THE FINANCIAL INTUITION BEHIND A LOGNORMAL INTEREST RATE DIFFUSION PROCESS

THOMAS J. MERFELD

1. INTRODUCTION

Life insurance companies can enhance the rigour of their financial management by using an option adjusted spread model. Companies writing annuity or interest sensitive life products are probably heavy writers of interest rate driven financial options. Such products typically have guaranteed rates, which have the characteristics of interest rate floors, and rate sensitive lapses, which have some characteristics of interest rate puts. On the asset side, companies acquiring corporate bonds and many types of mortgage-related securities are also heavy writers of financial options. Pricing tools capable of reflecting the cost of embedded options must be applied on both sides of the balance sheet in order to expect rigorous financial decisions.

Adequate access to such models includes intellectual access by management of the company. Prudent use of the power of one of these models requires an adequate feel for the inner workings of the model itself. When is it appropriate to use the model? What is reasonable input? When do we change our inputs? What is believable output? Can we apply the technology to cash flow structures that have never been subjected to option-adujsted pricing assumptions? Although an adequate commercially developed code may be available, appropriate use of it is often precluded by its "black box" nature - nobody in management has been able to pierce the veil of full utility. In addition, linkage with cash flow routines may not be feasible with commercial models.

The interest rate process is a primary component in an option-

I would like to acknowledge Mr. Dale R. Oldham, FSA for helpful discussions and Mr. Eric C. Rouchka for programming assistance.

adjusted spread model. Interest sensitive cash flow projection begins with the projection of an interest rate path drawn from a specified distribution of rates. The characterization of the distribution of interest rates is a highly technical matter. However, each step of the characterization is subject to study in a clear, but largely non-technical manner.

The purpose of this paper is to present the financial intuition supporting a lognormal interest rate diffusion process. The model described is fundamentally of the Black-Derman-Toy[1] class, which is, in most ways, a superset of many good lognormal models. Most publications do not address the issue of gaining a feel for the notion of an interest rate process. Academic authors often focus on closed-form solutions to option pricing (which rarely apply to options embedded in path-dependent cash flows) or are highly technical. Professional authors often desire to retain much of the proprietary nature of their process. This paper will proceed stepwise through the development of a lognormal interest rate process in a practical and intuitive manner. Each step has the primary objective of transmitting intuition rather than of demonstrating mathematical elegance.

Following a simple example of the effect of interest rate volatility, the paper has four sections. Section One discusses issues related to risk-free rate estimation. Section Two develops the elements needed for projecting short-term rates under arbitrage-free conditions. This section discusses, for purposes of illustration, the use of drift term estimates. Although their use in projections introduces errors, they are instructive to consider. The section ends with a discussion of fitting the distribution to the data without errors. Section Three describes projection of such rates. Section Four discusses aspects of the diffusion process.

2. A SIMPLE EXAMPLE AND OVERVIEW

Consider a high quality corporate bond selling at a price of 95 and currently callable at par (assume, for the moment, no cost to the issuing company of calling the bond, that the market is pricing no expectations of rate movements, that the spread to Treasury of the bond does not change over its remaining life and that the yield curve is flat).

[1] F. Black, E. Derman, W. Toy, "A One-factor Model of Interest Rates and Its Application to Treasury Bond Options" *Financial Analysis Journal*, January-February 1990, pp. 33-39.

If interest rates experience absolutely no perturbations until the maturity of the bond, then the market price of the bond will follow smoothly its level yield accretion schedule until maturity and the call option cannot be used by the issuer. That is, the market was not pricing any interest rate fluctuations, there were no such fluctuations and the value of the option was zero.

Now assume that rates are agitated in a random fashion only slightly and that the market has priced the bond accordingly. It is now conceivable, although unlikely, that at some point prior to maturity the market price will wander above the strike price and the issuing firm will enjoy the benefit of calling it at the cost of the holder of the bond. Although the central price tendency remained at the smooth accretion schedule absent the call, the presence of the call truncated the price distribution at all times at the level of the strike price. So the middle of the distribution indicated a market price, in the presence of the option, below the market price absent the option. Alternatively stated, a lower discounting spread to Treasury exists for the bond that would, once again, equate the central tendency of the price distribution at each period until maturity and the market path toward par. The amount by which the discounting spread must be lowered to achieve such price tracking is the cost, in basis points, of the call feature of the bond under the assumption of interest rate perturbation.

