HEDGING GENERAL CLAIMS

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ABSTRACT

The optimal mean–variance hedging strategy is derived for the case that the stochastic process of traded assets satisfies an autonomous stochastic differential equation. The result extends work of Duffie and Richardson (1991) and Schweizer (1992).

KEY WORDS: Continuous trading, minimizing squared error, martingale measure.

1. INTRODUCTION AND SUMMARY

1.1. THE PROBLEM

Let T be some finite planing horizon and L a stochastic liability. Let F_t, $0 \leq t \leq T$, be the price process of some asset which can be traded. We neglect interest rates or assume that all prices are discounted. We neglect transaction costs as well. If ϑ_t is the number of pieces of asset F_t held at time t then

$$G_T(\vartheta) = \int_0^T \vartheta_t dF_t$$

is the accumulated gain for the dynamic trading strategy ϑ in $[0, T]$. We shall consider the problem of identifying the trading strategy ϑ for which the gain $G_T(\vartheta)$ matches L best, in the sence that the squared error

$$E(L - G_T(\vartheta))^2 \tag{1}$$

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is minimized. This is an old problem for which, surprisingly, a complete solution is not yet given.

The problem dates back at least to the famous paper of Black and Scholes (1973). There a geometric Brownian motion F_t is considered, and it is shown that for $L = (F_T - K)^+ - L_0$ there exists a trading strategy ϑ^* for which [1] is zero. Here L_0 is an appropriate centering constant: it is the mean of $(F_T - K)^+$ under the equivalent martingale measure for F_t. The trading strategy ϑ^* clearly minimizes [1]. The centering constant L_0 is the equilibrium price for the call option $(F_T - K)^+$, the *Black–Scholes price*, and ϑ^* is the *perfect hedge* for L. If L is not centered or if there is more than one source of randomness, then the risk [1] cannot be eliminated in general.

Strategies of this type can be used to reduce the risk of a portfolio with the help of futures. For this application, $L = C + S_T$, where C represents costs (or some fixed desired gain) and S_T is the value of the portfolio at the end of the planning horizon $[0, T]$, and F_t is the futures price at time t.

1.2. The probabilistic setup

Let V_t, W_t, $0 \le t \le T$, be two standard Wiener processes on a probability space $(\Omega, \mathcal{F}, P)$ which are stochastically independent, and let $(\mathcal{F}_t, 0 \le t \le T)$ be the filtration generated by (V, W). Let F_t, $0 \le t \le T$, be a square integrable continuous semimartingale admitting the representation

$$dF_t = a_t dt + b_t dW_t \tag{2}$$

with predictable processes a_t, b_t for which

$$E \int_0^T (a_t^2 + b_t^2) dt < \infty \ .$$

We assume that the process a_t/b_t is bounded, uniformly in $0 \le t \le T$ and $\omega \in \Omega$. Since we assume that F_t is given, we need no Lipschitz conditions on a_t, b_t which guarantee the existence of a solution for [2] (see Rogers and Williams (1987), p. 265). Write Θ for the set of all possible strategies, i.e. for all predictable processes ϑ_t for which

$$E \int_0^T \vartheta_t^2 b_t^2 dt < \infty \ .$$

This assumption implies that for all possible trading strategies the gain $G_T(\vartheta)$ is square integrable with respect to P. Write $L^p(P)$ for the set of all $\mathcal{F}_T$–measurable random variables with finite absolute moment of order p under P. The liability L is assumed to be in $L^p(P)$, for some $p > 2$,

$$[3] \qquad EL^p < \infty, p > 2 \ .$$

Let Q be the probability measure with P–density

$$Z = \exp\left(-\int_0^T \frac{a_t}{b_t}\, dW_t - \frac{1}{2}\int_0^T \frac{a_t^2}{b_t^2}\, dt\right) \ .$$

The probability measure Q is a martingale measure for F_t in the sense that under Q the process $F_t : 0 \leq t \leq T$, is a square integrable martingale. The martingale measure for F_t is not unique in the given setup. Notice that $Z \in L^p(P)$ for all $p \geq 1$, and hence $L \in L^2(Q)$. Furthermore, $1/Z \in L^p(P)$ for all $p \geq 1$.

