

Self-Annuitization, Ruin Risk in Retirement and Asset Allocation: The Annuity Benchmark

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Abstract

The present paper considers a retiree of a certain age with an initial endowment of investable wealth facing the following alternative investment opportunities. One possibility is to buy a single premium immediate annuity-contract. This insurance contract pays a life-long constant pension payment of a certain amount, depending e.g. on the age of the retiree, the operating cost of the insurance company and the return the company is able to realize from its investments. The alternative possibility is to invest the single premium into a portfolio of mutual funds and to periodically withdraw a fixed amount, in the present paper chosen to be equivalent to the consumption stream generated by the annuity. The particular advantage of this self annuitization strategy compared to the life annuity is its greater liquidity. However, the risk of the second opportunity is to outlive the income stream generated by this investment. The risk in this sense is specified by considering the probability of running out of money before the uncertain date of death. The determination of this personal ruin probability with respect to German mortality and capital market conditions is the objective of the following paper.

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1. Introduction

In a number of contributions Milevsky/Robinson et al. (1994, 1997, 2000) and Milevsky (1998) consider the ruin risk of self-annuitization. A self-constructed annuity consists of investing at retirement an initial endowment of wealth amongst the various asset categories (e.g. equity, bonds, real estate) represented by mutual funds, earning a stochastic rate of return, and withdrawing a fixed periodic amount for consumption purposes. The financial risk of this strategy is that retirees can outlive their assets in the event of long-run low investment returns connected with longevity. This is in contrast to purchasing a life annuity, which is an insurance product that pays out a life-long income stream to the retiree in exchange for a fixed premium charge. As Mitchell et al. (1999) pointed out, the main characteristic of the life annuity is that it protects retirees against the risk of under-funding in retirement by pooling mortality experience across the group of annuity purchasers. The particular advantage of the self annuitization strategy compared to the life annuity is the greater liquidity and the chance of leaving out money for their heirs in the case of an early death, but it is at the expense of running out of money before the uncertain date of death.

The personal ruin risk of self-annuitization is crucially dependent on the amount periodically withdrawn from the fund as well as the fund asset allocation, i.e. the investment weights in equity, bonds and real estate assets. The choice of a risk minimizing asset allocation with respect to a suitable benchmark for the amount of withdrawal still is an open question. In the present contribution we have chosen as a benchmark the amount generated by the single premium immediate annuity contract itself. Generally we assume representative conditions as to be found in the German life annuity market as well as in the German mutual funds market. In particular, in our framework the usual products of annuities offered by life insurance companies as well as products offered by the investment industry are taken into consideration. Another specific feature to be considered in the analysis stems from the fact that only a certain part of the annuity is guaranteed, the remaining part is at least in principle subject to the amount of profit participation and therefore depends on the investment performance (surplus strength) of the insurance company offering the life-annuity contract.

2. Data Base and Design of the Analysis

With regard to the specification of the biometrically dependent parameters of the insured, we take the DAV (German Actuarial Association) basic mortality table 1994 R (c.f. appendix A) as a starting point. Hereby, a man with an entry age of 60, 65 and 70 was considered, respectively. For each case, a constant yearly pension, working on the assumption of a second order rate of return of 4%, 5,5% and 7 % respectively, was calculated for a one-time amount of DM 100 with the aid of the calculation formula in Appendix A. Firstly, through this variation in the second order interest rate, we are able to take into consideration the various surplus strengths of the insurance companies offering the annuities. Additionally, the problem that the envisioned pension payments cannot be guaranteed in full, can be covered. A reduction in the pension amount can be represented in the model by using a lower second order interest rate.

Regarding the cost-structure of the insurance company, it is assumed that $\alpha = 40\%$, $\beta = 1,25\%$ and $\gamma = 1,5\%$, which corresponds to the usual market conditions in Germany for single premium annuity contracts (see Appendix A for technical details). Accordingly, one obtains the following life-long yearly annuity payments (before taxes):

Interest Rate	4%	5.5%	7%
Entry Age	Life Annuity (DM p.a.)		
60	6.23465	7.17664	8.14253
65	7.06501	7.99189	8.93636
70	8.24026	9.15922	10.0885

From the table it can be observed that a 60-year old retired person, at a second order interest rate of 4% receives about 6.23% of his available old age security-capital as a life-long yearly pension. The significantly positive difference of 2.23465 percentage-points between relative pension payment and a concurrent consumption to initial wealth rate of 4%, expressed through the second order interest rate for annuity calculation, can be seen as resulting from two factors. First of all, it is assumed, for the calculation of the life annuity, that a complete consumption of the initial capital occurs. The remaining part of the identified pension-spread is due to the collective nature of the insurance. When a pensioner dies, this benefits the remaining pension-group and results in a significant rise in the yearly pension representable

by the insurance company, for example in comparison to a conversion of the initial capital into pension-payments on a purely individual basis. As can be observed from table 1 this pension-spread increases, the higher the entry-age of the pensioner is.

In order to maintain the comparability of the two investment strategies for retirement, the life annuities shown in Table 1 will serve as benchmarks for the evaluation of capital-exhaustion risks of alternative investment funds withdrawal-plans in the further course of this paper. For an identical initially invested capital (especially considering the issuing surcharge in the case of an investment in a fund) the annual withdrawal from the fund corresponds to those of the respective parameter-constellations of the life annuity as in Table 1 („insurance equivalent“ fund-withdrawal plan).

