TOWARDS A QUANTITATIVE MATCHING PHILOSOPHY

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ABSTRACT

The concept of matching involves choosing assets and liabilities whose values are likely to behave in the same way. While either assets or liabilities may appear risky in isolation, the net position is then much more stable. Matching asset allocations have usually been determined in practice using trial and error techniques, often with the aid of simulated outcomes from a stochastic model. This paper proposes a general algebraic approach to matching which may be applied in the context of virtually any chosen stochastic model. The optimal asset allocation is determined explicitly.

1. INTRODUCTION

The idea of choosing assets and liabilities with common statistical properties underlies the actuarial approach to financial risk management. It has been described in its various guises as asset- liability matching, immunisation, hedging or actuarial funding. The theory has been widely applied in investment, pension fund work, life and general insurance. Sometimes "academic" matching objectives have appeared to conflict with overall management strategies involving valuation, investment policy and smoothing of results. This paper presents a unified approach in which all four issues can be held together.

The following notation will be used throughout the paper:

C_t outward cash flow.

D_t discount function.

H_t an investment index. For all indices it is assumed that investment income and capital gains are reinvested net of any applicable taxes.

I_t the specific investment index indicating the returns achieved on the asset portfolio actually held.

J_t the specific investment index indicating the returns achieved on the asset portfolio representing retained surplus.

L_t life table function describing the retention of surplus within the

fund.

S_t distributed surplus.

u initial capital

$\overset{\circ}{V}_t$ prospective reserve = unsmoothed reserve.

V_t reserve actually held = prospective reserve plus retained surplus = smoothed reserve.

W_t weight function.

These items will be defined more rigorously during the course of the paper. All these quantities are *stochastic processes* so that they vary with time t and are random variables for each t. Further notation is introduced in the appendix.

2. The Fundamental Framework

The concept of matching as originally postulated by Wise [1984] could be described as organising investments so that the cash flows arising from assets are approximately equal to the cash flows from the liabilities under a wide variety of possible economic scenarios. An ultimate surplus is defined by rolling up all differences between asset and liability cash flows at a given rate of interest up to some time horizon. This technique provides the useful simplification of eliminating the effect of reserves. The ultimate surplus should be small in a wide variety of scenarios. In fact, Wise defined the optimum match as that strategy which minimised the expected squared ultimate surplus under an appropriate stochastic asset liability model.

We shall broaden this concept in two ways. Firstly we shall permit the sale and purchase of assets at market value (in addition to the use of investment income) to provide asset cash flows and to allow reallocation between asset classes. Secondly, we will not cumulate surpluses to some ultimate horizon but instead we will minimise a weighted sum of the squared surpluses emerging at each point. The emergence of surplus will then depend not only on the assets and liabilities but also on the reserving technique used. The optimisation will therefore comment on reserving bases as well as investment strategy.

We work in discrete time with a finite horizon: $t = 0, 1, 2, \ldots, T$. We must reserve to meet net cash flows outwards of C_t at each time t (comprising benefits plus expenses less premium income). To this end, a reserve V_t is set up at each time t to meet those cash flows arising at times $t+1$ and after. Note in particular that according to our convention the reserve at time t does not include a credit for premiums paid at time

t. This reserve is invested in investments whose performance (income and capital gains rolled up net or gross of tax as appropriate) is described by an index I_t. The *distributed surplus* at time t is a balancing item defined by

$$\frac{I_t}{I_{t-1}} V_{t-1} = C_t + V_t + S_t \,.$$

This means that the excess of the fund with investment returns over the cash flow and required reserves is released as surplus. We will for the moment not comment on how the reserve V_t may be calculated. In particular we will not rule out the possibility that it may be adjusted to hide the retention of surplus or to smooth the emergence of profits. However we shall insist that no reserves are allowed after the time horizon so that $V_T = 0$.

We will define a matched situation as one where the surplus distributed at each stage is small. We therefore try to minimise the expected sum of squares:

$$\mathbf{E}\left(\sum_{r=0}^{T} W_r S_r^2\right)$$

where the W_i are positive weights.

It would be a fair objection to point out that the above optimisation says nothing about maximising profit, simply neutralising it. This is not such a distortion as it may at first seem, since increasing emerging profit would generally involve putting more capital in at the start. The concept of matching encapsulates the need to avoid tying up more capital than is necessary to back liabilities.

