A GENERALIZATION OF THE FUZZY ZOOMING OF CASH FLOWS

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ABSTRACT

The concept of fuzzy zooming of cash flows introduced in (1) will be generalized in this paper, allowing for any division of the period of a cash flow into disjunctive partial periods which unite to the cash flow's total length of time and for the corresponding partial cash flows which unite to give the total cash flow.
In the introduction we shall clarify when and why fuzzy models of cash flows are preferable to stochastic models in classical probability theory. Later on, we shall derive some important results on equivalent fuzzy payments and on geometrically growing annuities which we shall use, after introducing equivalent two-payment cash flows, for interesting empirical analyses of geometrically growing annuities. We shall illustrate the excellent matching qualities of general fuzzy zoomings of the 5th order in the case of very long-term cash flows with different interest and growth rates. We shall see that these qualities remain quite robust even in the event of interest shocks during the cash flows.
Finally, we shall propose how generalized fuzzy zoomings could be practically applied for the description of "cash flows" of exchange indexes for the evaluation of assets.

KEY WORDS: Fuzzy arithmetics, triangular fuzzy numbers, equivalent fuzzy payment, generalized fuzzy zooming of the k-th order, equivalent two-payment cash flows, immunization, generalized cash-flow matching.

1. INTRODUCTION

In ordinary Boolean algebra an element is either contained or not contained in a given set (6), i.e. the element is either a full member or not a member, but certainly not partially a member, of the given set. Consequently, in traditional probability theory *probabilities* represent *uncertainty* but not *degrees of partial truths*, which are used for the *description of imprecision by degrees of membership*. Contrary to ordinary Boolean algebra, the transition of an element from the outside into a set, i.e. from non-membership to membership is not *abrupt* but *gradual.*

"As the complexity of a system increases our ability to make precise and yet significant statements about its behaviour diminishes..." (7). The increasing imprecision calls for a fuzzy treatment of the system. If the sources of uncertainty are non-statistical in nature, it is wrong in principle to deal with the problems that arise by traditional probability theory methods (6).

For such cases the tool of *fuzzy set theory* is at hand. If imprecision is the state of nature of a situation and the resulting uncertainty is *possibilistic* rather than *probabilistic*, then the situation is said to be *fuzzy* (3). *Fuzzy set theory* should then be used rather than probability theory.

If we denote a collection of objects $X = \{x\}$, a fuzzy set A in X is a set of ordered pairs

$$A = \{x, U_A(x)\} \ , \quad x \in X$$

where $U_A(x)$ is the "*grade of membership*" of x in A and $U_A : x \longrightarrow M$ is a function from X to the "*membership space*" M. The special fuzzy sets that are used in the fuzzy mathematics of finance are fuzzy numbers (4).

A fuzzy number is a fuzzy subset of the real line whose highest membership values are clustered around a given real number. The membership function is monotonic on both sides of this real number.

Arithmetics are defined on fuzzy numbers that take care of the fuzzy qualities required. Fuzzy numbers may, for example, be required to be associative and commutative on addition and multiplication and distributive on multiplication with regard to addition (5).

The remarks about fuzzy set theory are specially valid for fuzzy numbers with regard to describing complex financial, mainly long–term financial flowings off, (1), (4), i.e. the application of fuzzy arithmetics in finance aims at modeling situations described in vague or imprecise terms. Such imprecise terms are inherent in long–term cash flows. If we consider cash flows, the cash amounts due, the future interest rates and even the times of maturity often comprise inherent imprecision. Much of the uncertainty which is intrinsic in future cash amounts of cash flows and future interest rates is rooted in the imprecision of the underlying probabilities due to fuzziness of information or feelings on future developments. Fuzzy numbers provide a better framework than probability theory for modeling such problems of inherent imprecision (6).

Imprecision is even more inherent in predicting the development of a portfolio of shares. Fuzzy numbers should be used rather than probability theory when modeling such developments. A proposal for modeling the development of a stock exchange index for the evaluation of assets will be given in part 5 of this paper.