For $0 \leq t \leq T$ and $X \in L^1(P)$ write $E_t(X)$ for the conditional expectation of X, given $\mathcal{F}_t$.

Since we are dealing with two different probability measures, expectations have to be labelled: we write E and E_t for expectations and conditional expectations under the probability P, and $\hat{E}$ and $\hat{E}_t$ for the corresponding quantities under the probability Q.

1.3. The solution of Schweizer

Schweizer (1992) presents a solution ϑ^* for the case that

$$[4] \qquad a_t/b_t \quad \text{is deterministic}$$

i.e., doesn't depend on ω. The solution can be given in terms of the present gain at time t:

$$G_t(\vartheta^*) = \int_0^t \vartheta_s^* dF_s$$

and Ito's representation for L:

$$L = L_0 + \int_0^T g_t dF_t + \int_0^T h_t dV_t \ ;$$

it is

$$\vartheta_t^* = g_t + (L_t - G_t(\vartheta^*))a_t/b_t^2 \ . \tag{5}$$

Here,

$$L_t = \hat{E}_t L = L_0 + \int_0^t g_s dF_s + \int_0^t h_s dV_s \ .$$

The case [4] covers the situation considered in Duffie and Richardson (1991). The computation of g_t is not straightforward, it is based on Clark's formula which is discussed below.

In order to show the effect of optimal trading, we compute the risk for ϑ^*. It is given by

$$\begin{aligned} E(L - G_T(\vartheta^*))^2 = L_0^2 \exp\left(-\int_0^T \frac{a_t^2}{b_t^2}\, dt\right) + \\ + E\int_0^T \frac{h_t^2}{Z_t} \exp\left(-\int_t^T \frac{a_s^2}{b_s^2}\, ds\right) dt \ . \end{aligned} \tag{6}$$

Here,

$$Z_t = \hat{E}_t Z = \exp\left(-\int_0^t \frac{a_s}{b_s}\, dW_s - \frac{1}{2}\int_0^t \frac{a_s^2}{b_s^2}\, ds\right) \exp\left(\int_t^T \frac{a_s^2}{b_s^2}\, ds\right) \ . \tag{7}$$

If L is constant (or $h_t \equiv 0$) and a_t is bounded away from zero, uniformly in t and ω, then the risk is decreasing exponentially. The asymptotic behaviour of the risk in case $h_t \neq 0$ is less transparent because of $Z_T \to 0$.

1.4. The results

In this paper we derive the optimal trading strategy for the case that in the Ito representation of Z under $\hat{P}$ the second source of randomness is not present:

$$Z = z_0 + \int_0^T z_t dF_t \ . \tag{8}$$

This case covers Schweizer's situation [4]. Furthermore, it covers the case of an autonomous stochastic differential equation for F_t in which

the coefficients a_t, b_t are functions of F_t alone. The situation [8] is even more general: the predictable process z_t may depend on the process V_t as well.

In the case of [8], the optimal trading strategy is given by

$$\vartheta_t^* = g_t - (L_t - G_t(\vartheta^*))z_t/Z_t \ . \tag{9}$$

Here, $Z_t = \hat{E}_t Z = z_0 + \int_0^t z_s dF_s$. The risk for this trading strategy is given by

$$E(L - G_T(\vartheta^*))^2 = \frac{L_0^2}{\hat{E}Z} + \hat{E}ZEZ \int_0^T \frac{h_t^2}{Z_t^2}\, dt \ . \tag{10}$$

For this solution, not only g_t but both, g_t and z_t, have to be computed (or simulated) with Clark's formula.