The quadratic expression used could also be criticised. It has been adopted primarily for convenience in that it produces a tractable problem which can be analyzed algebraically. It has the further advantage that the results produced are linear in the initial capital available and subsequent cash flows (provided the weights are not changed). This means for example that the matching exercise for a book of business would produce the same result as if each policy were analyzed separately and the results then aggregated.

In order to obtain neat results I have made a number of simplifications. I have ignored any regulatory or practical constraints on the required reserves and on the assets which may be held or sold short. I have also ignored transaction costs, the complications which arise from the ability to defer taxation on unrealised capital gains, and the delay in obtaining tax relief on unrelieved capital losses. The answer obtained

from a matching exercise may in practice have to be modified to take account of some or all of these factors.

3. A Simple Case - Deterministic Model

We now examine a simple world where all cash flows and investment returns are known in advance with certainty. In this case the optimum can be found in a fairly straightforward manner. First, define a prospective reserve by

$$\overset{\circ}{V}_t = I_t \sum_{r=t+1}^{T} \frac{C_r}{I_r}$$

which is the amount of cash required now to meet the liability cash flows. Then define L_t by

$$L_t = \sum_{r=t+1}^{T} I_r^2 W_r \,.$$

The optimum choices for the adjusted reserve V_t and the distributed surplus S_t are

$$V_t = \overset{\circ}{V}_t + \frac{L_t}{L_{-1}} \frac{I_t}{I_0} (u - \overset{\circ}{V}_0)$$

$$S_t = \frac{L_{t-1} - L_t I_t}{L_{-1} I_0} (u - \overset{\circ}{V}_0)$$

where u is the initial amount of capital. It is instructive to view L_t as a life function (analogous to l_x) in the sense that at time t as proportion L_t/L_{-1} of the initial surplus $u - \overset{\circ}{V}_0$ is still retained together with the investment return achieved on that surplus.

In the deterministic situation, the main effect which has been achieved is that the surplus is spread out over the duration of the policy. A large weight W_t at time t leads to a smaller distribution of surplus at that point.

Needless to say the stochastic model which we will now consider is more complex. The complexity arises not just from the randomness but also the information structure. At each time t more information becomes available on which future decisions may be based. It is in general not possible to specify optimal asset allocations explicitly more

than one step ahead, since the decision will depend on information not yet known.

4. Investment Strategy

One major change facilitated by the incorporation of stochastic effects is the possibility of several alternative investments. Asset allocation then becomes an issue; indeed one which is arguably well addressed by matching techniques. With this choice of assets also comes ambiguity in valuation. We shall describe some of the investment issues.

Investment strategy can be divided into two strands. Firstly there is a need to determine an appropriate strategy for those assets backing liabilities. Secondly, there will be a need to invest additional 'free reserves' in a way preferred by the beneficiaries of distributed surplus. In general, the investment of free reserves may have little to do with the liabilities and much more to do with preferences of shareholders, participating policyholders or pension scheme members. The actual practice followed, if rational, can be a useful guide to the preferences of these agents. We shall define J_t to be an index formed from cumulating actual achieved returns on the free reserves. Given an asset distribution and the performance of each asset sector, the index J_t can easily be calculated, for example in a simulation environment. We shall make no requirement that the asset distribution underlying J_t should be stable over time. Indeed it may respond to new information in any manner required by the office.

We shall use I_t as the index representing the return on all assets, including those used to back liabilities. While the index J_t may be determined solely by agents preferences, I_t will contain an element of matching. The determination of an optimal I_t is one of the aims of this paper.

5. Valuation Techniques

How should future cash flows be valued? The traditional actuarial techniques involve a discount function D_t so that the value at time s of a cash flow C_t at time t is given by

$$\text{value} = \frac{D_t}{D_s} C_t \,.$$

This is not satisfactory in a stochastic setting where C_t may not be known at time s. Instead, for a stochastic model, using $\mathcal{F}_s$ to denote information available at time s we might calculate

$$\text{value } = \frac{1}{D_s}\,\mathbf{E}\,(D_t C_t | \mathcal{F}_s)\,.$$

We will require consistency with the market value of assets. Let H_t be a portfolio index for an arbitrary investment strategy. Let s be the time now, and suppose the portfolio is to be realised at time t. The value of the future cash flow arising should be the market value of the portfolio now, so that

$$H_s = \frac{1}{D_s}\,\mathbf{E}\,(D_t H_t | \mathcal{F}_t)\,.$$

It may not seem obvious that such a process D_t exists. It is shown in the appendix that mild continuity conditions imply the existence of a suitable stochastic discount function. Note that in general, D_t will not be the reciprocal of a portfolio index, although this may happen by chance. The use of stochastic discount functions has become accepted in modern financial economics, particularly in option pricing theory. It has yet to find favour in orthodox actuarial circles.