By the introduction of the "*equivalent fuzzy payment of a cash flow*" in part 2 and of an "*equivalent two-payments cash flow*" to a given equivalent fuzzy payment, and, last but not least, by introducing a "*generalized fuzzy zooming of the n-th order* in part 3, we shall introduce tools that lead to zoomings of very good matching for complex long-term inherently imprecise cash flows. These terms will be based on cash-flow durations and dispersions for which we shall prove in part 3 important specific properties that will be used in part 4. The test cash flow, which we shall expose to different kinds of "interest shocks" that we shall examine empirically in part 4, will grow geometrically.

As we shall see, the harder the "interest shock", the higher the order of generalized cash flow zooming needed for a sufficiently good description.

Generally, we can say that aside from the description of financial terms on which little information and inherent imprecision exists, general fuzzy zoomings can be successfully used to considerably reduce the complexity of long-term cash flows, getting reliable and informative results at the same time. A primary tool that can be used to arrive at these goals is "triangular fuzzy numbers" (TFN) that describe very well the word "about". Thus, if we say that in a vague situation a quantity has about the size x, a TFN with the highest membership value at x well describes that statement.

Summing up the recommendations for using fuzzy models versus stochastic models in finance, we can state that the more reliable and comprehensive statistics are the less significant are imprecision and fuzziness and the more reasonable it becomes to describe the situation via stochastic models. Stochastic models have a long tradition in financial theory. When, on the one hand, the information available is reliable and sufficient and, on the other hand, the inherent stochastic nature of the process is "safe" (with little or no inherent imprecision) stochastic models are and should be exclusively used.

On the other hand, it is doubtful if, for example, long-term cash flows with remarkably inherent imprecisions should be described in principle solely by stochastic models. In such cases, a fuzzy theoretical treatment should be considered (1), (4). An advantage of such a treatment is that calculations using fuzzy arithmetics with regard, for example, to vaguely known future rates and vaguely known liabilities are of a uniform nature. An example is given in (1).

In the "generalized fuzzy zooming" described below, a partial cash flow is replaced by a payment of a precise or vague amount at a vague point in time (time point as TFN).

The calculation procedures in fuzzy arithmetics are very different from those of classical probability theory (5). The fuzzy and stochastic theories do not, however, contradict but rather complement each other when it is not clear principally which theory should be used to describe a partially uncertain and partially imprecise situation.

2. The fuzzy zooming of cash flows

2.1. Fuzzy arithmetics and TFN

The theory of fuzzy numbers is developed and explained in the seminal paper of L.A. Zadeh (7) and in many books and papers like (1), (5), (8). Roughly speaking, a fuzzy number is characterized by a quadruplet of real numbers (a_1, a_2, a_3, a_4), where $a_1 < a_2 \leq a_3 < a_4$, and a continuous membership function $U_A(x) = U_A(x; a_1, a_2, a_3, a_4)$ that is equal to 0 for $x \leq a_1$ and $x \geq a_4$, equal to 1 for $a_2 \leq x \leq a_3$, strictly increasing for $a_1 \leq x \leq a_2$ and strictly decreasing for $a_3 \leq x \leq a_4$.

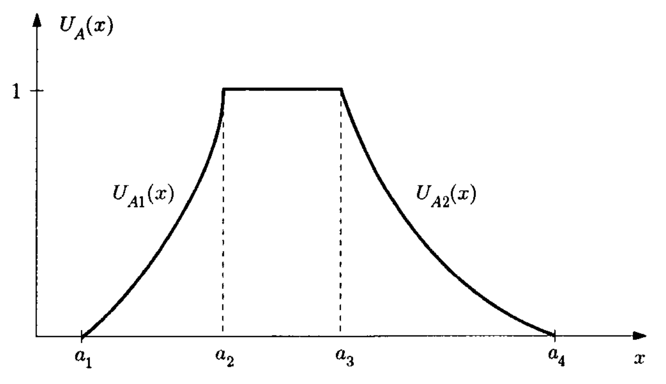


Fig. 1

The increasing part of $U_A(x)$ is denoted $U_{A1}(x)$ and its decreasing part $U_{A2}(x)$. On the set of fuzzy numbers a sum, difference, product and division are defined such that they fulfill required properties like keeping fuzzy numbers associative or commutative with respect to addition and multiplication. A recursive application n times of the product's operation leads to the n–th power of fuzzy numbers which is needed for example for the calculation of future values after n periods of time.

