SOME FURTHER INVESTIGATIONS INTO CASHFLOW MATHCHING

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ABSTRACT

This paper investigates in greater detail some of the cashflow matching ideas described in a paper presented to the AFIR Colloquium in April 1991.

The new method allows for higher interest rates to be charged on the cash fund when it is in deficit than when it is in credit. This requires the use of pairs of variables to represent the surplus or deficit cash balances in each stochastic run. As a result, there is no need to allow any of the variables to become negative, a device which was necessary in the 1991 paper to represent cash fund deficits.

Consequently, the linear programming logic is considerably simplified. This effect, however, is achieved at the expense of much longer run-times and a substantial increase in the amount of computer immediate-access memory required.

1. INTRODUCTION

1.1. In April 1991 I wrote a paper for the AFIR Colloquium at Brighton which investigated the actuarial problem of identifying, for the investment managers of a life or pension fund, the proportion of current assets that should be held in fixed-interest stocks, with the remainder of the fund held in potentially higher-yielding but more volatile assets, in order that there would be a specified probability of meeting the future liability payments as they arose.

1.2. The method of solution used a modified version of Linear Programming, in which some of the variables, representing the accumulated cash funds at specified test points in a set of stochastic future projections, were allowed to become negative, up to a maximum number of the variables being negative at any one time.

1.3. The method required the cash fund to roll up at a rate of interest which was linked to the future fixed interest yields arising under the stochastic economic projections. However, it was necessary to use those same rates of interest even at times when the cash fund was in deficit. This was clearly an unreal assumption and in order to minimise its effect, a regular and fairly frequent number of test points, spread over the full term of the liabilities, was found to be necessary. Otherwise, the computer would seek to optimise the asset-mix by investing in the cash fund if short term interest rates were too high, or alternatively by investing long (in higher-yielding equities) and running a long-term overdraft if borrowing rates were too generous.

1.4. In section A2.7 of the 1991 paper, a possible approach was briefly described that would enable the artificial cash fund interest rate assumption to be extended to allow for higher overdraft interest rates. This approach has been developed in the current paper and some of its consequences investigated.

1.5. The paper starts by describing the amended logic required to cater for the separate overdraft interest rates.

It then investigates some of the results produced by the amended logic, testing their sensitivity to variations in the overdraft rates and in the number of test points used.

Some further possible lines of development, stemming from the revised approach, are then briefly described. As was identified in the earlier paper, some of these results will need to wait for more powerful computer facilities to become readily available if they are to be realistic.

1.6. As in the previous paper, the work is based on the original developments of Wise (1984), using the Wilkie stochastic model (1986) with the parameters used in the paper by Daykin and Hey (1990).

Wise's original work projects the ultimate surplus of a fund as a stochastic variable and seeks to allocate the current assets of the fund in such a way that the variance of that ultimate surplus is minimised. It controls the required risk level implicitly by selecting an appropriate variance level for the ultimate surplus. Thus all the projected values of the ultimate surplus, including those at the extremes of the frequency distribution, are taken into account.

1.7. The methods that I have used in the 1991 paper and in the current paper differ from that earlier approach in that they examine the probability of cash fund deficits occurring at a series of future inspection points, not just at the final point, but make no attempt to minimise the variance of those deficits. The approach is therefore not concerned with the size of the projected deficits or surpluses, only with the probability of them occurring.

1.8. The logic underlying the build up of the cash fund, at interest rates dependent on whether the cash fund at the time is in deficit or surplus, depends directly on the algorithms published in 1988 by Kocherlakota, Rosenblum and Shiu.

I am grateful to all the above authors, without whose work the ideas in this current paper could not have been developed.

2. Revised logic for increased overdraft interest rates

2.1. The algorithm in Section IV of Kocherlakota et al (1988) is used to allow for a higher interest rate roll-up of the cash fund when it is in deficit than when it is surplus. This provides a more realistic basis for the underlying logic than that used in the 1991 AFIR paper, where no distinction was made in the interest rates applied to the cash fund, whether it was in deficit or surplus.

2.2. Let $\Delta(t, k)$ be the net increase in the cash fund from all sources, including dividend and interest receipts and proceeds from asset sales and redemptions, over the interval $t-1$ to t, for stochastic projection k.