The stochastic nature of D_t can be interpreted as follows:

i) Allowance for market inefficiencies. If there is an effect in the chosen stochastic model which the market is not perceived to reflect, then D_t can be made small when this effect is manifest. The effect is thus given low weight when performing valuations so that market values are reproduced. Conversely, we may use the weights to adjust a standard model to give low weights to scenarios which are believed to be over represented.

ii) Application of caution. It is possible to define D_t to be large in unpleasant scenarios, for example those involving large expense overruns. Relatively greater weight is then given to these scenarios in valuations, giving the same effect as a conservative valuation basis but within the framework of a realistic model.

iii) Adjusting rates of return for risk. The discount factor applied to a zero coupon bond would be $\mathbf{E}\,(D_t|\mathcal{F}_s)/D_s$. Now consider the identity

$$\frac{1}{D_s}\,\mathbf{E}\,(D_t C_t | \mathcal{F}_s) = \frac{\mathbf{E}\,(D_t|\mathcal{F}_s)}{D_s}\,\mathbf{E}\,(C_t|\mathcal{F}_s) + \frac{1}{D_s}\,\text{Cov}\,(D_t, C_t | \mathcal{F}_s)\,.$$

We can see that the discount factor applied differs from the bond factor by a covariance term. Cash flows negatively correlated with D_t will be

more deeply discounted. If we choose D_t to be a decreasing function of the performance of the market as a whole, the investor is then rewarded for taking on market risk by a higher rate of return. In this way, the same D_t can be applied to all asset classes, in contrast to the variety of rates of return assumed by actuaries when D_t is deterministic. This is a multiperiod generalisation of the familiar capital asset pricing model.

On this basis, a prospective valuation of future liabilities at time t would be

$$\overset{\circ}{V}_t = \frac{1}{D_t}\,\mathbf{E}\left(\sum_{r=t+1}^{T} D_r C_r | \mathcal{F}_t\right).$$

There are of course numerous other plausible valuation techniques. The one here would seem to be unusual in its forced agreement with market values. Judging by insurance company regulations, consistency between the valuation of assets and liabilities is not seen as important by the supervisory authorities. On the other hand, consistency with market values is usual for pension fund valuation only in the context of a wind-up. However, the approach used here where both items of consistency are essential turns out to be particularly useful when considering matching.

6. Smoothing

In many commercial situations there is a need to smooth performance results and the subsequent distribution of surplus. Smoothing is obtained in practice by not releasing all the surplus as it emerges, but retaining some of it. This may be explicit, or it may be hidden within the reserve calculation. Such smoothing, for example, is implicit in the net premium method of reserving for life assurance contracts.

A simple method of smoothing involves distributing at each stage a proportion of the accumulated surplus, the remainder being retained. We can describe this in terms of a life function L_t indicating the proportion of original surplus retained by time t. Thus at each time t, a proportion L_t/L_{t-1} (corresponding to p_{t-1} in a life table) of the surplus brought forward is retained, the remaining proportion $(L_{t-1} - L_t)/L_{t-1}$ (corresponding to q_{t-1}) being distributed.