Now assume increasing levels of interest rate volatility. At each increment of volatility, the call provision of the bond exacts additional cost from the holder. Stated otherwise, at each increment of volatility, the discounting spread to Treasury, adjusted for the cost of the option, is lower.

In pricing a financially-contingent claim or obligation under simple one-factor lognormal assumptions, a full option adjusted spread model: 1) estimates the zero coupon rates associated with each future term, 2) estimates the short-term forward rates associated with each future term, 3) estimates the universe of arbitrage-free forward rates associated with each term given a term structure of volatility, 4) projects a random path of short-term forward rates over the term of the claim or obligation, 5) generates a cash flow vector as a function of the rate path projected and the provisions of the claim or obligation, 6) discounts the cash back to the present on a recursive basis using the forward rates associated with each term, which are actually estimates of one-period zero coupon rates at each term, plus a given spread as month-to-month discount rates, 7) repeats this path projection, averaging each result with earlier projections, until requisite price convergence has been achieved.

The remainder of this paper focuses on the third and fourth steps, with some discussion of steps one and two.

3. Estimating risk-free prices and associated rates

Risk-free interest rates for use in the model must be interpolated and transformed based on current market prices. The interest rate process projects risk-free rates of interest under arbitrage-free conditions. In doing so, it requires the fundamental rates of interest, that is, the pure time value of money, for each period of the projection. We assume a month-to-month projection. In accomplishing this, we adjust market prices in four ways.

First, we interpolate between on-the-run Treasury rates. Other methodologies use all Treasury issues as input. An advantage of the latter approach is that more rates are used and the interpolation distances are not so great. We believe, however, that the risk-free assumption is compromised in this approach because some liquidity risk is present relative to the on-the-run rates. Instead, we choose simply to interpolate between the 10 on-the-run rates (i.e. including the twenty year rate) using non-linear regression smoothing techniques. With this approach, we have a smooth, well-fitting curve of par rates.

Second, we transform this vector of par rates into vector of zero-coupon rates for every assumed future coupon payment period. The par rate is that rate of interest assumed to be borne by an instrument trading at par at a given term. It is assumed to pay interest at every future coupon date. Consequently, the price and effective term information it provides is confused by early coupon payments. By stepping out in six month increments, all pre-term cash flows can be adjusted by the zero-coupon rate associated with the terms of such cash flows. We then solve for the zero-coupon rate at the term itself. That is, given that we know all pre-term and term cash flows, and given that, relative to each term, we have already solved all pre-term zero coupon rates, we can solve the zero coupon rate at the term. Specifically, it is that rate that, after giving effect to the pre-term cash flows discounted by their respective zero coupon rates, discounts the term cash flow such that the present value of the entire vector is equal to par. With this approach we have a smooth vector of zero coupon rates spaced at six month intervals.

Third, we then apply further smoothing and interpolating procedures to estimate the zero coupon rates for months between coupon payment dates. We then have a smooth, well-fitting, monthly series of zero coupon rates.

Fourth, we then transform the zero coupon rates to pure month-to-month rates of interest. Consider that the zero coupon rate at any term is the rate of interest prevailing over the entire period to term. The rate at any term, however, can be further conceived as the geometric average of all monthly rates to term. This monthly interest rate vector is the pure time value of money for any given period. We can call this rate the forward rate of interest at such term. For example, if the month one zero coupon rate $r_1 = 8.0\%$ and $r_2 = 8.05\%$ then the month one forward rate is $f_1 = 8.0\%$. Now, considering p_1 the current price of \$ 1 received in one month and noting that Treasury conventions use semi-annual coupon payments, since:

$$p_1 = \left(1 + \frac{r_1}{2}\right)^{-1/6} = .99348453\,,$$

and:

$$p_2 = \left(1 + \frac{r_2}{2}\right)^{-2/6} = .98693244\,,$$

and:

$$f_1 = \left[\left(\frac{p_0}{p_1}\right)^6 - 1\right]2 = 8.0\%$$

where:

$$p_0 = 1.0\,,$$

then:

$$f_2 = \left[\left(\frac{p_1}{p_2}\right)^6 - 1\right]2 = 8.100012\,.$$

In general:

$$f_t = \left[\left(\frac{p_{t-1}}{p_t}\right)^6 - 1\right]2\,;$$

more generally yet:

$$f_{n,t} = \left[\left(\frac{p_{t-n}}{p_t}\right)^{6/n} - 1\right]2 \qquad [1]$$

where: $n =$ the number of months forward the rate incorporates zero coupon prices, since a forward rate can be of any term, provided, however, the zero coupon prices are extant at t and $t - n$.