The probability of an individual outliving his wealth (PoR) will be regarded as the central risk index for the previously depicted “insurance equivalent” fund-solution. For the quantification of this probability we use the methodology presented in Appendix B. Hereby, the stochastic dynamics of the (uncertain) market values of the considered investment fund units are modeled as a (three-dimensional) geometric Brownian motion with constant drift, diffusion and correlation parameters. For the estimation of these parameters, we use the historical investment returns (including capital gains and dividends) for German investment funds over the period 1980–1998. Using the data set described in Albrecht/Maurer/Schradin (1999) three classes of well diversified funds have been studied: stocks, bonds, and real-estate funds concentrating their assets mainly within the German capital and real estate market. Hereby the issuing surcharge is 5% for stocks and real-estate funds and 3% for bond funds, which corresponds to usual market conditions in the German mutual fund market. Proceeding from a totality of 17 stock funds, 23 bond funds and 7 real-estate funds the respective fund within the different asset categories has been chosen which, as regards the average return over the period 1980–1998, took the median position. The yearly time-series returns from the years 1980 till 1998 give the following estimates for the (continuous) mean return p.a., the volatility and the correlation-coefficients:

Table 2: Descriptive Statistics for Stocks, Bonds and Real-Estate Funds

Class of Fund	Mean return (% p.a.)	Volatility (% p.a.)	Correlations		
			Stocks	Bonds	Real-estate
Stocks	11.78	16.78	1		
Bonds	7.52	5.02	0.335	1	
Real-estate	6.62	1.78	-0.247	0.353	1

Due to the complexity of the payment structures, we use Monte Carlo simulation to obtain the personal probability of ruin in retirement (PoR) for the self-annuitization strategy using an investment fund withdrawal-plan. Hereby, from the respective entry-age (i.e. 60, 65 and 70) up to the end-age of 110 in total 100.00 simulation runs were generated.

3 Results of the Simulation Studies

3.1 Withdrawal-plans for Individual Funds

We start with the special case of a 100% investment in the stock fund. The corresponding PoR depending on the amount of second order interest rate are included in the following figure.

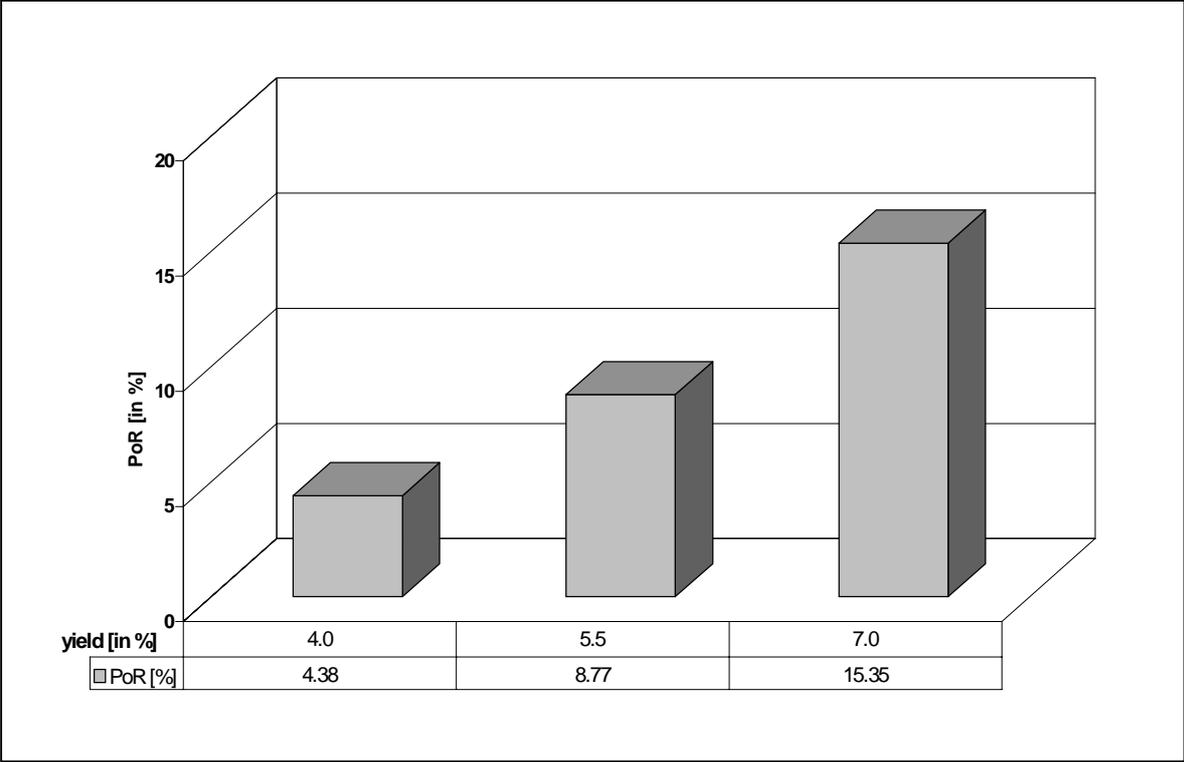


Figure 1: Probabilities of ruin in retirement (PoR) of a 60-year old man in the case of a equity fund investment dependent on interest rate for annuity calculation

It is recognized that the probability of ruin in retirement is correspondingly higher, the higher the second order interest rate chosen for the calculation of the annuity is. In other words, the better the investment performance of the insurance company (as measured by the second order interest rate for annuity calculation), the higher the risk for a comparable investment withdrawal-plan. It is evident that for a (hypothetical) second order interest rate of 4%, the

corresponding probability is a rather moderate 4.38%. This probability can be interpreted such that in 438 of 10.000 possible future capital market situations the 60-year old pensioner who invested his initial capital 100% in a stock fund, survived the exhaustion of the fund capital.