At time t, the surplus brought forward based on a prospective valuation is

$$\text{surplus brought forward } = \frac{I_t}{I_{t-1}} V_{t-1} - C_t - \overset{\circ}{V}_t.$$

The surplus distributed will be

$$S_t = \frac{L_{t-1} - L_t}{L_{t-1}} \left(\frac{I_t}{I_{t-1}} V_{t-1} - C_t - \mathring{V}_t \right)$$

so the smoothed reserve carried forward to justify this surplus will be

$$V_t = \frac{I_t}{I_{t-1}} V_{t-1} - C_t - S_t = \frac{L_t}{L_{t-1}} \left(\frac{I_t}{I_{t-1}} V_{t-1} - C_t \right) + \frac{L_{t-1} - L_t}{L_{t-1}} \mathring{V}_t .$$

This is, of course, consistent with the deterministic case discussed earlier. The smoothed reserve is a weighted average of the value of available assets and a prospective valuation of the liabilities. This corresponds to the practice in pension funds of spreading a surplus over several future years. A further contribution to the effect is achieved by the use of 'assessed value' in place of the market value of assets. In life assurance the smoothing effect may be achieved in a somewhat more clandestine manner by weakening the reserving basis when times are harsh and strengthening it following periods of high investment returns. In general, we may allow L_t to be stochastic, so that for example, distribution of surplus may be held back on account of anticipated lean times ahead.

7. The Common Link

Given a weight process W_t it can be shown (see appendix) that there exists a unique discount function D_t, portfolio index J_t and decreasing process L_t with $L_T = 0$ (up to multiplication by a constant) such that

$$W_t = \frac{D_t}{J_t(L_{t-1} - L_t)} .$$

Furthermore, with the interpretation given to these processes in the preceding sections, the reserving procedures outlined are optimal. (There is a further technical caveat that all these processes must be adapted, that is, they can be calculated in real time without having to anticipate future uncertain events).

The optimal procedure for matching as outlined so far appears reasonable, conforming both to actuarial intuition and practice. This lends weight to the suggestion that the same optimisation might give sensible answers for the allocation of investments between different asset categories.

In practice the weights W_t may be hard to determine, as they encapsulate the subjective preferences of the various beneficiaries of any released surplus. If W_t are known, it is still a difficult exercise to decompose them in the fashion illustrated above. Even if the decomposition has been obtained, to evaluate surplus it is necessary to calculate unsmoothed reserves $\overset{\circ}{V}_t$ which are defined in terms of conditional expectation. These are notoriously hard to calculate numerically except in certain special algebraically tractable cases. It would appear that so far we have investigated a problem which is difficult to formulate and virtually impossible to solve in practice.

The trick that is required is to formulate the problem in terms of prescribed processes J_t, D_t and L_t arising from assumed optimal investment, valuation and smoothing policies. The corresponding weights W_t representing the underlying preferences may then be inferred from the known process J_t, D_t and L_t.

8. Determining the Optimal Portfolio

Suppose now that there are finitely many investment vehicles at time t. Let F_{t+1} be the vector of return factors for these vehicles. Define a vector A_t and a matrix B_t by

$$A_t = \sum_{r=t+1}^{T} \mathbf{E}\left[\frac{D_r C_r}{J_{t+1}} F_{t+1} | \mathcal{F}_t\right]$$

$$B_t = \mathbf{E}\left[\frac{D_{t+1}}{J_{t+1}} F_{t+1} F_{t+1}^T | \mathcal{F}_t\right]$$

Then the optimal asset allocation vector Y_t for the reserve corresponding to the prospective value of future liabilities is given by

$$Y_t = B_t^{-1} A_t \,.$$

Any retained surplus above this allocation is then invested in the portfolio J_t. Notice in particular that only D_t and J_t are used in this calculation; the surplus distribution strategy described by L_t has no effect on the optimal portfolio.

9. The Role of the Stochastic Model

Let the stochastic model be described by a probability $\mathbf{P}$. Another stochastic model is proposed, describing the same economic quantities but with different probability $\tilde{\mathbf{P}}$. We shall suppose however that the possibilities are the same under the two models, so that any event with positive probability under $\mathbf{P}$ also has positive probability under $\tilde{\mathbf{P}}$ and vice versa.

The discount function D_t will in general no longer reproduce market values under $\tilde{\mathbf{P}}$. However, it can be shown (appendix) that there is a unique adjusted discount function $\tilde{D}_t$ which will reproduce under $\tilde{\mathbf{P}}$ the valuations carried out under $\mathbf{P}$ using D_t. Retaining the same processes J_t and L_t we have the surprising result that the matching portfolio under $\tilde{\mathbf{P}}$ is exactly the same as the matching portfolio under $\mathbf{P}$. The procedure outlined in this paper is thus perfectly robust with respect to changing the probability model. If two models are similar, even if the possibilities are not identical it is to be anticipated that the best match will not be highly sensitive to the stochastic model chosen.