Our result doesn't given a complete solution to the hedging problem, since there are cases in which the Ito representation

$$Z = z_0 + \int_0^T z_t dF_t + \int_0^T x_t dV_t \tag{11}$$

has non–vanishing process x_t. For the discrete case, the computation of an optimal strategy is always possible; see below.

2. Proofs and technical details

2.1. Proof of optimality

Recall that a possible trading strategy ϑ^* is optimal if and only if for all possible trading strategies ϑ we have

$$E(L - G_T(\vartheta^*))G_T(\vartheta) = 0 \ . \tag{12}$$

Let $k_t = h_t/Z_t$. Then

$$\hat{E}\int_0^T k_t^2 dt < \infty \ . \tag{13}$$

For this, use $L \in L^p(P)$ and the Burkholder–Davis–Gundy inequalities (see Rogers and Williams (1987)). For some constant c to be determined later let $A_t = c + \int_0^t k_s dV_s$. If ϑ is a possible trading strategy, then

$$EG_T(\vartheta)ZA_T = \hat{E}G_T(\vartheta)A_T = 0 \ . \tag{14}$$

Hence the trading strategy given in [9] is optimal provided

$$L - G_T(\vartheta^*) = ZA_T \ .$$

Define $X_t = L_t - G_T(\vartheta^*) - Z_t A_t$, where $L_t = \hat{E}_t L$. We choose c such that $X_0 = 0$; this is the case for

$$c = \hat{E}L/\hat{E}Z \ . \tag{15}$$

The process X_t is a semimartingale which satisfies the stochastic differential equation

$$dX_t = (g_t - \vartheta_t^*)dF_t + h_t dV_T - Z_t k_t dV_t - A_t z_t dF_t \ . \tag{16}$$

Here we used the fact that Z_t and A_t are orthogonal $\hat{P}$–martingales, and hence $Z_t A_t$ is a $\hat{P}$–martingale, too. Inserting ϑ^* yields

$$dX_t = \frac{z_t}{Z_t} X_t dF_t \ .$$

The only solution to this equation with initial condition $X_0 = 0$ is $X_t = 0$ according to Doleans theorem (see Rogers and Williams (1987)). This proves optimally of ϑ^*.

From our representation

$$L - G_T(\vartheta^*) = ZA_T$$

we obtain the equation [10]:

$$E(L - G_T(\vartheta^*))^2 = EZ^2 A_T^2 = \hat{E}Z\left(c + \int_0^T k_t dV_t\right)^2 \tag{17}$$

$$= z_0 c^2 + z_0 \hat{E} \int_0^T k_t^2 dt \ . \tag{18}$$

2.2. Schweizer's case

If a_t/b_t is deterministic, then the computation of z_t and Z_t is easy.

Let

$$U_t = \exp\left(-\int_0^t \frac{a_s}{b_s}\, dW_s - \frac{1}{2}\int_0^t \frac{a_s^2}{b_s^2}\, ds\right)$$

which is a P–martingale but not a $\hat{P}$–martingale. Then

$$U_t \exp\left(\int_0^t \frac{a_s^2}{b_s^2}\, ds\right)$$

is a $\hat{P}$–martingale and hence

$$Z_t = U_t \exp\left(\int_t^T \frac{a_s^2}{b_s^2}\, ds\right) .$$

We obtain

$$dZ_t = dU_t \exp\left(\int_t^T \frac{a_s^2}{b_s^2}\, ds\right) - U_t \frac{a_t^2}{b_t^2} \exp\left(\int_t^T \frac{a_s^2}{b_s^2}\, ds\right) =$$

$$-\frac{a_t}{b_t} U_t dW_t \exp\left(\int_t^T \frac{a_s^2}{b_s^2}\, ds\right) - U_t \frac{a_t^2}{b_t^2} \exp\left(\int_t^T \frac{a_s^2}{b_s^2}\, ds\right) dt = -Z_t \frac{a_t}{b_t^2}\, dF_t$$