Also:

[2] $$p_{n,t} = \left(1 + \frac{f_{n,t}}{2}\right)^{-n/6}$$

That is, since the zero coupon rate at month two was higher than the zero coupon rate at month one, then the forward rate at month two had to increase by a larger amount to drag the full zero coupon rate up. This process is then repeated for the entire vector of zero coupon rates. We now have a smooth schedule of rates comprising the pure time cost of money on a month-to-month basis in the future. These are the building blocks of the process.

We can also describe a schedule of interest rate volatility, albeit with less market precision. We are interested in what is, in effect, the forward rates of volatility for this process. We can look at various well-functioning option markets to find the market's assessment of short term interest rate volatility. However, no such longer-term market exists. Many practitioners simply use an historical average to estimate long term volatility. We look to the market for short term volatility and then phase in our view about longer-term volatility. Having done this, we estimate the forward rate schedule of volatility.

We now have the elemental building blocks to estimate the parameters of a lognormal interest rate process.

4. Fitting the Distribution

4.1. A useful rate parameter approximation - estimating drift terms

A provisional and instructive way to conceive of interest rate diffusion is to estimate and use parameters to describe the drift, that is, the expected dynamic of rates from one period to the next. The procedure provides an approximation that is accurate enough for some practitioners. We can better investigate the elements of a precise diffusion process by examining the problems that arise in using the drift term method.

Immediately following is the development of a simple concept of drift. In a later section a richer notion is presented.

Consider again, as a limit case, a state in which the interest rate options market priced interest rate volatility at zero percent. The market would be contending that future interest rates were known with certainty and, in fact, it would be pricing no possibility of a change in rates. The options market would be priced as though the forward curve would be realized precisely. In such case we could characterize the forward rate at any month as a function of the forward rate for the previous month and the expected percent change in the rate as implied by Treasury prices.

That is:

[3] $$f_{1,t} = f_{1,t-1}e^{d_t}$$

where:

[4] $$d_t = \ln\left(\frac{f_{1,t}}{f_{1,t-1}}\right),$$

a measure of the drift of the central tendency of the short term forward rates.

Referring to the previous example:

$$f_{1,2} = f_{1,1}e^{d_2}\,.$$

Since:

$$d_2 = \ln\left(\frac{.08100012}{.08}\right) = .01242400\,,$$

$$f_{1,2} = .08e^{01242400} = 8.100012\%\,.$$

In fact, the entire term structure could be defined by $f_{1,1}$ and the vector of drift terms. Note that, in any case, $f_{1,1}$ is known with certainty.

4.2. One basic problem

The expression could also be placed in the context of the expected rate under conditions of uncertainty:

[5]
$$f_{1,t}^{up} = f_{1,t-1}e^{(\sigma_t\sqrt{1/12})+d_t}$$
$$f_{1,t}^{dn} = f_{1,t-1}e^{(-\sigma_t\sqrt{1/12})+d_t}$$

where: σ_t = annual interest rate volatility at time t.

Since σ_t is assumed to be zero, both up and down nodes of the lattice produce a rate of 8.100012 percent, the expected rate for $f_{1,2}$.

Consider now the effect of using a measure of volatility greater than zero while retaining the same drift terms. Note, in the context of a binomial rate process, that at $t = 2$, only two possible rates exist.

$$f_{1,2}^{up} = .08e^{(.09\sqrt{1/12})+.01242400} = 8.313214\%$$

$$f_{1,2}^{dn} = .08e^{(-.09\sqrt{1/12})+.01242400} = 7.892278\%$$

These are apparently reasonable rates, except when they are subject to a test to determine whether they combine accurately to price the yield curve. Consider the average one month forward prices they imply:

$$p_{1,2}^{up} = \left(1 + \frac{.08313214}{2}\right)^{-1/6} = .99323541$$

$$p_{1,2}^{dn} = \left(1 + \frac{.07892278}{2}\right)^{-1/6} = .99357031$$

for an expected price of .99340286; recall that, since we had only two possible rates for $f_{1,2}$, each of these prices was assigned a weight of .5.