This is in contrast to many optimisation techniques used with stochastic models. A frequent observation is that the use of a stochastic model can provide a substantial apparent improvement in performance. On closer inspection, it often turns out that the optimum proposed does not exploit niches in the market, but rather exploits flaws in the model. The procedure thus homes in on areas where the model is least appropriate. This would be expected to occur in the current situation if the W_t were proposed directly. However, under the procedure outlined here the supposedly optimal D_t, J_t and L_t would be unlikely to exploit flaws in any particular stochastic model. If in fact the model does contain exploitable flaws not exploited by the supposed optima, it must follow that the weight given to the anomalous scenarios is very small. In other words, the process supposed here is naturally self correcting, paying little attention to any flaws in any proposed model.

It is sometimes even convenient to choose a stochastic model which is unrealistic on account of its simplicity, confident in the knowledge that it would give the same answers as a more accurate but complex model. The best example of this is the so-called risk-neutral transformation used in option pricing, where all assets have the same expected return over any interval with no reward for risk. Such models have an appeal because they are easier to construct than models giving a return accurately reflecting risk.

APPENDIX

In order to simplify matters we will assume $\mathcal{F}_0$ is trivial, so that all variables known at time 0 are treated as constants. We then define the Hilbert space $\mathbf{H}$ of processes $X_t : t = 0, 1, 2, \ldots, T$ such that

$$\mathbf{E} \sum_{t=0}^{T} W_t X_t^2 < \infty .$$

For such X and Y we define the inner product $\langle X, Y \rangle$ by

$$\langle X, Y \rangle = \mathbf{E} \sum_{t=0}^{T} W_t X_t Y_t .$$

Let H be an investment index, with $\mathbf{E}(W_t H_t^2) < \infty$ for each t. (From now on we shall tacitly exclude investment indices not satisfying this condition). Let F be a random event whose outcome is known at time r. Let $s > r$. An *elementary investment process* X_t is defined to be a process of the form

$$X_t = \begin{cases} -H_r & t = r\,, F \quad \text{occurs} \\ H_s & t = s\,, F \quad \text{occurs} \\ 0 & \text{otherwise}. \end{cases}$$

This describes the cash flows from investing in the index at time r if the event F occurs, the investment being realised at time s.

We define $\mathbf{M}_0$ to be the vector space spanned by elementary investment processes. Elements of this space are cash flow sequences which can be obtained with no initial capital by investing and realising portfolios contingent on observed events. $\mathbf{M}_0$ is a subspace of $\mathbf{H}$.

The first Lemma will characterise the orthogonal complement of $\mathbf{M}_0$.

LEMMA 1. – *Suppose $Y \in \mathbf{H}$. Then the following are equivalent:*

i) $Y \in \mathbf{M}_0^{\perp}$

ii) $W_r Y_r H_r = \mathbf{E}(W_s Y_s H_s | \mathcal{F}_r)$ *for times $r < s$ and any portfolio index H.*

Condition ii) is equivalent to asserting that $\{W_t Y_t H_t\}$ is a martingale, or that $W_t Y_t$ is a discount function. The proof is immediate from the Kolmogorov definition of conditional expectation.

We shall also want to consider possible cash flow sequences when there is some initial investment. For an initial investment u we therefore define the set $\mathbf{M}_u$ as

$$\mathbf{M}_u = \mathbf{M}_0 + \{u, 0, 0, 0, \ldots\}.$$

So that $\mathbf{M}_u$ consists of cash flows in $\mathbf{M}_0$ plus an initial injection of u. Of course it is quite possible that this quantity is entirely invested so that no cash flow actually occurs at time 0. We write $\mathbf{M}$ for the union of all the $\mathbf{M}_u$.

An important element which we have not yet included is the market valuation of a portfolio. The following lemma provides the key:

LEMMA 2. – *Suppose $X \in \mathbf{M}$. The market value P_t of the portfolio immediately following the cash flows at time t satisfies*

$$W_t Y_t P_t = \sum_{r=t+1}^{T} \mathbf{E}\left(W_r Y_r X_r | \mathcal{F}_t\right)$$

for any $Y \in \mathbf{M}_0^{\perp}$.