Since – as we have mentioned in part 1 – for the long-term cash flows dealt with in this paper, the information is sparse and the imprecision relatively great, we restrict ourselves to especially simple fuzzy numbers, the so–called *"Triangular Fuzzy Numbers" (TFN)* for which a_2 is equal to a_3 and $U_{A1}(x)$ and $U_{A2}(x)$ are linear functions.

Adding but not multiplying TFN again results in TFN. Therefore, future values or present values as they were calculated for example in (1) are no longer TFN.

Insurance and financial quantities with inherent imprecision elements that can only be estimated with a very large relative standard deviation in classical probability theory qualify for modeling by TFN. Such quantities are for example:

- future cash flow amounts of a project;
- claims with deficient information;
- present values of IBNR claims for determination of IBNR reserves;
- future exchange rates;
- future interest rates;
- future quotation of shares.

Aside from describing the development of such quantities in the future, the TFN can be successfully used to reduce the complexity of cash flows (1).

2.2. The equivalent fuzzy payment of a cash flow

This concept was introduced in (1) and uses the basic terms of classical financial mathematics "*duration*" and "*dispersion*" and their interpretation in the concept of fuzzy arithmetics.

The duration of a cash flow can be interpreted as the time distance between the present time and the centre of gravity in time of the present values of the cash flow's future payments (2). The duration is an important characteristic of the cash flow. If we look, for example, for an *immunization* of debits by a cash flow of a bond portfolio, we should at least fulfill:

a) the *present value principle*, which demands the present value of the bond cash flow to be at least as large as the sum of present values of the debits;

b) the *duration principle*, which demands the absolute values of the durations of credits and debits to be equal.

These two principles effect a relatively high level of insensitivity in providing security for debits by credits with regard to changes of the interest curve parallel to the time axis (2).

The dispersion of a cash flow characterizes the deviation of a cash flow from a zero coupon bond of the same present value and duration (2). The dispersion's dimension of 2 is that of a variance and not of its square root, the standard deviation, in probability theory.

Let us consider a cash flow with payments $G_k \rightarrow$ (index) at points of time t_k, $k = 1, \ldots, n$, where $0 < t_1 < t_2 < \ldots < t_n$. We denote the present time $t_0 = 0$ and the inverse inflation factor, which we assume to be constant, $v = (1+i)^{-1} = 1/r$.

The present value PV, the duration D and the dispersion M^2 are defined as follows:

$$PV = \sum_{k=1}^{n} G_k \nu^{t_k}$$

$$D = \frac{1}{PV} \cdot \sum_{k=1}^{n} t_k G_k \nu^{t_k}$$

$$M^2 = \frac{1}{PV} \cdot \sum_{k=1}^{n} t_k^2 G_k \nu^{t_k} - D^2 .$$

If payments are due at the end of each time unit, i.e. if $t_k = k$, $k = 1, \ldots, n$, then we arrive at

$$PV = \sum_{k=1}^{n} G_k \cdot \nu^k$$

$$D = \frac{1}{PV} \cdot \sum_{k=1}^{n} k \cdot \nu^k \cdot G_k$$

$$M^2 = \frac{1}{PV} \cdot \sum_{k=1}^{n} k^2 G_k \cdot \nu^k - D^2 .$$

This simplifying special case can be generally used in practice.

The *convexity* Co can be calculated from D and M^2 in the following way:

$$\text{Co} = \nu^2 \cdot (M^2 + D^2 + D) \text{ (DDK-identity) (2).}$$

The convexity is another basic term in financial mathematics. The *convexity principle* — which demands the convexities of credits and debits to be equal — in addition to the present value principle and the duration principle, causes even large changes of the relative interest rate levels to result in only a small deviation of the present value of all credits from the present value of all debits. The convexity principle increases the level of immunization, at least in the case of parallel shifts of the yield curve (2).