As the second special case, the 100% investment in the bond funds has been studied. The corresponding probabilities of asset-exhaustion during the lifetime of the pensioner, once again dependent on the second order interest rate chosen, are shown in figure 2.

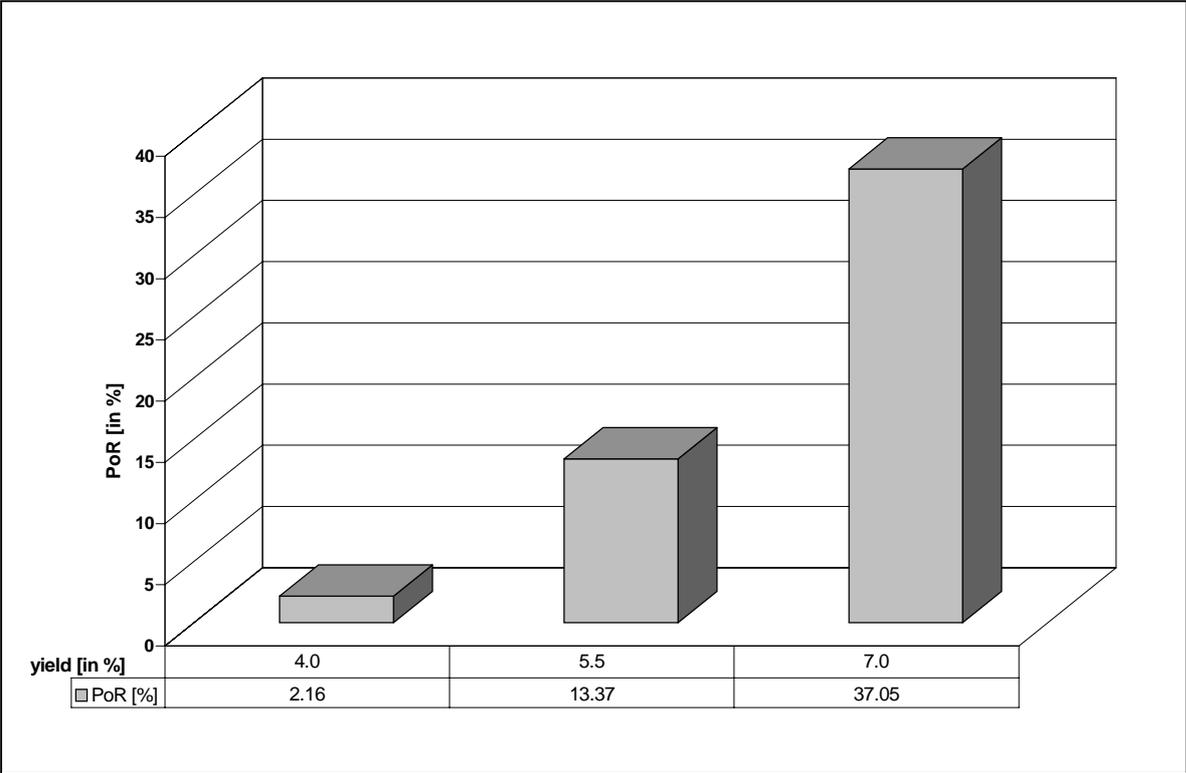


Figure 2: Probabilities of Ruin in Retirement (PoR) of a 60-year old man in the case of a bond-fund investment dependent on interest rates for annuity calculation

In comparison to a 100% investment in a stock fund, there is a significantly lower risk of asset exhausting for a bond fund with a second order calculated interest rate of 4%. However, the corresponding probabilities for a yield of 5.5% with a 100% investment in the bond fund are clearly higher than a 100% investment in a stock fund. This may be explained as follows: the probability of outliving wealth in the fund case depends on the expected return of the fund, the extent of return volatility as well as the amount of the insurance equivalent withdrawals (as represented by the second order interest rate for annuity calculation).

In the case of the stock fund, the risk of asset-exhaustion during the lifetime of the pensioner is dominated by the volatility of fund returns. In the case of the significantly less volatile bond

fund, the likelihood of asset-exhaustion is much more strongly determined by the expected fund return and less by its volatility. From a withdrawal amount represented by a second order interest rate of 5.5% onwards a 100% investment in a bond fund is therefore – at least with regards to the PoR - less advantageous relative to a 100% stock investment. This amount of the pension withdrawal implies competitive advantages for the investment alternative which is riskier, but also generates higher expected returns. In the case of lower pension withdrawals (represented by a second order interest rate of 4%) the conditions are however reversed, as the less risky and lower return investment gains in attractiveness.