Note that this expected price indicates a rate for $f_{1,2}$ of 8.102622 percent (by equation [1]), which is .26 basis points in error of the market rate of 8.100012. If volatility is assumed to be 15 percent, the error increases to .72 basis points. The price errors, $p_{1,2}$ at volatilities of 9 and 15 percent are .00000217 and .00000600 respectively. Use of these parameters would establish conditions for arbitrage where we have assumed that none can exist. The reason for the error is that the second and higher derivatives of the price function are not zero; that is, prices change with respect to interest rates in a curvelinear fashion, a condition called convexity in the investment literature.

4.3. A SIMPLE APPROXIMATE SOLUTION

An iterative process can be performed to solve the unique drift term that eliminates the price error. The drift term is a variable without economic significance in itself that can be adjusted to eliminate pricing errors caused be the effect of the stochasticization of rates. In the above example, a drift term of .01210186 eliminates virtually all price error.

$$f_{1,2}^{up} = .08\,e^{(.09\sqrt{1/12}+.01210186)} = 8.310536\%$$

$$f_{1,2}^{dn} = .08\,e^{(-.09\sqrt{1/12}+.01210186)} = 7.889736\%$$

$$p_{1,2}^{up} = .99323754$$

$$p_{1,2}^{dn} = .99357234$$

These prices indicate an expected price of .99340494, eliminating error at an acceptable threshold.

4.4. A GENERAL DISTRIBUTION FITTING SOLUTION

The period two iteration process is made simpler because only two possible rates exist. Consequently, we simply weigh them equally.

At more distant periods two apparent complexities impinge. First, the number of possible paths increases geometrically. At period two, two paths were possible; by periods 3, 4 and 5, the possible paths are 4, 8, and 16 paths respectively. This would appear to present an eventual computation problem at distant periods. However, one feature of the lognormal approach is that, for any given rate path and its mirror path (i.e. a mirror path would be one that moved in the opposite direction of the main path at every period so that it constituted a mirror image of the main path), the same number of up and down movements will produce the same rate. Consequently, at any period, the number of possible rates is small relative to the number of possible paths leading to the rate. That is, at any term, there are fewer nodes than possible paths to arrive at such node. This condition holds regardless of the shape of the forward rate and volatility curves. In fact, the number of possible rates at month t is equal to t. This simplifies our calculation.

Secondly, the weighting of a particular rate becomes more complex. At $t = 2$, the weights were simply .5 for each rate. Beyond that, we need to find the number of paths ending at a given rate as a proportion of all paths at such term to use as weights. Fortunately, a simple method indicates this ratio.

Chart One depicts graphically the first eight periods of a lattice estimation based on forward market rate, price and volatility information. The chart provides basic input data. In this case, we have assumed them all (with the exception of drift) rather than having calculated them.

Graphic One shows the number of paths that can end at any node at a given term. The pattern is, perhaps, obvious from the chart. The total possible path endpoints at a given term is equal to 2^{t-1}. Furthermore, at $t \geq 4$, the graphic provides information, at any node, with respect to the relative proportion of rates that rose or fell from the preceding term to arrive at such node. For example, at $t = 8$ and the third node down from the top, with a 6/21 probability, the rate will fall to arrive at the node and with a 15/21 probability the rate will rise to arrive at the node.

Graphic Two provides weights associated with any node. This is the number of paths that can end at such node as a proportion of total possible path endpoints at the term. This information is rounded to five places. Again, perhaps the pattern is intuitively obvious.

Estimates in Graphics Three, Four and Five must be derived simultaneously by an iterative process.

At any term, estimate the unique set of forward rates such that the following conditions are met:

1) the distance between any two nodes can be described as:

$$\sigma_t = \frac{.5 \ln \left(\frac{f_{1,t}^{up}}{f_{1,t}^{dn}} \right)}{\sqrt{1/12}} . \tag{6}$$

In this way, both rate and volatility conditions are satisfied. Note that, at any term, adjacent nodes (where such nodes are f^{up} and f^{dn} relative to each other) have the following relationship (by rearranging equation [6]):

$$f_{1,t}^{up} = f_{1,t}^{dn} e^{2\sigma\sqrt{1/12}} .$$

For example, in Graphic Three, $t = 7$, consider the fourth and fifth rates down from the top:

$$.08608125 = .07839506\, e^{(2).162\sqrt{1/12}} ,$$

2) the difference between the number of rates above and below $f_{1,t}$ is equal to zero in even numbered month and one negative one in odd numbered months. That is, the lattice must be centered on the forward rate curve, and

3) the sum of the weighted zero coupon prices must equal (within a certain tolerance for error) the zero coupon price indicated by the

term structure interpolation. We derive a given zero coupon price by multiplying the estimated forward prices by the two zero coupon prices of the preceding term leading to such position in the lattice. For example, at $t = 8$ and the third node down from the top:

$$.94292628 = \Big[(.99191501)(.94891238)(6/21)\Big] + \\ + \Big[(.99191501)(.95129179)(15/21)\Big],$$

where

$$.99191501 = \left(1 + \frac{.09982552}{2}\right)^{-1/6}$$

by equation [2].