Define interval $t-1$ to t as period t.

Let $D(t, k)$ = Cash fund deficit at end of period t in stochastic run k.

$S(t, k)$ = Cash fund surplus at end of period t in stochastic run k.

D and S are constrained to be non-negative at all times.

2.3. Let $L(t, k)$ = Cash fund overdraft rate over period t in stochastic run k.

$B(t, k)$ = Cash fund deposit interest rate over period t in stochastic run k.

L and B are found by taking constant percentages of the gilt yields in each period t for each stochastic run k.

The percentage for L is chosen to be greater than that for B.

2.4. Then, using the algorithm referred to in 2.1, we get:

$$S(t,k) - D(t,k) = (1+B(t,k)) \cdot S(t-1,k) - (1+L(t,k) \cdot D(t-1,k) + \Delta(t,k)$$

For convenience, the subscript k, referring to the relevant stochastic run, is omitted from future equations.

2.5. $\Delta(t)$ is built up from the cashflows over period t. For initialisation, we take $D(0)$ as zero and the initial cash fund surplus $S(0)$ as equal to the existing assets at outset.

We assume, as in the 1991 paper, that all cashflows occur at the end of each year.

2.6. Let $DPG(I,J,T)$ represent, for each stochastic run k, the accumulated interest payments received over the period T from an investment of 1 made at the end of year I in a gilt stock redeemable at the end of year J.

Similarly $DPI(I,J,T)$ and $DPE(I,J,T)$ refer respectively to the interest payments and equity dividends received in the period T from an investment of 1 made at the end of year I in an irredeemable gilt or an equity shareholding planned to be sold at the end of year J.

The interest and dividend payments are assumed to accumulate to the end of year T at the cash fund deposit interest rate in force in period T.

2.7. Let $GPV(I,J)$ = the redemption proceeds of an investment of 1 made at the end of year I in a gilt stock which will be redeemed at the end of year J.

Similarly $GIPV(I,J)$ and $EPV(I,J)$ refer respectively to irredeemable gilts and equities planned to be sold at the end of year J.

For simplicity, we assume that these assets can only be redeemed or sold at an inspection point. This assumption could be widened to include other years if desired.

2.8. Then, for inspection point t (in stochastic run k) we have:

$$\Delta(t) = \sum_{i=0}^{M'} \left\{ \sum_{j=t}^{N} \Big[g_{i,j} \cdot DPG(i,j,t) + h_{i,j} \cdot DPI(i,j,t) + e_{i,j} \cdot DPE(i,j,t) \Big] + \right.$$

$$\left. + g_{i,t} \cdot GPV(i,t) + h_{i,t} \cdot GIPV(i,t) + e_{i,t} \cdot EIPN(i,t) \right\} CP(i) - CN(t)$$

where $M' = \min\{M; t\}$ and M is the latest positive cashflow period.

$g_{i,j}$ is defined, as in the 1991 AFIR paper, as the proportion of the positive cashflow invested at the end of period i in a gilt stock redeemed in period j, with $h_{i,j}$ and $e_{i,j}$ similarly the proportions of irredeemable gilts and equities bought in period i and planned to be sold in period j.

$g_{i,i}$ is defined as a cash holding; $h_{i,i}$ and $e_{i,i}$ are zero for all i.

As in the 1991 paper, $CP(i)$ is the cash flow (under stochastic run k) receivable and $CN(i)$ is the cashflow payable at the end of period i. CN and CP can either be gross or net cashflows. They do not include any investment income or sale and redemption proceeds.

2.9. By introducing dummy variables $Y(t,k)$ into the equations in 2.4, we obtain equations in the form:

$$Y(t) + S(t) - \big(1 + B(t)\big) \cdot S(t-1) - D(t) + \big(1 + L(t)\big) \cdot D(t-1) = \Delta(t)$$

for each test point t within each stochastic run k.

In this form, the equations fall naturally into a Linear Programming (LP) format, with the various g, h and e included in the $\Delta(t)$ function being the variables required to be determined.