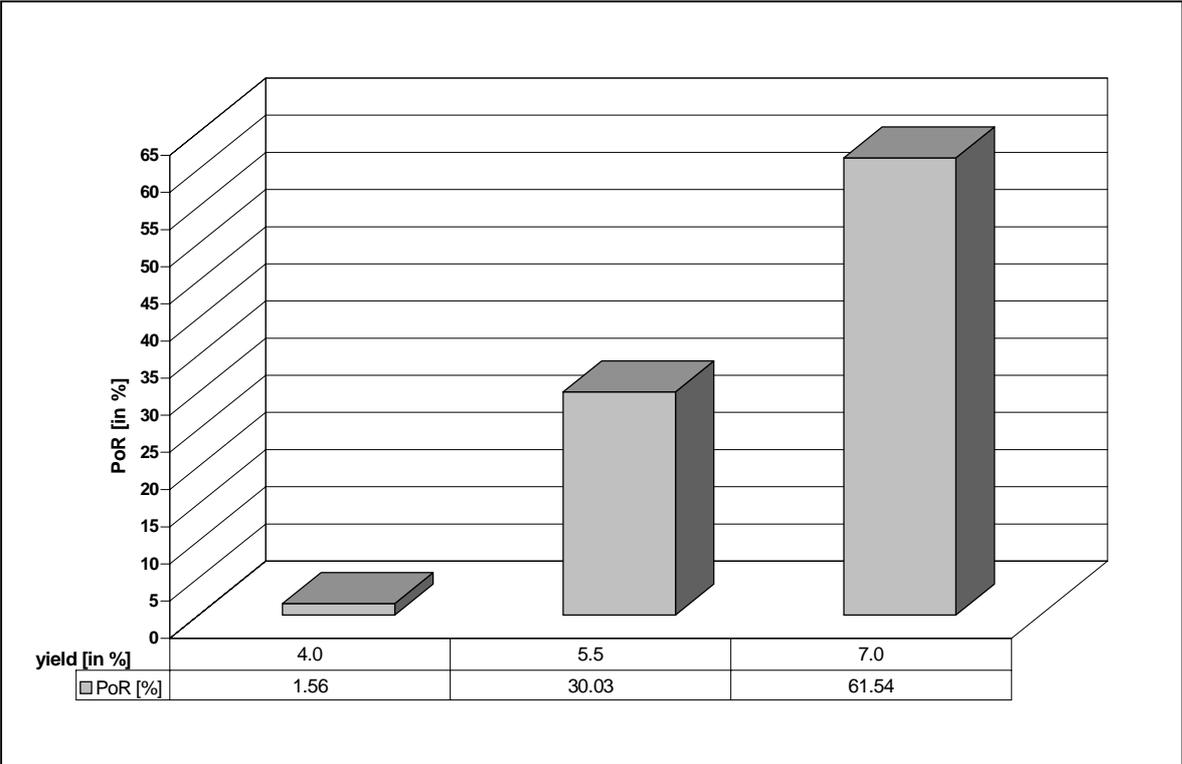


Figure 3: Probabilities of Ruin in Retirement (PoR) of a 60-year old man in the case of a real estate investment dependent on interest rates for annuity calculation

The special case shown in Figure 3 of a 100% investment in real-estate fund shows that in the case of a high second order interest rate, a similar situation appears as in the previous example. Here there is an even lower risk (1.56%) for a pure real-estate based insurance equivalent withdrawal-plan and an interest rate of 4.0%, than in the previous evaluations. However, this risk rises strongly at a second order interest rate of 5.5%. In the case of a second order interest rate of 7%, there is even a capital-exhaustion risk of 70%. These extreme values can be explained as follows: as Table 2 shows, the expected average return on a real-estate fund share lies at 6.62% p.a. whereby the very low volatility of 1.78% shows that these returns are very stable over time. As well, there is, according to Table 1, using an interest rate of 4% an amount of annuity for a 60-year old man of only ca. 6.23% referring to

the initial investment capital. The very stable interest paid on fund-capital lies then 39 points above the annuity payments. Thereby the determined benchmark returns can in most cases (i.e. simulated paths) be realized from the current interest on the real-estate fund shares, without having to reduce the existing fund capital. For second order interest rates the expected real-estate returns are however consistently below the benchmark. Thereby, in order to make up the difference, the fund capital must be resorted to regularly.

3.2 Diversification Effects through Asset Allocation

The previous evaluations have neglected the central result of modern portfolio theory according to which, under certain conditions, risk diversification effects may be taken advantage of through the appropriate mixture of investment alternatives. A diversification occurs when the risk of the entire portfolio is lower than that of the risk-minimizing individual investment. A necessary condition for the appearance of a diversification effect is that the respective individual investments are not completely positively correlated with one another. A glance at Table 3 shows that the correlation coefficients between the individual classes of funds are in all cases significantly different than one and in some cases are even negative.

The realization of a diversification effect does not occur in all kinds of fund mixtures. Rather, in order to attain an optimal diversification potential, i.e. a maximal risk-reduction relative to the individual investments, it is necessary to have a systematic selection procedure for the relative investment weights in the individual funds. For the determination of the risk-minimizing (static) asset mix of stocks, bonds and real-estate funds, the investment weights are varied by increments of 5%.

To graphically illustrate the diversification effect, the probability of an asset-exhaustion during the lifetime of the pensioner is shown in the following figures for alternative fund allocations from at first only two classes of funds (i.e. stocks and bonds). Hereby the assumed entry-age was 60 years, and the second order interest rate chosen for the calculation of the annuity was varied between 4%, 5.5% and 7%.

