THE INVESTMENT RETURN FROM A CONSTANTLY REBALANCED ASSET MIX

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ABSTRACT

A study has been made of the investment return which results from a portfolio when the various asset sectors are constantly rebalanced to fixed proportions by market value. The mathematical analysis has been written in an article which is due to be published in the Journal of the Institute of Actuaries in 1993. This paper sets out the main points from that article and continues investigations of the topic.

The main points are illustrated using examples taken from the UK stock market and also with international diversification of equities. The theoretical results, which are based on the geometric Gauss-Wiener model of short term stock market returns, give an excellent fit to the observations.

The conclusion is that when the mean returns on the various asset sectors are disparate, there can be significant differences between a constant mix strategy and a strategy of rebalancing the portfolio proportions at regular intervals. The mean and standard deviation of the difference can be evaluated theoretically to give a guide as to the likely effect of choosing a particular rebalancing interval.

1. INTRODUCTION

The following question arises in relation to the management of a diversified portfolio:

Given a policy of investment in a particular set of holdings, what is the effect on the investment return if the portfolio is constantly rebalanced in order to maintain fixed proportions by market value of the various holdings in the fund?

Consider for example a mature pension fund whose trustees wish to instruct their fund manager to target a notional fund comprising 50% UK equities and 50% gilts, such that the performance of the fund will be measured relative to the 50:50 benchmark. To enable effective management relative to the trustees' objectives, and accurate monitoring by the trustees of the fund manager's results, an unambiguous specification of the notional benchmark fund is required.

The main requirement is to represent the two types of holding, equities and fixed interest, for which appropriate indices could be chosen. A secondary but nonetheless important point to decide is whether the asset proportions are to remain a constant 50:50 by market value at all times, or whether they are to follow the relative behaviour of the two investment sectors. The question "What is the effect of maintaining constant asset proportions?" is therefore directly relevant to setting investment objectives and measuring performance relative to a notional benchmark fund.

This question was addressed in an article which, at the present time of writing, is due shortly to be published in the Journal of the Institute of Actuaries (JIA). This paper sets out the main points from that article and continues investigation of the topic.

2. NOTATION

Consider two stocks or two asset sectors called S_1 and S_2. Let $P_i(t)$ be the index of total return on S_i at time t. Set $P_1(0) = P_2(0) = 1$ for a convenient scale, so that $P_i(t)$ is the total return between times 0 and t with income reinvested in S_i.

Denote the asset proportions by market value by α_i where $\alpha_1 + \alpha_2 = 1$.

Different investment strategies over time will lead to different outcomes, but we are interested in two situations.

- **a.** With a passive investment strategy, all income is reinvested in the sector from which it derives and there is no switching between sectors. The asset proportions are allowed to drift away from their initial values α_i
- **b.** With a "constant mix" strategy, there is constant rebalancing to maintain fixed sector proportions α_i by market value at all times.

Let $Q = Q(t)$ denote the portfolio value at time t which results from the constant mix strategy. Use Q_0 to denote the value resulting from the passive strategy.

Let us consider asset returns over a specified period of time t.

Write

$$P_1(t) = P_1 \quad \text{and} \quad P_2(t) = P_2 \quad \text{so that} \quad Q_0 = \alpha_1 P_1 + \alpha_2 P_2 .$$

Our purpose is to consider the value of Q which results from constant rebalancing between asset sectors to maintain constant asset proportions α_1 and α_2 by market value at all times during the period. How does the result for Q compare with the value of Q_0 which results from the passive strategy?

3. Examples from the UK stock market

Returns on the UK equity market over the last 20 years provide a practical basis for looking at this question. The Financial Times - Actuaries All-share indices of market value and dividend yield provide the data base for calculating total returns over any required period. This equity index will represent asset sector S_1. Asset sector S_2 will be taken for our first case as cash not earning interest, so that $P_2 = 1$. Take equal sector proportions $\alpha_1 = \alpha_2 = 0.5$.

The following table shows the accumulated return P_1 on equities, the result $Q_0 = 1/2P_1 + 1/2$ of the passive strategy, and the result Q of the constant mix strategy. It has been assumed for this purpose that monthly rebalancing gives a close enough approximation to constant mix, but this will be a point for later scrutiny. Dealing expenses have been disregarded.

Table 1 - 1/2 UK equities and 1/2 cash without interest

Period	10 years 1972 to 1981	10 years 1982 to 1991	20 years 1972 to 1991
Equity return P_1	2.801	5.974	16.734
Passive return Q_0	1.901	3.487	8.867
Constant mix return Q	1.847	2.560	4.729

Thus over the 10 years 1982 to 1991 an investment in UK equities with gross dividends reinvested would have produced a total return of:

$$100 \times (5.974 - 1) = 497\% .$$

The 50:50 passive portfolio would have produced half this:

$$100 \times (3.487 - 1) = 248\% .$$

The 50:50 constant mix portfolio would have produced significantly less:

$$100 \times (2.560 - 1) = 156\% .$$

It can therefore be said with confidence that the passive and constant mix strategies can produce very different results over time. The distinction between the two approaches to setting an investment benchmark and measuring investment performance relative to a notional benchmark fund should not be blurred.