2.10. The variables $S(t,k)$ and $D(t,k)$ were compressed in the 1991 AFIR paper into the one set of variables $V(t,k)$, which were constrained so that, at each test point t, only a certain number could be negative. This provided the probability at each test point that the cash fund would be in deficit.

Under the revised method, we examine the number of $D(t)$ that are positive at each test point t. Negative values of S and D are not allowed and this simplifies the LP logic considerably.

The original algorithm given in the Kocherlakota paper allows the pair of variables $S(t,k)$ and $D(t,k)$ to be positive at the same time

while the LP process is running, since, once the final feasible solution is reached, one or other of each pair of these variables will need to be zero for an optimum result to have been obtained. However, the additional constraints included in the current method, that only up to a set number of the $D(t)$ variables can be positive at each test point t, mean that the above assumption is no longer true, and the pivoting logic for solving the simplex algorithm requires the additional constraint to be included that, at any time in the pivoting process, only one of each of the pairs $S(t,k)$ and $D(t,k)$ can be positive (the other element of each pair therefore necessarily being zero).

3. Further amendments to the 1991 logic

3.1. Future investment assumptions

In constructing the model, a decision has to be taken regarding the investment of future positive cashflows.

There are various assumptions that can be made to deal with the asset-mix of future investments:

a) Future investment in the same types of assets as those available for the initial assets, but with the mix of assets unconstrained. This was the approach used in the 1991 AFIR paper.

b) Future investment each year having the same overall proportion in equities, possibly with that overall proportion being specified at outset.

c) Current assets and future investments each year having the same overall proportion in equities.

Assumption (a) gives the widest investment freedom and hence the lowest initial assets or widest range of initial investment mix. But, where future positive cashflows are large compared with the initial assets, the initial asset-mix will be quite sensitive to changes in the risk assumptions. It seems unrealistic to ask investment managers to commit at outset to a possibly widely varying set of future investment assumptions and some future constraints are therefore desirable. I have therefore used assumption (c) in this current paper.

Assumption (c) merely requires inclusion of the extra constraints

$$\sum_{i=t}^{N} e_{t,i} - VZ = 0 \quad \text{for} \quad t = 0 \quad \text{to} \quad M$$

where VZ is an additional slack variable (representing the total proportion each year to be invested in equities), N is the full term of the liability outgo and M, as in 2.7, is the latest positive cashflow period.

By introducing further dummy variables into these equations, they fit naturally into the full LP simplex matrix.

3.2. Gross or net cashflows

As in the 1991 paper, it is possible to assume either free investment of the gross positive cashflows, with gross cashflow payments out coming from the cash fund, or alternatively to assume that, as far as possible, cashflow payments out are met firstly from cashflow income at that time, with only the residual net cashflow income, if any, being available for free investment. In any event, interest, sale and redemption proceeds from the assets are kept separate from these cashflows.

The restrictions of the 1991 paper, which assumed a single set of cash fund interest rates, required the use of the net cashflow method to produce reasonably stable sets of results. However, the improved control available from the separate overdraft interest rates in the current paper allows greater investment freedom and hence the use of the gross cashflow method is more acceptable. Comparisons of results from the two methods are shown later in the paper, in 4.2.

3.3. Varying future risk levels

The logic of the 1991 paper allowed a fixed number R of the stochastic runs to be insolvent at each test point. However, an alternative approach would be to allow for an expanding funnel of doubt by replacing the constant R by a function $R(t)$, increasing monotonically with test point duration t.

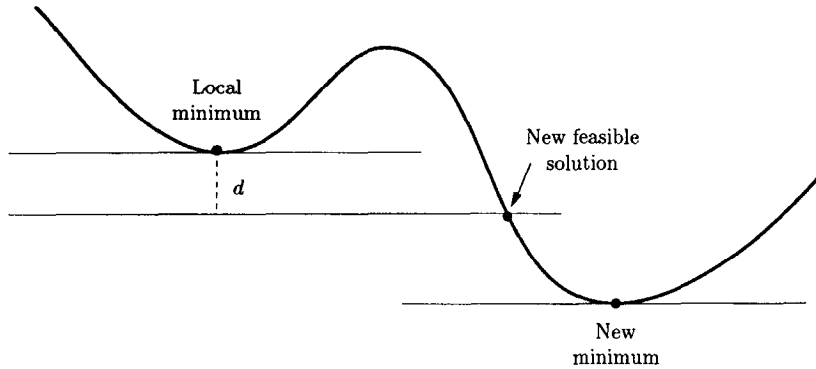
The function can of course be adapted as required; for instance, it can be made more conservative at future points that are known, for external reasons, to be particularly sensitive to capital requirements.