This follows directly from Lemma 1 if X is an elementary investment process, and follows for other $X \in \mathbf{M}_0$ by linearity. The extension to $\mathbf{M}_u$ is trivial.

In order to obtain important existence results from Hilbert space theory, we will need to complete the spaces under consideration. We then extend results to completions by continuity arguments. This of course assumes that the underlying functions determining market values form cash flows are continuous.

We now define $\overline{\mathbf{M}}_u$ u to be the topological closure of $\mathbf{M}_u$. Similarly, $\overline{\mathbf{M}}$ is the closure of $\mathbf{M}$. This adds to the list of possible strategies the limits of all such strategies. We shall look for the optimum among this class, understanding that if the optimum is a limit rather than a feasible strategy then it is possible to get arbitrarily close to the optimum. The last remaining limiting definition is that of a limiting portfolio: We say that $H \in \mathbf{H}$ is a *limiting portfolio index* if for each $Y \in \mathbf{M}_0^{\perp}$ the process $\{W_t Y_t H_t\}$ is a martingale. This augments the set of portfolio indices by allowing linear combinations and limits of such combinations.

We are now moving towards the crucial decomposition result. We define a *life process* L_t to be an adapted process defined on $t = \{-1, 0, 1, 2, \ldots, T\}$ which is positive on $t < T$, decreasing and $L_T = 0$. We say that $X \in \overline{\mathbf{M}}$ is a *depreciating process* if $X_t P_t > 0$ for each $t < T$, P_t denoting the market value as in Lemma 2. We can now present the following factorisation result:

LEMMA 3. – *The following are equivalent:*

i) X_t *is a depreciating process*

ii) X_t *factorises as* $J_t(L_{t-1} - L_t)$ *where* J *is a limiting portfolio index and* L *is a life process*

iii) X_t *factorises uniquely in the form ii), up to multiplication of the factors by a constant scalar.*

Proof: i) $\Longrightarrow$ ii): Pick $L_{-1} > 0$ and define L_t and J_t by

$$L_t = \frac{P_t}{X_t + P_t} L_{t-1}, \quad J_t = \frac{X_t}{L_{t-1} - L_t}.$$

It is plain that L_t is a life function. Using Lemma 2, we have for any $Y \in \mathbf{M}_0^{\perp}$,

$$W_t Y_t P_t = \mathbf{E}\left(W_{t+1} Y_{t+1} (X_{t+1} + P_{t+1}) | \mathcal{F}_t\right).$$

Now from the definitions of L and J we have

$$L_t J_t = P_t, L_t J_{t+1} = X_{t+1} + P_{t+1}$$

so on substitution and cancellation by L_t we have

$$W_t Y_t J_t = \mathbf{E}\left(W_{t+1} Y_{t+1} J_{t+1} | \mathcal{F}_t\right)$$

which is the required martingale property and so J is a limiting portfolio index.

Proof: ii) $\Longrightarrow$ iii) and ii) $\Longrightarrow$ i) We show that L_t must necessarily satisfy the inductive relation in the first part of the proof.

For, assuming the form of ii) for X, by Lemma 2 we have

$$\begin{aligned} W_t Y_t P_t &= \sum_{r=t+1}^{T} \mathbf{E}\left(W_r Y_r J_r (L_{r-1} - L_r) | \mathcal{F}_t\right) \\ &= \sum_{r=t}^{T-1} \mathbf{E}\left(W_{r+1} Y_{r+1} J_{r+1} L_r | \mathcal{F}_t\right) - \sum_{r=t+1}^{T-1} \mathbf{E}\left(W_r Y_r J_r L_r | \mathcal{F}_t\right) \\ &= \sum_{r=t}^{T-1} \mathbf{E}\left(W_r Y_r J_r L_r | \mathcal{F}_t\right) - \sum_{r=t+1}^{T-1} \mathbf{E}\left(W_r Y_r J_r L_r | \mathcal{F}_t\right) \\ &= W_t Y_t J_t L_t \end{aligned}$$

and the results follow.

