present value of pension fund's Normal Cost (NC) is the summation over all ages which members of the scheme pay contributions. It is given by:

$$NC = \sum_{x=25}^{64} n_x NC_x. \quad (40)$$

## 7.2 Simulation results for the MA(1) and AR(1) models

### 7.2.1 Introduction to simulation results

Our principal aim in analysing the simulated results is to compare them with the theoretical results obtained by Owadally and Haberman (2000a) for the case where rates of investment return are independent and identically distributed.

The model of the pension scheme uses rates of return obtained from performing 2000 simulations of the stochastic investment return models, which follow the MA(1) and AR(1) processes, described in section 6. Each simulation is performed for 300 years. Also the model is used to calculate the contribution rate for each year of the simulation. Plan asset gains and losses emerge and supplementary contributions are paid to spread them equally throughout the period by paying a proportion  $1 - K$  of the unfunded liability as discussed in section 4. For the simulations, we specify the mean of the force of interest to be 5% (equal to the valuation rate of interest) and the standard deviation to be 20%. The different parameters of the MA(1) and AR(1) models, to be used in the simulations, are calculated, as described in section 6.2 and 6.3.

The actuarial liability, normal cost and yearly benefit outgo are calculated using the results in section 6.1. The standard deviation of the funding level and contribution rate are then calculated, and results are tabulated to show how these variances change when the spread period “ $m$ ” and smoothing weight “ $\lambda$ ” vary with time and for different parameters of the MA(1) and AR(1) models. For each spread period, in both models, the standard deviation of the funding level and that of the contribution rate are calculated. The range of parameters investigated is as follows:

$\lambda = 0, 10, 20, 30, 40, 50, 60, 70, 80, 90\%$ ;  $m = 1, 3, 5, 10, 15, 20, 25, 30$ ;  
 $\phi$  and  $\varphi = \pm 0.1, \pm 0.3$ . Note that  $\phi = 0$  and  $\varphi = 0$  correspond to the IID case.

Because we are using simulations for a specified range of parameter values ( $\lambda$ ,  $m$ ,  $\phi$  and  $\varphi$ ), it is not possible to determine  $m^*$  and  $\lambda^*$  definitively as achieved by Owadally and Haberman (2000a) using a theoretical approach for the IID case (see Propositions 1, 2 and 3).

### **7.2.2 Results for the Standard Deviation of Funding Level**

From the results in Tables 7.1 to 7.4, it can be observed that, as the length of spread period increases, the limiting value of the standard deviation of the funding level increases. The same trend has been observed for all the parameters tested for the MA(1) and AR(1) models. This means that the funding level become more variable as the spread period is increased.

Another feature observed from the analysis is that, as the value of  $\lambda$  increases, the standard deviation of the funding level increases. This is similar to what is observed when rates of return are IID over time.

### 7.2.3 Results for the standard deviation of the contribution rate

From the results in tables 7.5 to 7.12, it can be observed that, as  $\lambda$  increases, the limiting value of the standard deviation of the contribution rate decreases. But, eventually increasing  $\lambda$  beyond a certain point causes the standard deviation of the contribution rate to increase.

The results show that, for most cases investigated, the standard deviation of the contribution rate decreases and reaches a minimum and then starts to increase as the spread period ( $m$ ) increases. This has been observed when the parameter ( $\phi$ ) of the AR(1) takes the values  $-0.3$ ,  $-0.1$ ,  $+0.1$ , and  $+0.3$  and when the parameter ( $\phi$ ) of the MA(1) takes the value of  $-0.3$ ,  $-0.1$ , and  $+0.1$ . But for the case of  $\phi = +0.3$  in table 7.8 for the MA(1) model, when  $\lambda$  takes the values from 0 to 0.4, the standard deviation of contribution rate decreases throughout the range of spread periods tested. Therefore, from these results, we conclude that more stable contribution rates emerge if asset gains and losses are not spread over periods exceeding 10 years. Spread periods should be even shorter than this if asset values are being smoothed and whenever asset gains and losses are spread over long periods then excessive smoothing must be avoided, because this could result in an unstable funding process.

The pattern of the variance of the contribution rate when plotted against  $K$  for different levels of  $\lambda$  shows that, for lower levels of  $\lambda$ , the variance of contribution rate decreases and reaches a minimum, and then starts to increase as  $K$  increases. But for higher levels of  $\lambda$ , which in our case is  $\lambda \geq 0.7$  for the MA(1) case when  $\phi = -0.3$ ,  $\lambda \geq 0.8$  for the AR(1) case when  $\phi = +0.1$ , and  $\lambda \geq 0.7$  for the AR(1) case when  $\phi = +0.3$  and  $\lambda \geq 0.9$  for all other values of  $\phi$  and  $\phi$  in both models, the variance of the contribution rate increases for all values of  $K$ .