3.4. Local Minima

Appendix 2 of the 1991 AFIR paper drew attention, due to the introduction of negative variables, to the possibility of local minimum solutions arising.

A modification can be made to the program to test the solution to see if it is truly a global optimum solution or only a local minimum. This is done by making a small reduction in the final value of the Objective Function and then, using this as an additional constraint, looping back into the original pivoting process to see if a further feasible solution can be identified. If so, the process continues until a further (lower) optimum feasible solution is found.

In simple graphical terms, we have



Since the program could pass through many local minima before reaching the true global minimum, a limit on the number of tests allowed is built into the process.

3.5. Treatment of zero values

The multiplication of a sequence of small numbers in the LP pivoting process, coupled with the framework of the simplex matrix, leads to many elements in the matrix, particularly in the later pivot stages, that are very close to zero.

Because the LP logic of the current paper limits the numbers of positive D variables, regardless of their size, care is needed in the pivoting logic to remove those D values that are close to zero from the array of Basic Variables, if the arrays of positive D variables at each test point are not to become silted up with a lot of very small D cases. If this happens, further pivoting may be prevented, resulting in local minima or even non-feasible solutions occurring.

Merely leaving those zero values as Basic Variables but ignoring them in counting the number of positive D cases at each test point

is dangerous, since subsequent pivoting may well make them positive, resulting in too many D variables at certain test points.

It is therefore safer to pivot out the zero D variables into a non-basic state as soon as possible. In doing this, however, some controls need to be included to avoid continuous recycling in the pivoting process, which might otherwise occur.

3.6. Availability of capital

A basic feature of the assumptions underlying the two models is that the risk involved depends on the number of stochastic surpluses that are negative, but not on the size of those deficits.

Since a fund will not have access to unlimited capital, it may be felt necessary to limit all the D variables in the current model to a maximum size. This can easily be included in the simplex matrix by including equations of the form

$$X(t,k) + D(t,k) = DMAX$$

for all test points t and stochastic runs k, where $DMAX$ is the maximum deficit allowed and the $X(t,k)$ are further dummy variables which need to be eliminated. This will ensure that $0 \leq D(t,k) \leq DMAX$ for all the D variables, but this is achieved only at the expense of broadly doubling the size of the simplex matrix.

4. Some practical results

4.1. Memory size and run times

The current model, as the equation in 2.9 demonstrates, will need a much larger simplex matrix than did the one used in the 1991 paper, albeit one with most of its elements equal to zero.

The examples in this paper use five test points and one hundred stochastic runs and, for these, the current model requires about sixteen times as much core memory as did the 1991 program. In addition, the run time is increased by a multiple of about twenty.

However, to offset this, there are the advantages of

a) the ability to use different rates of interest for cash fund surpluses and deficits, and

b) the simplicity of the program logic, which enables further constraints to be added relatively easily to the main model. An example of this is described in section 3.6.

Although for most normal purposes the size of the current model has to be severely limited, it should not be many years before far more computer power will become readily and cheaply available. At that stage, the simplicity of the current model will far outweigh the efficiency of the 1991 model.

4.2. Comparisons of gross and net cashflows

Using the pension scheme data from 5.4 of the 1991 AFIR paper, with 5% risk probability and 100 simulations, we use the new model with cash fund interest rates at 75% of the full gilt yields while the fund is in surplus and 125% while it is in deficit, to calculate the minimum initial assets

a) with free investment of the gross contribution income, and
b) with investment only of the residual contribution income net of pension outgo.

Testpoints are taken at quinquennial intervals.

The results are:

	Initial assets
Gross cashflows	277
Net cashflows	316

It can be seen that, by investing only the net residual cashflows, some investment freedom has been given up, resulting in the need for higher initial assets than in the gross cashflow situation.