In the next example cash is replaced by fixed interest stocks (FT-Actuaries over 15 year gilt index) for the second asset class.

Table 2: 1/2 UK equities and 1/2 UK fixed interest stocks

Period	10 years 1972 to 1981	10 years 1982 to 1991	20 years 1972 to 1991
Equity return P_1	2.801	5.974	16.734
Gilt return P_2	1.985	3.945	7.832
Passive return Q_0	2.393	4.960	12.283
Constant mix return Q	2.538	5.074	12.878

The differences between Q_0 and Q are much less pronounced in Table 2, and it is interesting to note that the constant mix strategy out-performed the passive strategy, unlike the comparison in Table 1.

What are the factors which determine whether the constant mix strategy is better or worse than passivity? The UK equity market showed major growth over the last 20 years. The implication of Table 1, namely that it was better not to sell out of a rising market, seems unsurprising. But Table 2 shows the opposite effect even though equities generally out-performed fixed interest stocks over the period. Rebalancing can result in additional returns if there is a pattern of selling after a price rise in one sector and buying into a cheaper market in the other.

It is clear that the comparison between Q_0 and Q will depend upon relative price behaviour of the two assets, especially over short intervals at the timescale of the rebalancing operations. It is not immediately clear whether one strategy is likely to out-perform the other, and what magnitude of differences can be expected.

4. FORMULA FOR THE RETURN ON A CONSTANT MIX STRATEGY

A mathematical expression for Q can be derived on the assumption that the returns on each asset sector follow a geometric Gauss-Wiener process:

$$dP_i/P_i = \mu_i(t)dt + \sigma_i dz_i(t) \tag{1}$$

where dz_i is $N(0, dt)$.

It is shown in my JIA article that:

$$Q = P_1^{\alpha_1} P_2^{\alpha_2} e^{1/2kt}$$

where $k = \alpha_1\alpha_2(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)$
and ρ is the correlation between dz_i and dz_2.

This formula can be checked against the values of Q in Table 1 above. In that case $\sigma_2 = 0$ and so:

$$Q = \sqrt{P_1} e^{1/2kt} \quad \text{where} \quad k = 1/4\sigma_1^2 .$$

For each of the three periods, the value of Q is calculated using the observed standard deviation of annual equity returns over the identical period as the value of σ_1.

Table 3 - Check of formula for Q against Table 1

Period	10 years 1972 to 1981	10 years 1982 to 1991	20 years 1972 to 1991
Equity return P_1	2.801	5.974	16.734
P_1 standard deviation	0.281	0.187	0.238
k	0.0197	0.0087	0.0142
Observed Q	1.847	2.560	4.729
Calculated Q	1.847	2.553	4.713

The agreement between theory and observed results is quite good in this case, as in the next where the results of Table 2 are compared.

Table 4 - Check of formula for Q against Table 2

Period	10 years 1972 to 1981	10 years 1982 to 1991	20 years 1972 to 1991
P_2	1.985	3.945	7.832
P_2 standard deviation	0.165	0.107	0.139
Correlation	0.501	0.265	0.433
k	0.0149	0.0090	0.0118
Observed Q	2.538	5.074	12.878
Calculated Q	2.541	5.077	12.886

5. GENERALISATION

The two asset formula for Q can be extended to n assets according to the following:

[2] $$Q = (\Pi P_i^{\alpha_i})e^{1/2kt}$$

where

[3] $$k = \sum \alpha_i \sigma_i^2 - \sigma^2$$

[4] $$\sigma^2 = \sum \alpha_i \sigma_i \rho_{ij} \sigma_j \alpha_j$$

The result is derived in my JIA paper using stochastic calculus - a mathematical technique which seems to be necessary to deal with the chaotic behaviour of stock markets over arbitrarily short periods of time.

To illustrate the results of rebalancing a more complex asset mix, consider this four-asset portfolio, based on FT-Actuaries UK and World indices:

70% in UK equities
10% in European (ex UK) equities
10% in USA equities
10% in Japanese equities

Based on the indices, the cumulative returns, standard deviations and correlations over the four year period from September 1988 to September 1992 are as shown below.

Table 5 - Four sector portfolio

	UK	Europe	USA	Japan
Cumulative return P_i	1.549	1.420	1.632	0.660
Annual standard deviation of P_i	0.181	0.170	0.193	0.279
Correlations:				
UK	1.00	0.67	0.61	0.45
Europe		1.00	0.61	0.49
USA			1.00	0.27
Japan				1.00

In this case we find that:

$$(\Pi P_i^{\alpha_i})e^{1/2kt} = 1.4174 \times 1.01947 = 1.4450\,.$$

This is in very close agreement with the observed value $Q = 1.4451$, ie, + 44.5% over the four years (calculated using monthly rebalancing as a proxy for constant mix).

The close agreement observed in this and other examples gives support to the geometric Gauss-Wiener model of stock market behaviour.