These results suggest that, for the parameter values investigated, with the objective of minimizing  $\lim Var_f(t)$  and  $\lim Var_c(t)$ , that is for efficient asset valuation smoothing and asset gain /loss spreading, it is recommended that:

- (a) if there is moderate autocorrelation in the asset model and if the spread period is less than 3 years, then  $\lambda$  should be at most 80% (so that the weight placed on the current market values of assets is in the range 20-100%);

- (b) if there is positive autocorrelation in the AR(1) asset model (so that  $\phi \geq 0.3$ ) and if the spread period is less than 3 years, then  $\lambda$  should be at most 10%;
- (c) if there is moderate autocorrelation in the asset model and if the spread period is less than 5 years, then  $\lambda$  should be at most 60%;
- (d) it is not efficient use spread periods longer than 5 years for the case of moderate autocorrelation unless values of  $\lambda$  smaller than 60% are used.

For the case of  $\phi = +0.3$ , we note that the standard deviation of the contribution rate decreases for the range of spread periods investigated for  $\lambda \leq 0.40$  which implies that the optimal spread period is likely to be greater than 30 years.

#### **7.2.4 General comments on results**

When the logarithmic rate of return process follows the MA(1) model, the standard deviation of both funding level and contribution rate decrease as the autocorrelation at lag one decrease that is as  $\phi$  increases. For the case when logarithmic rate of return follows the AR(1) model, the standard deviations of the funding level and that of contribution rate increase as the autocorrelation at lag one increases, that is as  $\phi$  increases.

The results obtained from the simulation of the MA(1) and AR(1) models are consistent with those for independent and identically distributed rates of return reported by Owadally and Haberman (2000a), except that the acceptable ranges for the parameters ( $m, \lambda$ ) are more constrained when we have dependent models for investment rates of return than when we have the IID case.

This is particularly true for the AR(1) asset return model. In this case, the more positive is  $\phi$ , the narrower is the range of efficient choices for  $\lambda$ , so that it becomes efficient to reduce the smoothing of market values of assets (e.g  $\phi = 0.3$  and  $\lambda \leq 0.1$ ).

## **8. Moments of the Pension Funding Process Using the Wilkie Model**

### **8.1 Asset-liability Modelling Simulation Cases and Assumptions Employed**

The objective of using the Wilkie model in this paper is to represent the future stochastic behaviour of the investment returns for the pension scheme being modelled so that standard deviations of the pension fund levels and contribution rate can be calculated.

The same pension scheme model will be used as in our earlier analysis, except that pensions are not indexed with inflation. This will be more realistic because nominal rates of return have been more volatile than real rates of return. This adds to the volatility in the funding of benefits that are not indexed with inflation.

Stochastic projections are carried out 2000 times over 300 years, and investment returns are generated for each year in each simulation by using the Wilkie(1995) stochastic investment model. Then the pension scheme model calculates the fund level and contribution rate for each year in each simulation, and finally the standard deviations of the funding level and contribution rate are calculated. In all the projections, smoothed asset values (based on the parameter  $\lambda$ ) are used and the Unit Credit Funding method is used to value the liabilities of a pension scheme.

The following assumptions will be considered in the analysis:

- Two asset classes will be considered and they include equities, and government bonds (Consols). A proportional rebalancing between the two asset types is assumed with income from an asset being re invested in that asset.
- Pensions in payment are not indexed with inflation.
- Stochastic inflation on prices and wages is assumed and pensions are a fraction of final salary
- Economic projection assumption. The pension fund is invested in two asset classes, U.K equities, and Government bonds.
- Economic valuation assumptions. Wage inflation is deterministic and it is assumed to be 6.5% and a real discount rate of 4.5% is assumed.
- Demographic projection assumptions:
  - Population follows the A1967-70 life table.
  - The entry age is 25 years.
  - Retirement age is 65 years.
  - No early retirement, no withdrawals.

- No salary scale.
- Earned salary equals unity that is  $S = 1$
- Demographic valuation assumptions. Demographic valuation and projection assumptions are identical in the sense that the assumptions do not deviate from the actuarial valuation basis and gains and losses emerge only as a result of unforeseen economic experience.

## 8.2 Calculation of Actuarial Liability, Normal Cost and Benefit Outgo

The Unit Credit Funding method is used in calculating the expected present value of Actuarial Liability at time “t”.