and this implies $z_t = -Z_t a_t / b_t^2$ or

[19] $$-z_t / Z_t = a_t / b_t^2 .$$

2.3. Clark's formula

The bivariate Ito representation theorem states that for two orthogonal P–martingales V, W generating the filtration $\mathcal{F}_t$ and for any square integrable $\mathcal{F}_T$–measurable random variable X there exist predictable processes x_t, y_t for which

$$E\left(\int_0^T x_t^2 d[V]_t + \int_0^T y_t^2 d[W]_t\right) < \infty$$

and a constant x_0 satisfying

$$X = x_0 + \int_0^T x_t dV_t + \int_0^T y_t dW_t .$$

Here $[V]$ and $[W]$ are the quadratic variation processes of V and W, respectively.

The computation of the integrands x_t and y_t in Ito's representation is not straightforward. Since they are defined through an infinitesimal relation, they cannot be simulated directly, since simulations of stochastic processes are possible in discrete time only. Here Clark's formula is extremely useful. We shall give an ad hoc application of it. A precise formulation and proof can be found in Rogers and Williams (1987).

Let δ be a generic positive number converging to 0. For stochastic processes Y_t we write $Y_\delta = o(\delta)$ if $Y_\delta/\delta \to 0$ in $L^2(P)$. So, e.g.,

[20] $$F_{t+\delta} - F_t = a_t\delta + b_t(W_{t+\delta} - W_t) + o(\delta)$$

$$E_t F_{t+\delta} = F_t + a_t\delta + o(\delta) \ .$$

With this notation the integrands are defined via

[21] $$E_t X(V_{t+\delta} - V_t) = x_t[V]_t\delta + o(\delta)$$

[22] $$E_t X(W_{t+\delta} - W_t) = y_t[W]_t\delta + o(\delta) \ .$$

Assume now that X can be differentiated with respect to small increments of V and W in the following sense:

$$X = X_t(V) + (V_{t+\delta} - V_t)X_t'(V) + R_t(V)$$
$$X = X_t(W) + (W_{t+\delta} - W_t)X_t'(W) + R_t(W)$$

where

$$E_t(V_{t+\delta} - V_t)R_t(V) = o(\delta)$$
$$E_t(W_{t+\delta} - W_t)R_t(W) = o(\delta)$$

where $X_t(V)$ and $(V_{t+\delta} - V_t)$ as well as $X_t(W)$ and $(W_{t+\delta} - W_t)$ are stochastically independent and $X_t'(V)$, $X_t'(W)$ are square integrable. If interchanging of limits and integrals can be justified, then

$$x_t = E_t X_t'(V)/[V]_t$$
$$y_t = E_t X_t'(W)/[W]_t \ .$$

For example, if $X = f(V_T)$ with smooth function f, and if

$$V_t = \int_0^t v_s^2 ds \ ,$$

then

$$X = x_0 + \int_0^T x_t dV_t \quad \text{with} \quad x_t = E_t f'(V_T)/v_t^2 \ .$$

If, e.g., $a_t/b_t^2 = e(F_t)$ and $a_t^2/b_t^2 = f(F_t)$ for some smooth functions e and f, then $Z = z_0 + \int_0^T z_t dF_t$ with

$$[23] \qquad z_t = \hat{E}_t Z \left(-e(F_t) - \int_t^T e'(F_s) dF_s + \frac{1}{2} \int_t^T f'(F_s) ds \right) \Big/ b_t^2 \ .$$

The quantity in [23] can be simulated.

For the computation of g_t in Ito's representation for L

$$L = L_0 + \int_0^T g_t dF_t + \int_0^T h_t dV_t$$

we have to swich from increments of F_t to increments of W_t. Our starting point is the identity

$$g_t b_t^2 \delta = \hat{E}_t L(F_{t+\delta} - F_t) + o(\delta) \ .$$

Using [20] we obtain

$$g_t b_t^2 \delta = a_t \delta L_t + b_t \hat{E}_t L(W_{t+\delta} - W_t) + o(\delta) \ .$$

Using the relation

$$\hat{E}_t X = E_t Z X / U_t$$

and the product rule, we finally end up with

$$[24] \qquad b_t^2 g_t = a_t L_t + b_t E_t (Z_t'(W) L + Z L_t'(W)) / U_t \ .$$

Here, $Z_t'(W)$ and $L_t'(W)$ have the same interpretation as $X_t'(W)$ above. Also the expression in [24] can be simulated. For the art of simulating solutions of stochastic differential equations see the monograph of Kloeden and Platen (1992).