These zero coupon prices are then weighted according to information from Graphic Two to produce the weighted zero coupon prices in Graphic Five.

In accomplishing this fitting, we use, by convention, a drift term to establish the center rate as a function of the rate at the top of the lower one-half of the distribution of the even-numbered month that preceded it. By equation [5], for example, at $t = 5$,

$$.08442919 = .07920589\ e^{(.168\sqrt{1/12})+.01536501}.$$

In even-numbered months, we establish a phantom rate, equal to the forward rate at such term, and establish the "center" rate as the rate at the bottom of the upper one-half of the distribution. By a transformation of equation [6], for example, at $t = 6$,

$$.08941112 = .0857\ e^{(.165\sqrt{1/12})-.00523921}.$$

The use of the drift term in centering the distribution forces us to characterize it differently in odd and even months. Consequently, the drift term series, as shown in Chart 1, is discontinuous, although it shows continuity across odd-numbered months and even-numbered months.

Chart 1

t	$f_{1,t}$	$p_{1,t}$	Zero Coupon Prices_t	Possible Path Endpoints_t	$\sigma_{12,t}$	Drift_t
0	n.m.	n.m.	n.m.	n.m.	n.m.	n.m.
1	8.0%	.99348453	0.99348453	1	n.m.	n.m.
2	8.100012	.99340494	0.98693244	2	18.0%	-0.00128785
3	8.24	.99329361	0.98031368	4	17.5	0.01744927
4	8.35	.99320618	0.97365361	8	17.1	-0.00343246
5	8.48	.99310294	0.96693826	16	16.8	0.01536501
6	8.57	.99303150	0.96020016	32	16.5	-0.00523921
7	8.66	.99296010	0.95344045	64	16.2	0.1054404
8	8.75	.99288874	0.94666028	128	16.0	-0.00677904

n.m.=Not meaningful.

Graph.1 - Path Endpoints

							1
						1	
					1		7
				1		6	
			1		5		21
		1		4		15	
	1		3		10		35
1		2		6		20	
	1		3		10		35
		1		4		15	
			1		5		21
				1		6	
					1		7
						1	
							1

Graph. - 2 Endpoint Weights

							.00781
						.01563	
					.03125		.05469
				.0625		.10938	
			.125		.15625		.16406
		.25		.25		.23438	
	.5		.375		.3125		.27344
1		.5		.375		.3125	
	.5		.375		.3125		.27344
		.25		.25		.23438	
			.125		.15625		.16406
				.0625		.10938	
					.03125		.05469
						.01563	
							.00781

Graph. 3 - Forward Rate Distribution

							12.008213%
						11.396414%	
					10.817720%		10.948635%
				10.250410%		10.378829%	
			9.649629%		9.834757%		9.982552%
		9.094046%		9.302870%		9.452103%	
	8.521047%		8.742468%		8.941112%		9.101713%
8.000000%		8.220110%		8.442919%		8.608125%	
	7.679974%		7.920589%		8.128668%		8.298598%
		7.430159%		7.662461%		7.839506%	
			7.175976%		7.390049%		7.566349%
				6.954148%		7.139516%	
					6.718544%		6.898711%
						6.502029%	
							6.289984%

Graph. 4 - Zero Coupon Prices

							0.93723706
						0.94638947	
					0.95517109		0.94016340
				0.96359386		0.94891238	
			0.97165442		0.95730280		0.94292628
		0.97931524		0.96534206		0.95129179	
	0.98660004		0.97302815		0.95931078		0.94553487
0.99348453		0.98032929		0.96698682		0.95353601	
	0.98726484		0.97431914		0.96120242		0.94799792
		0.98128092		0.96853451		0.95565290	
			0.97553260		0.96298471		0.95032375
				0.096999114		0.95764995	
					0.96466427		0.95252025
						0.95953423	
							0.95459491