For a person aged  $x = 25$  to  $x = 64$  years, the expected present value of the pension liability is:

$$AL(x, t) = \frac{2}{3} \frac{l_{65}}{l_x} v^{65-x} \ddot{a}_{65} S(1+e)^{t+65-x} m_x \ddot{a}_x.$$

For a person aged 65 years and above, the expected present value of the actuarial liability is:

$$AL(x, t) = \frac{2}{3} \ddot{a}_x S(1+e)^{t+65-x}.$$

Then, the expected present value of Actuarial Liability at time t is therefore the summation of  $AL(x, t)$  over all ages:

$$AL(t) = \frac{2}{3} \sum_{x=25}^{65} n_x \frac{l_{65}}{l_x} v^{65-x} \ddot{a}_x S(1+e)^{t+65-x} + \frac{2}{3} \sum_{x=65}^{\infty} m_x S(1+e)^{t+65-x} \ddot{a}_x \quad (41)$$

where,

$n_x$  represents number of people aged between 25 and 64 years.

$m_x$  represents number of people who are 65 years and above.

$e$  is the rate of wage inflation, which in our case is 6.5%

and all annuities are calculated at a nominal rate of interest of 11%.

If we define  $NC(x, t)$  to be the present value of the Normal Cost for a person aged  $x$  at time “t”, then:

$$NC(x, t) = \frac{I}{60} {}_{65-x}| \ddot{a}_x S(1+e)^{t+65-x}$$

and the expected present value of the total Normal Cost is:

$$NC(t) = \frac{1}{60} \sum_{x=25}^{64} {}_{65-x}| \ddot{a}_x S(1+e)^{t+65-x} n_x. \quad (42)$$

The rate of annual payment of Pension Benefit is determined by Trowbridge's equation of equilibrium, comparable to (1):

$$B(t) = NC(t) + d * AL(t) \quad (43)$$

where  $d = \frac{i}{1+i}$  and is calculated at  $i = 4.5\%$ .

### 8.3 Simulation Results.

The results based on the Wilkie model have been derived from 2000 simulations with each simulation performed over 300 years (as in section 7). Investment returns have been generated in each simulation and then these investment returns have been used in the pension scheme model for calculating the fund level and contribution rate for each year of each simulation. Finally the standard deviations of the funding level and contribution rate have been calculated.

Two different asset allocation strategies have been used to investigate changes in the variability of the funding level and contribution rate: 80:20 and 60:40 for the balance respectively between equities and government bonds (consols) for different lengths of spread period ( $m$ ) and smoothing weights ( $\lambda$ ).

The variances for the funding level are shown in tables 8.1 and 8.2 for an 80:20 and a 60:40 equity: bond portfolio respectively, and those for the contribution rate are listed in tables 8.3 and 8.4 respectively.

Tables 8.1 and 8.2 show that the funding levels become more variable as gains and losses are spread over a longer period of time, that is when  $m$  increases. Another feature is that, as  $\lambda$  increases, the standard deviation of funding level also increases. These results are similar to those obtained from the simulations of the MA(1) and the AR(1) models.

It can be observed from tables 8.3 and 8.4 that the results are similar to those obtained when the investment rate of return follows a Moving Average MA(1) or Autoregressive AR(1) process. The variability of the contribution rate decreases and then increases (with  $\lambda$  fixed) when gains and

losses are spread over longer periods ( $m$ ) so that contribution rates become more stable as  $m$  is increased up to a critical value  $m^*$ .

As  $\lambda$  increases (for  $m$  fixed),  $\text{Var}(t)$  has a minimum, except for large values of  $m$  when  $\text{Var}(t)$  increases as  $\lambda$  is increased.

The results indicate (as before) that it is necessary to avoid over smoothing. This means that it is better to recommend that the spread period of asset gains and losses to be 5 years or less for the two asset allocation strategies investigated and that the smoothing parameter should be at most 60% (so that the weight placed on the current market value of the assets should lie in the range 40 - 100%).

These results are similar to those obtained when the rates of return are independent and identically distributed or are simulated from an MA(1) or AR(1) model. Apart from the differences in the spread periods recommended, the behaviour of the standard deviations of the funding level and contribution rate as  $\lambda$  and the spread period ( $m$ ) change is consistent.

These results are also similar to the theoretical results from Owadally and Haberman (1999, 2000a).

They show that excessive smoothing of asset gains & losses of the pension scheme and spreading them over long periods is not efficient to the funding process because they lead to unstable contribution rates and more variable funding levels over time. For the parameter case of  $\lambda = 0$  (i.e. market values of assets) the results are consistent with those of Owadally and Haberman (2000b) and Haberman and Smith (1997) obtained from simulations of the Wilkie model.

## **9. CONCLUSIONS**

Certain aspects of pension funding and asset valuation of a defined benefit pension schemes have been considered. The spread method of amortizing gains and losses is considered and used in the simulations for the calculation of contribution rates. Scheme assets are not only valued at pure market value but also at average market values.

The aim of this paper is to show the combined effect of changing the smoothing weights ( $\lambda$ ) and the spread period ( $m$ ) on the standard deviations of the funding level and contribution rate. Scheme Rates of return are modelled as MA(1), AR(1) models, in a similar manner to Owadally and Haberman (2000b), and also by using the Wilkie (1995) investment model where returns on equities and Government bonds are simulated and used in the valuation.