We wish to determine the factorisation:

$$W_t = \frac{D_t}{J_t(L_{t-1} - L_t)}$$

where D is a discount function, J is a limiting portfolio index and L is a life function. We can rewrite the factorisation as

$$\frac{D_t}{W_t} = J_{t-1}(L_{t-1} - L_t)\,.$$

The left hand side is then in $\mathbf{M}_0^{\perp}$ while the right hand side is a depreciating process by Lemma 3 so is in $\mathbf{M}$. This in fact determines the decomposition uniquely up to scalar multiples, by virtue of the following:

LEMMA 4. – *There is a unique element $N \in \mathbf{H}$ such that $\overline{\mathbf{M}}_u \cap \mathbf{M}_0^{\perp} = \{uN\}$. Furthermore, in the absence of arbitrage this element is non-zero.*

The existence and uniqueness are standard linear analysis. To show that N is not zero, we have to show that $\{u, 0, 0, \ldots\} \notin \overline{\mathbf{M}}_0$. But if it did, this would provide a cash flow u with no further liabilities, which is an arbitrage opportunity.

We think now of P_t as the market value of the portfolio corresponding to the limiting cash flow vector N. We apply Lemma 2 with $X = Y = N$ to give

$$W_t N_t P_t = \sum_{r=t+1}^{T} \mathbf{E}\left(W_r E_r^2 | \mathcal{F}_t\right).$$

From this we may deduce that the left hand side is positive, and that N_t and P_t have the same sign. Thus N is a depreciating process and so the decomposition exists.

Having proved the existence of the decomposition, it now remains to show that the suggested procedures are optimal. We can write $\mathbf{H}$ as a direct sum of orthogonal spaces: $\mathbf{H} = \overline{\mathbf{M}}_0 \oplus [N] \oplus \overline{\mathbf{M}}^{\perp}$. The matching problem can be rephrased as finding the element $\hat{C}$ of $\overline{\mathbf{M}}_u$ closest to C, the vector of liability cash flows. To find the optimum, we first decompose C uniquely as

$$C = C^M + \overset{\circ}{V}_0 N + C^R$$

with $C^M \in \overline{\mathbf{M}}_0$ and $C^R \in \overline{\mathbf{M}}^{\perp}$. The coefficient of N is obtained as $\langle C, N\rangle / \|N\|$, as is readily confirmed. We can also decompose $\hat{C}$ is this fashion. However, since $\hat{C} \in \overline{\mathbf{M}}_u$ we have $\hat{C} - uN \in \overline{\mathbf{M}}_0$ and so

$$\hat{C} = \hat{C}^M + uN\,.$$

We wish to minimise the squared distance between C and $\hat{C}$, which by Pythagoras is

$$\|C - \hat{C}\|^2 = \|C^M - \hat{C}^M\|^2 + (u - \overset{\circ}{V}_0)^2 \|N\| + \|C^R\|^2\,.$$

This is clearly minimised by setting $\hat{C}^M = C^M$. In this case, the vector $C - \hat{C}$ of differences (which are equal to minus the distributed surplus) will be in $\mathbf{M}_0^{\perp}$. It is this result which we will use to prove the optimality of the recommended procedures. Firstly, we turn to surplus distribution, where the following Lemma is crucial:

LEMMA 5. – *Let $Y \in \mathbf{M}_0^{\perp}$ and let N be as in Lemma 4. Then*

$$Y_t = \frac{L_{t-1} - L_t}{L_{t-1}} \frac{1}{W_t N_t} \sum_{r=t}^{T} \mathbf{E}\left(W_r N_r Y_r | \mathcal{F}_t\right).$$

Proof: Use Lemma 2 with $X = N$. Then add $W_t N_t Y_t$ to each side. The result then follows.

The significance of this is that the difference between the amount of cash available, i.e. $V_{t-1} I_t / I_{t-1}$ and the amount needed for liabilities,

i.e. $C_t + \overset{\circ}{V}_t$ is precisely the present value of future distributed surpluses, which in algebraic form is

$$\text{value of distributed surpluses} = \frac{1}{D_t}\sum_{r=t}^{T} \mathbf{E}\left(D_r S_r | \mathcal{F}_t\right) =$$
$$= \frac{1}{W_t N_t}\sum_{r=t}^{T} \mathbf{E}\left(W_r N_r S_r | \mathcal{F}_t\right).$$

Since $S \in \mathbf{M}_0^{\perp}$, Lemma 5 gives the proportion of surplus which is distributed, which fits with our claimed smoothing policy. Finally, it remains to show the optimality of the assumed investment process. The key for this is the martingale property.