Graph. 5 - Weighted Zero Coupon Prices

							0.00732216
						0.01478734	
					0.02984910		0.05141519
				0.06022462		0.08896054	
			0.12145680		0.14957856		0.15469884
		0.24482881		0.24133552		0.22295901	
	0.49330002		0.36488556		0.29978462		0.25854469
0.99348453		0.49016464		0.36262006		0.29798000	
	0.49363242		0.36536968		0.30037576		0.25921818
		0.24532023		0.24213363		0.22398115	
			0.12194158		0.15046636		0.15591249
				0.06062445		0.08977968	
					0.03014576		0.05209095
						0.01499272	
							0.00745777

5. Projecting short-term rates

A path of short-term forward rates can then be assembled by a simple random selection process. At any term, the rate moves either up or down, with a probability of .5 in either direction, in a random pattern from the previous term. The rate can move only from an adjacent node at the previous term; it cannot leap nodes. Referring to Graphic Three, a given path may include the following rate pattern:

t:	1	2	3	4	5	6	7	8
$f_{1,t}$:	8.0	8.521047%	8.220110%	7.920589%	7.662461%	8.128668%	7.839506%	8.298598%
Move:	n.m.	up	down	down	down	up	down	up

Adequate sampling will produce results that reconstitute the term structure of interest rates. Proper sampling ensures that rates are selected, at each term, roughly according to the weights shown in Graphic Two. Given such sampling, the rate implied by the weighted average price at each term is the Treasury forward rate. Given such correct prices, the zero coupon rate implied by the product of prices across terms is the Treasury zero coupon rate. This further confirms the fit of the model.

6. Certain aspects of the rate diffusion process

Note the following with respect to this projection.

First, path pairs can be projected in order that the average price converges more quickly. In theory, projecting paths without their respective mirrors will provide the same expected present value. Projecting paths and their mirrors, however, affords relatively quick convergence. Doing so is justified on the basis that, if the primary path represents a randomly determined set of observations, then so does its inverse. In fact, more sophisticated sampling techniques can further increase efficiency.

Second, bear in mind that each point represents the pure time value of money for that period. As such, the value at any period t of a cash flow received at any period $t+1$ can be described by equation [2]. Thus, cash generated at various future points can be discounted by equation [2] to the period before it, and so on, to the present.

Third, each path of short-term rates implies a path of long term rates as well. Consider the following:

$$p_{t,t+n} = \prod_{t+n}^{t} p_{t+1,t} \tag{7}$$

That is, the product of the one month prices over a given number of months is equal to the price for the entire period. Such prices imply an interest rate by equation [1].

Fourth, by using long-term implied rates on treasuries, cash flows from mortgages and insurance liabilities, for example, can be generated and future prices on callable corporate bonds can be projected and calls triggered.

Fifth, cash flows so generated can be discounted to the present using pure one month prices adjusted by any additional spread that might be appropriate. In each case, cash flows are discounted by the short term rates that produced such flows.

Sixth, the selection of the proper number of paths is an empirical issue. The appropriate number of paths is that number over which the expected present value has converged to a degree that is adequate for a given purpose. Certain complex mortgages, annuity or interest-sensitive life instruments take about 1000 paths to converge, some less than 100. Using the results to gain a feel for the spread offered by a given instrument may require only 25, whereas estimating derivatives of price with respect to a subtle shift in rates may require over 1000.

Seventh, extreme rates raise an issue of instrument convexity. Critics point out that rates projected under lognormal assumptions can become quite high or quite low. Graphic Two demonstrates how unlikely extreme rates are; after only eight periods, the most extreme rate appears in less than 1 out of 100 paths. Evidence is not clear that such possibility even convincingly argues against the lognormal distribution assumptions[(2)]. Furthermore, the normal negatively sloped term structure of volatility dampens distant rate amplitude in a way that is consistent with mean reverting rate processes. However, for the moment, suppose that some paths indeed wander to levels that could never be reached in practice. Proper sampling assures that the effect of each extreme is buffered. Plausible expected instrument prices result from two implausible, but largely offsetting paths. This sampling removes the first derivative effect from the expected price, leaving only the second derivative effect, that is, convexity, reflected in the expected price. It is not clear that such convexity does not represent an accurate reflection of reality.

[(2)] J.E. Murphy Jr., "The Random Character of Interest Rates", (Chicago: Probus Publishing Company), 1990.

7. Conclusion

This concludes discussion of a lognormal one factor interest rate process methodology. We have attempted to present, in a practical and accessible way, a standard approach to an arbitrage-free generalized interest rate diffusion process with a few words on term structure interpolation and estimation and use of the method of pricing financial claims or obligations containing embedded options.