4.3. Sensitivity to number of testpoints and overdraft rate differentials

This example takes the same data as in 4.2, with gross cashflows.

We calculate the initial assets required for solvency for the scenarios

a) testpoints for i) years 5, 10, 15, 20 and 25
ii) years 15 and 25
iii) year 25 only

coupled with

b) cash fund interest, when the fund is in surplus, at 75% of the underlying gilt yields, together with deficit overdraft rates at i) 75% or ii) 200% of gilt yields.

(In order to allow a constant range of possible investments, we allow 100% risk factors at any unused testpoint in (a ii) and (a iii) above).

The results are:

Overdraft rates%	Testpoint years	Initial assets
75	5, 10, 15, 20 and 25	277
	15 and 25	267
	25	252
200	5, 10, 15, 20 and 25	277
	15 and 25	277
	25	276

These results show that the results became more volatile as fewer testpoints are used. Where a wide spread is used, together with a low risk probability, the results are virtually unaffected by the choice of overdraft rate. Also, the severe level of those rates, at twice the fixed interest yields, makes, as one would expect, the results virtually independent of the choice of testpoints.

4.4. Summary of results

Since the above sets of results have only been derived from one hundred stochastic runs and up to five testpoints, they can only give a broad indication of the levels of initial assets and their investment mix. However, they do follow a logical pattern in that higher overdraft rates require more initial assets than do lower overdraft rates and net cashflows require more initial assets than do gross cashflows.

Also, the fewer the number of testpoints used, the more sensitive are the results, for the same level of risk probability, to the choice of cash fund overdraft rates.

5. Control Systems

As indicated in 5.8 of the 1991 paper, the real power of the system

comes from being able to introduce feedback control mechanisms into the cashflow projections.

This feature has already been used in the pension fund example, where future pension benefits and contributions are linked to rates of salary increase that vary within the stochastic process. Other work, external to this paper, has been carried out on life fund projections, linking future terminal bonus rates to recent past investment experience. This work can quite easily be extended to more slowly-moving reversionary bonus rates.

Where an office has existing contracts containing guaranteed surrender values, future lapse rates linked in some way to the stochastic investment model could also be incorporated, although this process is more complicated since varying lapse rates affect the future cashflows.

An attractive feature to build in would be an investment switch procedure depending on recent past investment experience. Where the decision process depends only on that experience, it is relatively straightforward to introduce (although considerably more variables would be needed in the simplex matrix). If, however, the decisions also depend on the current asset-mix at the time, then they involve knowledge of the original investment proportions, which are the unknowns of the process. It is not therefore clear that the optimisation mechanism could cope with such a decision process and a more straightforward simulation approach, inspecting the results from several possible initial asset distributions, may have to be used.

6. Conclusion

This paper is an extension of the ideas in the 1991 AFIR paper. Hopefully, in years to come, it will be possible at a sensible cost to examine the cashflows of a pension scheme or life fund using a wide range of testpoints and at least 5,000 simulations, in order to determine on a consistent basis the relevant asset-mix of those funds, or to test the robustness of proposed financial control mechanisms.

Bearing in mind that the optimisation process will seek out weak points in the stochastic model being used, it will probably be necessary also to run the simulation suite on more than one stochastic model.

Therefore this paper is intended to move a few steps further down the path briefly described in 1991, hopefully pointing the way to further developments in the future.

BIBLIOGRAPHY

(1) C.D. DAYKIN & GB HEY, *Managing Uncertainty in a General Insurance company*, JIA 117, 173, 1990.
(2) P.E.B. FORD, *Cashflow matching using modified linear programming*, 2nd AFIR international colloquium, vol 3 page 301, 1991.
(3) R. KOCHERLAKOTA, E.S. ROSENBLUM & E.S.W. SHIU, *Algorithm for Cash-Flow Matching*, TSA XL, 477, 1988.
(4) A.D. WILKIE, *A stochastic investment model for actuarial use*, TFA 39, 341, 1986.
(5) A.J. WISE, *The Matching of Assets to Liabilities*, JIA 111, 445, 1984.