It is an interesting feature that the calculation of return on a constant mix portfolio involves the statistic k. This item cannot be ignored: in the above case the value of k is just under 1/2% pa.

A practical conclusion from this might be that since the return on a constant mix benchmark fund is not a simple function of the individual sector returns, it is not sensible to specify a benchmark which is assumed to be constantly rebalanced to fixed asset proportions. Instead it is more straightforward to define a passive benchmark which is either not rebalanced or is rebalanced at some regular interval.

6. The rebalancing interval

This provokes questions about the relative performance of benchmarks which are rebalanced at different intervals. To illustrate the comparison with the above four asset portfolio over the four years to

September 1992, here are the results for alternative rebalancing periods:

Table 6: Rebalancing the four sector portfolio

Rebalancing period	Total return over 4 years
4 years (ie not rebalanced)	45.5%
1 year	43.9%
3 months	44.4%
1 month	44.5%

Thus, over the four year period, the differences between a constant mix and rebalanced portfolio would have been 1.0% for the 4 year hold, 0.6% for annual rebalancing, and about 0.1% for quarterly rebalancing. This shows that annual rebalancing cannot be regarded as equivalent to constant mix. Quarterly rebalancing might be an acceptable proxy for constant mix, but the difference could easily be more pronounced than in the above example.

Reverting to the simpler asset model of Table 1, where the difference between Q_0 and Q is more pronounced, these are the corresponding figures for the 10 years 1982 to 1991:

Table 7: Rebalancing the equity/cash portfolio

Rebalancing period	Total return over 10 years
10 years (ie not rebalanced)	248.7%
1 year	158.7%
3 months	158.3%
1 month	156.0%

7. General conclusions

These are the results for a particular ten year period, but using the asset model we can derive more general conclusions about the distribution of $Q_0 - Q$ for any chosen rebalancing period. In this example of 50% in equities, and 50% in cash with no interest, the expected value of $Q_0 - Q$ is:

$$1/2(\sqrt{\overline{P}} - 1)^2$$

where $\overline{P}$ is the expected value of P_1 for the chosen length of period.

This is a non-negative result which illustrates the more general conclusion which can be demonstrated that $E[Q_0] \geq E[Q]$ for any number of assets and in any proportions.

It can also be shown that the expected return is greater if the rebalancing period is longer. This feature is exemplified in Table 7 though not in Table 6. Using the theoretical model, formulae can also be developed for the standard deviation of $Q_0 - Q$ for alternative rebalancing periods. For a given market and asset mix this gives a way of deciding whether or not rebalancing at say monthly or quarterly intervals is equivalent for practical purposes to a true constant mix portfolio.

A general indication from this analysis is that the standard deviation $Q_0 - Q$ can be significant when the mean returns on the asset sectors are disparate, but becomes negligible when mean asset returns are the same for all sectors. The mean of $Q_0 - Q$ also vanishes when mean returns are all equal.

8. Footnote on Theoretical Development

Most of the theoretical results referred to above are developed in my JIA article which will be published in 1993, but I will be pleased to supply details on request. The main points which follow from equation [1] are as follows.

The value of P_i after any period of time has log-normal distribution. In particular after unit time:

$$\log P_i \quad \text{is} \quad N(\mu_i - 1/2\sigma_i^2, \sigma_i^2)\,.$$

Likewise Q is log normal. After unit time:

$$\log Q \quad \text{is} \quad N(\mu - 1/2\sigma_2, \sigma_2)$$

where $\mu = \sum \alpha_i \mu_i$ and σ_2 is as in equation [4].

Their expected values are $E[P_i] = \exp(\mu_i)$

$$\text{and} \qquad E[Q] = \exp(\mu)\,.$$

It follows that $E[Q] = \Pi E[P_i]^{\alpha_i}$

It can be shown that k as defined by [3] cannot be negative, and so it follows from [2] that:

$$Q \geq \Pi P_i^{\alpha_i}\,.$$

In general, inequality applies because k is non-zero except with very special conditions. The multiplying factor $e^{1/2kt}$ in the formula for Q is called for by the stochastic nature of the model, and does not appear in a non-stochastic analysis based on "force of returns" arguments.

The presence of the multiplying factor is logical for an efficient market. If the factor were not generally larger than 1, the constant mix return Q would be equal to the geometric mean of the sector returns P_i, and would generally be less than the arithmetic mean Q_0. Such a consistent inequality, irrespective of the actual outcomes of the asset returns, would point to a significant opportunity for arbitrage profits through asset switching, and to market inefficiency. Of course, it may be that some markets do show evidence of such inefficiency.

Finally, given the log normal distribution of Q as above, formula [3] for k can be verified in the following way. Taking time $t = 1$:

$$\begin{aligned}
E[\log \Pi P_i^{\alpha_i}] &= E\Big[\sum \alpha_i \log P_i\Big] \\
&= \sum \alpha_i E[\log P_i] \\
&= \sum \alpha_i(\mu_i - 1/2\sigma_i^2)\,.
\end{aligned}$$

Subtracting this from $E[\log Q] = \mu - 1/2\sigma^2$ gives the difference of $1/2k$

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