2.4. The discrete case

In discrete time, the construction of an optimal trading strategy can always be done by a backtracking procedure. In the discrete setup we assume that $\mathcal{F}_t$ are the sigma–fields of events observable up to time t, $t = 0, \ldots, T$, and that L is $\mathcal{F}_T$–measurable and F_t is $\mathcal{F}_t$–measurable, and L and F_t, $t = 0, \ldots, T$, are square integrable. Again we write E_t for the conditional expectation, given $\mathcal{F}_t$. We construct a trading strategy ϑ_t, $t = 0, \ldots, T-1$, which minimizes

$$[25] \qquad E\left(L - \sum_{t=0}^{T-1} \vartheta_t (F_{t+1} - F_t)\right)^2$$

recursively. Assume that $\vartheta_0, \ldots, \vartheta_{T-2}$ are chosen and we have to find the optimal ϑ_{T-1} minimizing [25]. We write P_t for the value of our portfolio at time t:

$$P_t = \sum_{s=0}^{t-1} \vartheta_s (F_{s+1} - F_s) \ .$$

Since [25] equals

$$[26] \qquad EE_{T-1}(L - P_T)^2 = EE_{T-1}(L - P_{T-1} - \vartheta_{T-1}(F_T - F_{T-1}))^2$$

and in the inner integral ϑ_{T-1} is deterministic, we obtain the optimal value for ϑ_{T-1} by differentiation, which yields

$$[27] \qquad \vartheta_{T_1} E_{T-1}(F_T - F_{T-1})^2 = E_{T-1}(L - P_{T-1})(F_T - F_{T-1}) \ .$$

With this choice and the shorthand notation $\Delta = F_T - F_{T-1}$ [25] reads

$$E(L - P_{T-1})^2 - E\vartheta_{T-1}^2 \Delta^2 =$$

$$E(L - P_{T-1})^2 - EE_{T-1}^2(L - P_{T-1}) - \Delta / E_{T-1}\Delta^2) =$$

$$EL^2 - E\frac{E_{T-1}^2 L\Delta}{E_{T-1}\Delta^2} - 2EP_{T-1}L\left(1 - \Delta\frac{E_{T-1}\Delta}{E_{T-1}\Delta^2}\right)$$

$$+EP_{T-1}^2\left(1 - \frac{E_{T-1}^2\Delta}{E_{T-1}\Delta^2}\right) \ .$$

Minimizing [25] is equivalent to the choice of [27] for ϑ_{T-1} and minimizing

$$\hat{E}(L_{T-1} - P_{T-1})^2$$

with

$$L_{T-1} = E_{T-1}L\left(1 - \Delta\frac{E_{T-1}\Delta}{E_{T-1}\Delta^2}\right)\Big/\left(1 - \frac{E^2_{T-1}\Delta}{E_{T-1}\Delta^2}\right)$$

and $\hat{E}$ is the measure defined by

$$\hat{E}X = EX\left(1 - \frac{E^2_{T-1}\Delta}{E_{T-1}\Delta^2}\right) .$$

Notice that $E^2_{T-1}\Delta \leq E_{T-1}\Delta^2$, and equality holds only if Δ is $\mathcal{F}_{T-1}$–measurable. In an arbitrage free market, $\mathcal{F}_{T-1}$–measurable of Δ implies $\Delta = 0$, and for $\Delta = 0$ any choice of ϑ_{T-1} is optimal. So the possibility $E^2_{T-1}\Delta = E_{T-1}\Delta^2$ may be excluded.

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