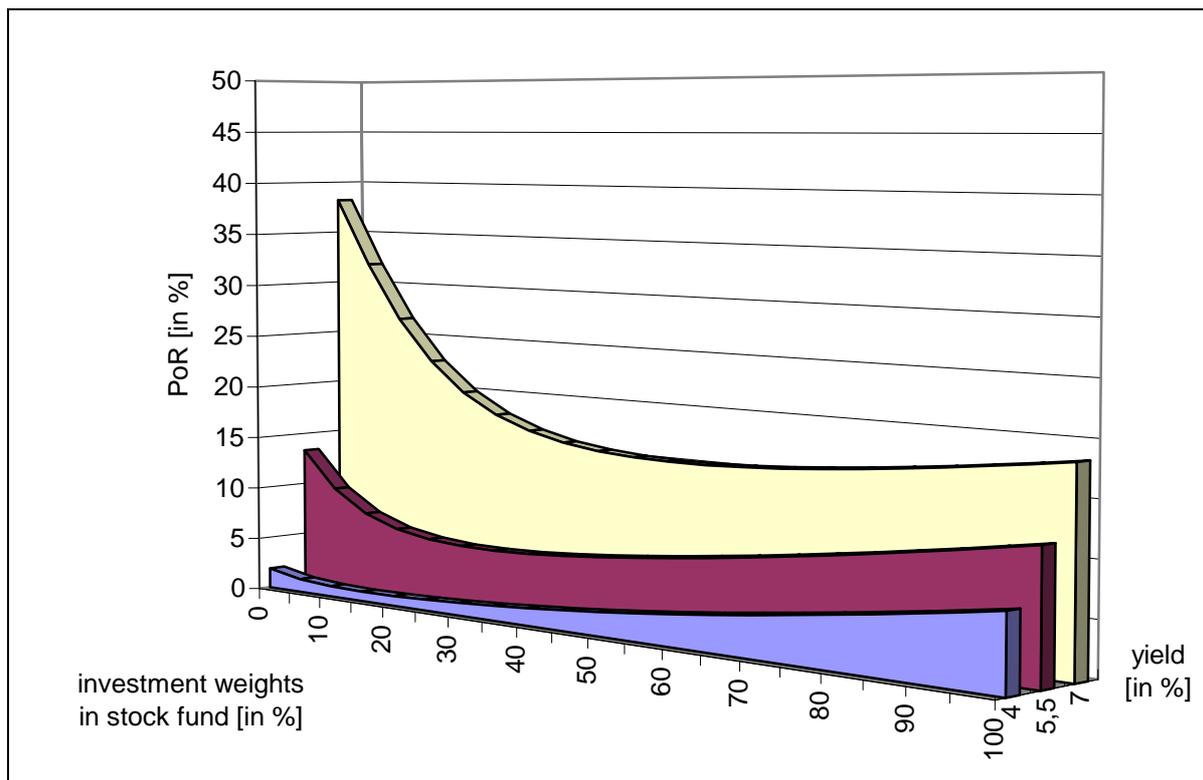


Figure 4: Probabilities of Ruin in Retirement for a 60-year old man with alternative stock and bond investments [in %] dependent on interest rates for annuity calculation of 4%, 5.5%, 7%

Figure 4 gives insight into the influence which a variation in the fund mixture has on the one hand, as well as a variation in the amount of the second order interest rate on the capital-exhaustion risk on the other. It is likewise evident that the probability of asset-exhaustion during the lifetime of the pensioner is higher, the higher the second order interest rate chosen for the calculation of the annuity. The shape of the curve shows additionally that there is an optimal stock fund proportion (and/or bond fund proportion) in which the risk of capital-consumption is minimal and/or where the diversification potential is exploited to the maximum. The “risk-minimizing” stock proportion is higher, the higher the second order interest rate is. The corresponding minimal PoR are presented in figure 5:

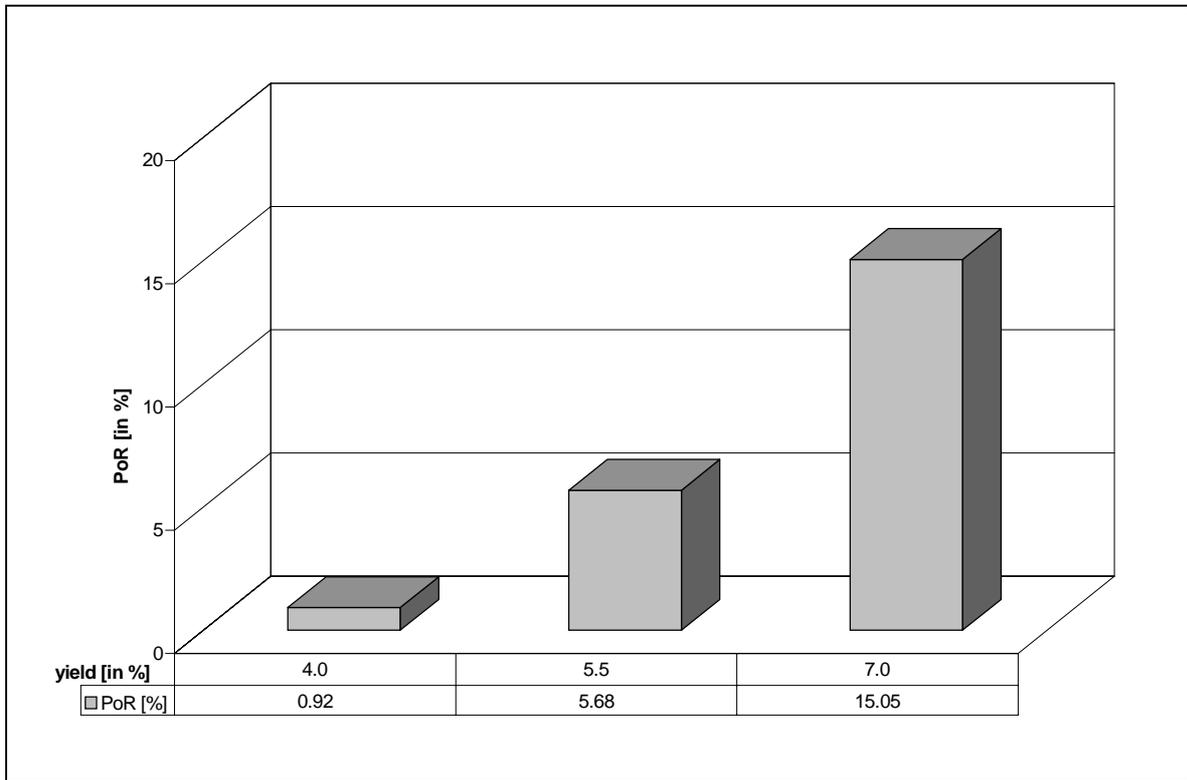


Figure 5: Probabilities of Ruin in Retirement for a 60-year old man with a risk minimizing stock/bond-fund allocation dependent on interest rates for annuity calculation