Duration and convexity have an important property. The duration and convexity of a portfolio of securities is equal to the present-value-weighted mean of the durations and convexities, respectively, of the individual securities. This property does not exist for the dispersion. A portfolio's dispersion can, however, be calculated by the DDK identity by using the portfolio's duration and convexity.

For a known cash flow G_t and a real inflation factor r we define as an *equivalent fuzzy payment* the ordered pair

$$\left\{PV \cdot r^D;\ F\left[D - \sqrt{M^2}, D, D + \sqrt{M^2}\right]\right\}$$

where $r = 1/v = 1 + i$ is the inflation factor over a unit of time which is usually one year. F is a TFN characterized by the three points $a_1 = D - \sqrt{M^2}$ $a_2 = a_3 = (a_1 + a_4)/2 = D$ $a_4 = D + \sqrt{M^2}$

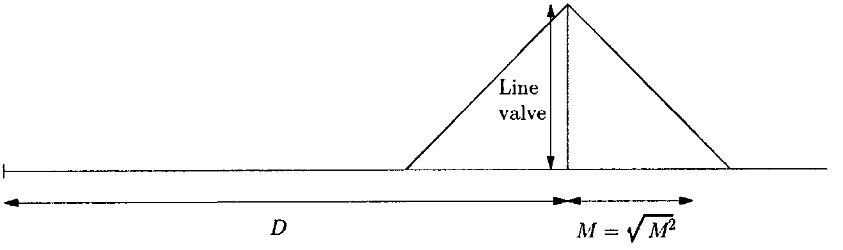


Fig. 2

In the general case when G_t is only vaguely known and r is a fuzzy number D and M^2 are fuzzy and calculated by methods similar to those mentioned in (1). The second components of the TFN, F, are then defined in such a way that a_t is the lower end of the interval of definition of $D - \sqrt{M^2}$ and a_4, the upper end of the interval of definition of $D + \sqrt{M^2}$. In the real case the two definitions coincide.

Two cash flows with identical equivalent fuzzy payments behave similarly with respect to changes in present values that are a consequence of smooth changes in levels of interest rates. If, therefore, one cash flow is on the assets and the other on the absolute values of the liabilities, then their common equivalent fuzzy payment ensures stability, i.e. the liabilities will also be nearly covered by the assets in case of smooth changes in interest rates.

2.3. The fuzzy zooming

An equivalent fuzzy payment usually well describes the changes of a cash flow in case of *parallel changes of the interest rate curve along the time axis*. In the case of non-parallel shiftings of the interest rate curve a description of a cash flow by an equivalent fuzzy payment — which is a fuzzy zooming of the first order — is often insufficient.

To guarantee in such a case that future liabilities will be more or less covered by assets we must approach a cash flow matching. This is done by analogous fuzzy zoomings of a higher order of the cash flows on liabilities and on assets where equivalent fuzzy payments are defined on appropriately defined partial cash flows (1).

A fuzzy zooming of the first order is a fuzzy payment equivalent to the total cash flow.

A fuzzy zooming of the second order is a pair of fuzzy payments characterizing the partial cash flows on the time intervals $[0, D]$ and $[D, n]$, where D is the cash flow's duration and n its life time.

A corresponding subdivision of the two partial intervals into four partial intervals and a corresponding assignment of four equivalent fuzzy payments leads to *a fuzzy zoomirlg of the fourth order*. Fuzzy zoomings of higher orders are defined correspondingly.

With every growing order a fuzzy zooming we move more and more from immunization to perfect cash flow matching (1). Though the expenditure on time for calculations becomes greater, comprehension of the total cash flow improves. The kind and size of changes in the interest curve to be expected and the quality of information necessitated may dictate the order of fuzzy zooming to be applied on long–term cash flows with a large expected number of payments.

The difference in the quality of information of generalized fuzzy zoomings of the first and fifth order, describing cash flows under different interest rate developments will be illustrated in numerical examples in part 4.