LEMMA 6. – *If H is a limiting portfolio index and $Y \in \mathbf{M}_0^{\perp}$ then we have*

$$\mathbf{E}\left[\frac{D_{t+1}H_{t+1}}{J_{t+1}}\frac{L_t Y_{t+1}}{L_t - L_{t+1}}|\mathcal{F}_t\right] = \mathbf{E}\left[\frac{D_{t+1}H_{t+1}}{J_{t+1}}\frac{J_{t+1}}{J_t}\frac{L_t Y_t}{L_{t-1} - L_t}\right]$$

Proof: Recalling that $N \in \mathbf{M}_0^{\perp}$) we have

$$\mathbf{E}\left(N_t W_{t+1} H_{t+1} Y_{t+1} | \mathcal{F}_t\right) = N_t W_t H_t Y_t = \mathbf{E}\left(N_{t+1} W_{t+1} H_{t+1} Y_t | \mathcal{F}_t\right).$$

Decomposing N_t into portfolio and life process, and rewriting W_t in its decomposed form, we find the Lemma after some rearrangement.

In order to apply this, we shall need to take into account what we already know about the way in which the distributed surplus is calculated. We shall assume that the retained surplus is invested in the portfolio J (without loss of generality) and suppose that the other assets are invested in some special portfolio $\overset{\circ}{I}$. We then express S_t in terms of the surplus retained at time t, and S_{t+1} in terms of the surplus carried forwards from time t to give

$$S_t = \frac{L_{t-1} - L_t}{L_t}(V_t - \overset{\circ}{V}_t)$$
$$S_{t+1} = \frac{L_t - L_{t+1}}{L_t}\left[\overset{\circ}{V}_t \frac{\overset{\circ}{I}_{t+1}}{\overset{\circ}{I}_t} + (V_t - \overset{\circ}{V}_t)\frac{J_{t+1}}{J_t} - \overset{\circ}{V}_{t+1} - \overset{\circ}{C}_{t+1}\right].$$

Applying lemma 6 with $Y = S$, substantial cancellations occur, and the remaining result is

$$\mathbf{E}\left[\frac{D_{t+1}H_{t+1}}{J_{t+1}}\mathring{V}_t\frac{\mathring{I}_{t+1}}{\mathring{I}_t}|\mathcal{F}_t\right] = \mathbf{E}\left[\frac{D_{t+1}H_{t+1}}{J_{t+1}}(\mathring{V}_{t+1} + C_{t+1})|\mathcal{F}_t\right].$$

Where there are finitely many investments between t and $t+1$ this readily converts to the matrix formulation of optimality as given.

Finally, we wish to confirm that the results are robust under a change of model. Suppose that we have two alternative models $\mathbf{P}$ and $\tilde{\mathbf{P}}$ with the same null sets. We may then define the Radon-Nikodym derivative

$$\rho_t = \frac{d\mathbf{P}}{d\tilde{\mathbf{P}}}|\mathcal{F}_t$$

so that for any random variable Z known at time t conditional expectations for $s \leq t$ under the two models are related by

$$\tilde{\mathbf{E}}(Z|\mathcal{F}_s) = \frac{1}{\rho_s}\mathbf{E}(\rho_t Z|\mathcal{F}_s).$$

Suppose that corresponding to the discount function D_t for $\mathbf{P}$ there is a discount function $\tilde{D}_t$ for $\tilde{\mathbf{P}}$ we would have on equating valuations

$$\frac{1}{\tilde{D}_s}\tilde{\mathbf{E}}(\tilde{D}_t Z|\mathcal{F}_s) = \frac{1}{D_s}\mathbf{E}(D_t Z|\mathcal{F}_s).$$

Using the derivative ρ_t to rewrite the conditional expectation $\tilde{\mathbf{E}}$ gives

$$\frac{\mathbf{E}(\rho_t\tilde{D}_t Z|\mathcal{F}_s)}{\rho_s\tilde{D}_s} = \frac{\mathbf{E}(D_t Z|\mathcal{F}_s)}{D_s}.$$

If this holds for each Z it follows that $D_t = \rho_t\tilde{D}_t$ (up to multiplication by a constant). The transformed weights $\tilde{W}_t$ satisfy $W_t = \rho_t\tilde{W}_t$. The two Hilbert spaces under the two models are then isometric, so in particular the optimal strategies are the same.

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