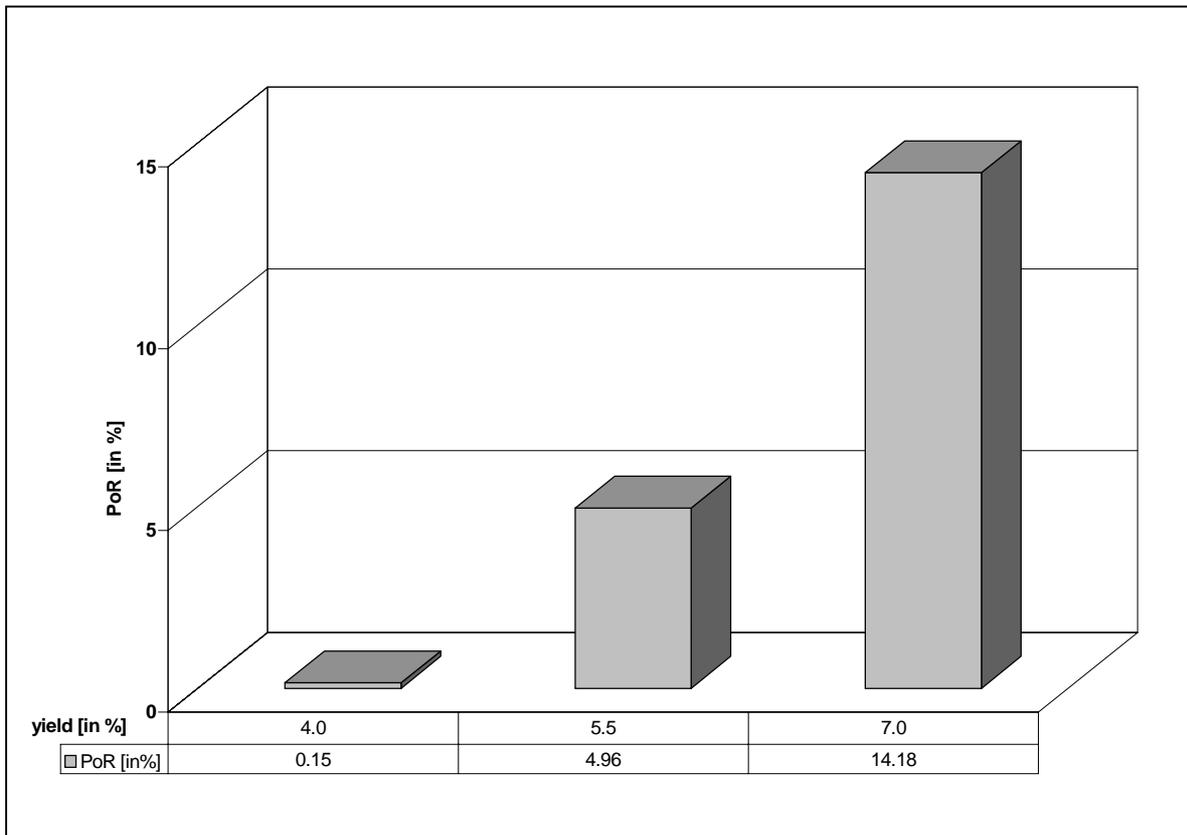


Figure 6: Probabilities of Ruin in Retirement for a 60-year old man with a risk minimizing stock/bond/real-estate fund allocation dependent on interest rates for annuity calculation

For further results regarding the fund allocation for the case of a 65 and 70-year old pensioner, see Albrecht/Maurer (2000/2001).

In Figure 6 the resulting PoR dependent on the second order interest rate for annuity calculation on the one hand and on the basis of a risk-minimizing fund allocation of stock, bond and real-estate funds on the other are shown. The results shown in the above figure make clear once again that the amount of the periodic pension-withdrawal and thereby the amount of the surplus strength of the insurance company offering the policy are the main determinants with regards to a substantial size in the probability of asset-exhaustion during the lifetime of the pensioner. In the area of a second order calculated interest to the amount of 4.0% resp. 5.5% the probability of asset-exhaustion is low to moderate, for 7% however, it is much more substantial.

In conclusion, the respective probabilities of outliving individual wealth as well as the risk-minimizing (concurrent) fund allocations for a 60-, 65- and 70-year old pensioner are represented in the following table.

Entry Age	Yield (%)	PoR (%)	Investment weights (%) in		
			Stocks	Bonds	Real-estate
60	4.0	0.15	10	0	90
	5.5	4.96	35	15	50
	7.0	14.18	50	30	20
65	4.0	2.16	25	10	65
	5.5	9.07	50	35	15
	7.0	17.50	80	20	0
70	4.0	7.14	50	35	15
	5.5	14.0	75	25	0
	7.0	21.39	100	0	0

Table 3: PoR and corresponding risk minimizing fund allocation for a pensioner with entry-age 60-, 65- and 70 dependent on interest rates for annuity calculation

On the whole it is noticeable that for the 65- and 70-year old pensioner, the respective probabilities of asset-exhaustion during his lifetime reach a uniformly higher level in comparison to the case of the 60-year old pensioner. It is plainly shown that independently of the chosen second order interest rate in the risk-minimizing situation, an increase in the stock-rate occurs in each case. Apparently the relatively high amount of the insurance equivalent pension, due to the shorter remaining lifespan of the 65 and 70-year old pensioner, implies a

higher involvement in the more profitable investment of the stock-fund in each case. This in turn involves however a higher risk-level in each case. The 70-year old pensioner has to accept substantial probabilities of asset-exhaustion within his lifetime using an insurance equivalent fund solution.