The results obtained after the simulations have been analysed and compared with earlier theoretical results for the case of independent and identically distributed rates of return, as obtained by Owadally and Haberman (2000a).

From the results, it has been observed that spreading asset gains and losses over long periods and over-smoothing should be avoided because they lead to more variable funding levels and unstable contribution rates. These results were consistent with those obtained by Owadally & Haberman (2000a).

Analyses indicate that for a spread period of 1-5 years, the efficient smoothing weight should range between 20 – 100% and when the spread period is more than 5 years then the weighting on market value of assets should be at most 60%. For efficiency it is suggested that gains and losses should not be spread for longer than 5 years unless values of  $\lambda$  less than 60% are used.

This work can be extended by looking at the combined effects of direct amortization of gains & losses and that of smoothing market values of assets, and at the more general cases of the Moving Average Process of order  $p$ , MA( $p$ ) or Autoregressive Process of order  $q$ , AR( $q$ ) for modelling the rates of investment return. For the case of the Wilkie (1995) model, different classes of assets other than equities and Government bonds that have been used in this dissertation could be considered for further investigation and their effects analysed.

## REFERENCES

ANDERSON, A.W. 1992. Pension Mathematics for Actuaries, 2<sup>nd</sup> ed. Winstead, Connecticut: ACTEX Publications.

BOOTH, P., CHADBURN, R., COOPER, D., HABERMAN, S. AND JAMES, D. 1999. "Modern Actuarial Theory and Practice". CRC Press, Boca Raton, United States.

DUFRESNE, D. 1988. "Moments of Pension Contributions and Fund Levels when Rates of Return are Random". *Journal of the Institute of Actuaries* vol 115. pp 535 – 544.

DUFRESNE, D. 1990. "Fluctuations of Pension Contributions and Fund Levels." *Actuaries Research Clearing House*, vol 1990.1, pp 111- 120.

EZRA, D.D. 1979. Understanding Pension Fund Finance and Investment. Toronto, Canada: Pagurian Press.

HABERMAN, S. 1994. "Autoregressive Rates of Return and Variability of Pension Contributions and Fund levels for a Defined Benefit Pension Scheme". *Insurance: Mathematics and Economics*, vol 14, pp 219-240.

HABERMAN, S. AND SMITH, D. 1997. "Stochastic Investment Modelling and Pension Funding: A Simulation Based Analysis". Actuarial Research Paper No. 102. Department of Actuarial Science and Statistics, City University, London.

HABERMAN, S. AND WONG, L.Y.P. 1997. "Moving Average Rates of Return and the Variability of Pension Contributions and Fund Levels for a Defined Benefit Pension Scheme". *Insurance: Mathematics and Economics*, vol 20, pp 115-135.

KEMP, M.H.D. 1996. "Asset/Liability Modelling for Pension Funds". Paper presented on October 15, 1996 at the Staple Inn Actuarial Society, London, England

KINGSLAND, L. 1982. "Projecting the Financial Condition of a Pension Plan Using Simulation Analysis". *Journal of Finance* vol 37: pp577-584.

OWADALLY, M.I. AND HABERMAN, S. 1999. "Pension Fund Dynamics and Gains and Losses Due to Random Rates of Return". *North American Journal*, vol 3 (no 3), pp 105-177.

OWADALLY, M.I. AND HABERMAN, S. 2000a "Asset Valuation and the Dynamics of Pension Funding with Random Investment Returns". Actuarial Research Paper No. 131. Department of Actuarial Science and Statistics, City University, London.

OWADALLY, M.I. AND HABERMAN, S. 2000b "Efficient Amortization Of Actuarial Gains/losses and Optimal Funding in Pension Plans". Actuarial Research Paper No. 133. Department of Actuarial Science and Statistics, City University, London.

TROWBRIDGE, L. 1952. "Fundamentals of Pension Funding". Actuarial Education and Research Fund. *Transaction of The Society of Actuaries*, vol 4, pp 17- 43.

TROWBRIDGE, C.L. AND FARR, C.E. 1976. *The Theory and Practice of Pension Funding*. Homewood, Illinois: Richard D. Irwin.

WILKIE, A.D. 1995. "More on a Stochastic Asset Model for Actuarial Use". *British Actuarial Journal*, vol 1, pp 777-964.

WINKLEVOSS, H.E. 1993. *Pension Mathematics with Numerical Illustrations*, 2<sup>nd</sup> ed. Philadelphia, Pennsylvania: University of Pennsylvania Press.

WISE, A.J. 1987 "Matching and Portfolio Selection: Part I". *Journal of Institute of Actuaries*, vol 4, pp 113-133.