If one observes the relative investment weights of stock-units, the following results are obtained: the higher the entry-age and the higher the chosen second order interest rate, the higher the proportions of stock-units in the risk-minimizing fund allocation. At a second order interest rate of 7% stocks are clearly the dominating class of investment. A similarly clear tendency, however in comparison with stocks displaying an opposite trend, may be observed for the real-estate and/or bond fund units. The lower the entry-age, and the lower the second order interest rate, the higher the proportions of shares are. To that extent the bond and real-estate funds take on the roles of risk stabilizers and stock funds take on the role of yield-increasers. The following table summarizes the results in condensed form.

Investment weights	Interest Rate ↑	Entry-age ↑
Stock	+	+
Bonds	+/-	+/-
Real-estate	-	-

Table 4: Stocks, bonds and real-estate fund proportion for varying entry-ages and second order interest rates chosen for annuity calculation

3.3 Empirical Scale of the Second Order Interest Rate for Annuity Calculation

The previous mainly hypothetical assumptions regarding the second order interest rate will now be supplemented with empirical information. The decisive factor for the order of magnitude of the second order interest rate is the long-term return from the capital investment of the respective insurance company. According to the study by Albrecht/Maurer/Schradin (1999, p. 49) the average nominal investment rate of return reached a value of 7.71% (measured by the net interest) in the time-period 1980-1997 in the market average of the insurance industry. Moreover the market is characterized by a very large degree of homogeneity. Among the 30 largest German life-insurers evaluated within the framework of the study, that with the lowest average return over this time-period showed a value of 7.34 %.

4 Conclusions

In the present paper, the probability of consumption shortfall during the lifetime of a pensioner for fund withdrawal plans with insurance equivalent withdrawal amount has been studied. Static asset mixtures of stock, bond and real-estate funds were considered. As a main result the following was ascertained: in comparison to private annuity products a self annuitization strategy using mutual fund withdrawal plans contains in particular for high entry-ages, a substantial risk of outliving the individuals wealth, as long as the benchmark (the annuity) is based on a competitive investment return. It should be kept in mind when assessing the results however, that in this study (only) matters of ruin risk were of primary concern. Possible return implications, such as the bequest potential resulting from fund investment in the event of an early death, or the possibility of additional withdrawals from the fund as well as the greater flexibility of the self annuitization strategy, were not considered.

Appendix A: Actuarial Basis

We consider the following general basic situation. An x -year old retiree concludes a life-long annuity insurance (taking immediate effect upon advance payment) having paid a single premium to the amount of C . In return, he receives at the points of time $t = x, x+1, \dots, w$ pension payments to the amount of R_0, R_1, \dots, R_{w-x} . Here, w (normally: $w = 110$) represents the highest age considered for premium calculation. The (risk free) rate of interest used by the insurance company to discount future expected cash flows is denoted by i ($q := 1 + i, v := q^{-1}$). With respect to a table with the death-probabilities q_x the quantity ${}_t p_x$, denoted the conditional probability that the a man aged x will attain age $x+t$.

The actuarial present value of the life annuity (PVA) is then given through (in probability-theoretical or alternatively, in actuarial notation):

$$\begin{aligned} \text{PVA}(i) &= \sum_{t=0}^{w-x} R_t \cdot {}_t p_x \cdot v^t \\ &= \frac{1}{D_x} \cdot \sum_{t=0}^{w-x} R_t \cdot D_{x+t} \end{aligned} \quad (\text{A1})$$

In the case of equal pension payments per unit of time $R_t \equiv R$ there results the expression

$$\text{PVA}(i) = R \cdot \ddot{a}_x, \quad (\text{A2})$$

where

$$\begin{aligned} \ddot{a}_{x:\overline{n}|} &:= \sum_{t=0}^{n-1} {}_t p_x \cdot v^t \\ &= \frac{1}{D_x} \cdot \sum_{t=0}^{n-1} D_{x+t} \end{aligned} \quad (\text{A3})$$

as well as

$$\ddot{a}_x := \ddot{a}_{x:\overline{w+1-x}|}. \quad (\text{A4})$$

The individual net single premium $\text{NSP} = \text{NSP}(i)$ corresponds to $\text{PVA}(i)$ and therefore the annuitization of the investment amount C results in an amount of $R = R(i)$ on the basis of the solution of the equation $\text{BWR}(i) = C$, and is given by

$$R = \frac{C}{\ddot{a}_x}. \quad (\text{A5})$$

Considering gross premiums, we have to modify the approach. The German cost system in life insurance traditionally is using α -costs (acquisition costs), β -costs (renewal commission) and γ -costs (management expenses) as expense loadings to the net premium. The α -costs resp. β -costs are in percent of the amount of the gross single premium GSP and the γ -costs are in percent of the annuity per year of the yearly pension payments. In total, the equation $\text{GSP} = \text{NSP} + (\alpha + \beta) \cdot \text{GSP} + \gamma \cdot \ddot{a}_x \cdot R$, results in a gross single premium of amount

$$\text{GSP} = R \cdot \frac{\ddot{a}_x \cdot (1 + \gamma)}{1 - \alpha - \beta}. \quad (\text{A6})$$

By using a rate of interest i to discount future expected cash flows on annuitization of the investment amount C leads to a life-long constant pension $R = R(i)$ to the amount of

$$R = \frac{C \cdot (1 - \alpha - \beta)}{\ddot{a}_x \cdot (1 + \gamma)}. \quad (\text{A7})$$

The DAV-mortality table 1994 R (c.f. Bundesaufsichtsamt für das Versicherungswesen 1995 as well as Schmithals/Schütz 1995) is conceived on the basis of q_x^B of the basic mortality table 2000 and the trend function $F(x)$ as a two-dimensional mortality table in the form

$$q_x^t = \exp\{-\bar{F}(x) \cdot (t + x - 2000)\} \cdot \bar{q}_x^B \quad (\text{A8})$$

Hereby q_x^t corresponds to the one-year death probability of an x -year old man. For the evaluations here undertaken, we have proceeded on the assumption that the studied person of a fixed pension entry-age (e.g. 60, 65, 70) will reach this age in the year 2000. The relevant q_x corresponds to the respective amounts in the basic table 2000, i.e. $q_x = q_x^B$.

Appendix B: Determination of the Probability of Ruin in Retirement

Letting $\{V_R(t); t \geq 0\}$ represent the stochastic process of the development in value of an initial capital amount C invested at the time $t = 0$ whereby from the accumulating wealth a constant yearly pension payment to the amount R is withdrawn at the beginning of each year (i.e. at the time-points $t = 0, 1, 2, \dots$). Let T_x represent for the following the curtailed remaining lifetime of an x -year old pensioner at $t = 0$, the time-point of the investment.

The event $V_R(u) \leq 0$ consists of an exhaustion of the invested capital at the point of time $t = u$. Formally, we are interested in the earliest of these points in time (the point in time when ruin occurs), i.e.

$$\tau_R = \inf \{u > 0; V_R(u) \leq 0\} \quad (\text{B1})$$

Obviously, we have $V_R(u) \leq 0$ for $u = \tau$. The time-point τ_R is uncertain and depends on the various respective development-path taken by the asset process.

At the centre of interest is the possibility $T_x > \tau_R$, i.e. the wealth-exhaustion has been realized at a point in time when the observed person is still alive. In particular, the probability $\text{PoR}(R)$ of this event is of relevance, whereby

$$\text{PoR}(R) := P(T_x > \tau_R) \quad (\text{B2})$$

For the derivation of a possibility of determining PoR we may look at the the following consideration which is based on the assumption that T_x and τ_R may be taken to be stochastically independent. We have:

$$\begin{aligned} P(T_x > \tau_R) &= P(T_x - \tau_R > 0) \\ &= \sum_{t=0}^{\infty} P(T_x - t > 0 \mid \tau_R = t) \cdot P(\tau_R = t) \\ &= \sum_{t=0}^{\infty} P(T_x > t) \cdot P(\tau_R = t) \\ &= \sum_{t=0}^{\infty} {}_t p_x \cdot P(\tau_R = t) \\ &= \sum_{t=0}^{w-x} {}_t p_x \cdot P(\tau_R = t) . \end{aligned}$$

If one, for the present purposes, excludes the case $R > C$, i.e. $\tau_R = 0$, then in total the following results:

$$\text{PoR}(R) = \sum_{t=1}^{w-x} {}_t p_x \cdot P(\tau_R = t). \quad (\text{B3})$$

The advantage in the representation (B3) consists in the separation of the mortality law on the one hand, and the constellation of asset-exhaustion at a fixed point in time on the other. The mortality law is given in the DAV-mortality table 1994 R. The probabilities $P(\tau_R = t)$ however, must be determined by Monte-Carlo-Simulation.

Appendix C: Process of Funds-Investment: The Case of Stocks and Bonds

Let $\{V_R(t); t \geq 0\}$ again represent the process of the asset development considering advance yearly withdrawals to the amount of R . The asset invested in $t = 0$ is to the amount C . This gives $V_0 = C - R$. Assuming additionally an initial issuing surcharge to the amount of $100 a_A$ % for stock funds and to the amount of $100 a_F$ % for bond funds, this results in the amount $x_A(C - R)/(1 + a_A) + (1 - x_A)(C - R)/(1 + a_F)$ available for investment, i.e. the starting value is in this case $V_0 = x_A(C - R)/(1 + a_A) + (1 - x_A)(C - R)/(1 + a_F)$.

The available capital at the beginning of the investment period (after withdrawal of the pension payment) is invested with the proportion x_A ($0 \leq x_A \leq 1$) in a stock fund and to a complementary proportion $1 - x_A$ in a bond fund. We take the standard-approach where the value development of stocks and bond funds follows a two-dimensional geometric Brownian motion. Letting $I_A(t)$ resp. $I_B(t)$ represent the respective continuous one-period returns, then for the process $\{S_A(t)\}$ of the further development of the stock fund we get

$$S_A(t) = S_A(t-1) \cdot \exp\{I_A(t)\} \quad (C1)$$

and correspondingly, for the process of the development of the bond fund

$$S_B(t) = S_B(t-1) \cdot \exp\{I_B(t)\} \quad (C2)$$

In general the following obtains:

$$V(t+1) = V(t) \cdot [x_A \cdot \exp\{I_A(t+1)\} + (1 - x_A) \cdot \exp\{I_B(t+1)\}] - R \quad (C3)$$

The generation of the paths of the process of development of the invested capital can therefore take place on the basis of the successive (simultaneous) generation of realizations of $[I_A(t+1), I_B(t+1)]$ ($t = 0, 1, 2, \dots$). In the case of a bivariate geometric Brownian motion it is known that $[I_A(t), I_B(t)]$ represent independent realizations of a bivariate normally distributed vector (I_A, I_B) of random variables. The corresponding parameters have to be identified empirically.

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