The division of cash flows and partial cash flows by durations is plausible but for certain applications it may be arbitrary. We shall therefore generalize the fuzzy zooming procedure in the next part.

3. Generalized fuzzy zooming

3.1. Definition of generalized fuzzy zooming

Let $t_0 = 0$ be the present time and t_n the time of maturity of the cash flow's last payment. Payments $G_1, \ldots, G_n$ are due at time points $t_1, \ldots, t_n$.

In *a generalized fuzzy zooming of the k-th order*, we calculate the equivalent fuzzy payments of k disjunctive partial cash flows which unite to the original cash flow. A generalized fuzzy zooming of the k-th order consists therefore of the following sequence of k equivalent fuzzy payments:

$$(PV_1 \cdot r^{D_1}, F_1), \ldots, (PV_k \cdot r^{D_k}, F_k) \ ,$$

where $F_1, \ldots, F_k$ are the TFNs defined by the durations and dispersions of the k disjunctive partial cash flows. $PV_1, \ldots, PV_k$ are the present values of the cash flows (real or fuzzy numbers) and r is the interest factor (real or fuzzy).

3.2. Important properties

Let us decompose a cash flow characterized by the duration D_P and dispersion M_P^2 into two partial cash flows with durations and dispersions D_1, M_1^2, and D_2, M_2^2, respectively. Using the DDK identity and the property of duration and convexity for the calculation of a portfolio's duration and convexity as mentioned towards the end of section 2.2, the following two decomposition functions can be derived (1):

$$D_2 = \frac{D_p \cdot PV_p - D_1 \cdot PV_1}{PV_p - PV_1}$$

$$M_2^2 = \frac{(M_p^2 + D_p^2) \cdot PV_p - (M_1^2 + D_1^2) \cdot PV_1}{PV_p - PV_1} - D_2^2$$

where PV_p and PV_1 are present values of the respective cash flows. We now turn to the important question of how shiftings of cash flows along the time axis influence equivalent fuzzy payments. An answer is given by the

Transformation Theorem for Equivalent Fuzzy Payments:

If a cash flow is shifted by m years into the future its equivalent fuzzy payment is transformed as follows:

$$\left\{PV \cdot r^D, F[D-\sqrt{M^2}, D, D+\sqrt{M^2}]\right\}$$

$$\downarrow$$

$$\left\{PV \cdot r^{D+m}, F\left[D+m-\sqrt{M^2}, D+m, D+m+\sqrt{M^2}\right]\right\} .$$

PROOF

We have to show that

a) $D \rightarrow D+m$

b) $M^2 \rightarrow M^2$, i.e. that M^2 remains invariant with respect to transformation in time.

a) $$D = \frac{\sum t_k G_k \nu^{t_k}}{\sum G_k \nu^{t_k}}$$

$$\begin{aligned} D_{(m)} &= \frac{\sum (t_k+m) G_k \nu^{t_k+m}}{\sum G_k \nu^{t_k+m}} = \frac{\sum t_k G_k \nu^{t_k} + m \sum G_k \nu^{t_k}}{\sum G_k \nu^{t_k}} \\ &= D+m \quad \text{q.e.d.} \end{aligned}$$

b) $$\begin{aligned} M^2_{(m)} &= \frac{\sum (t_k+m)^2 G_k \nu^{t_k+m}}{\sum G_k \nu^{t_k+m}} - D^2_{(m)} \\ &= \frac{\sum t_k^2 G_k \nu^{t_k} \cdot \nu^m}{\sum G_k \nu^{t_k} \cdot \nu^m} + 2m \frac{\sum t_k G_k \nu^{t_k} \cdot \nu^m}{\sum G_k \nu^{t_k} \cdot \nu^m} + \\ &+ m^2 \frac{\sum G_k \nu^{t_k+m}}{\sum G_k \nu^{t_k+m}} - (D+m)^2 \\ &= M^2 + D^2 + 2mD + m^2 - D^2 - 2mD - m^2 \\ &= M^2 \quad \text{q.e.d.} \end{aligned}$$

This transformation theorem shows that an equivalent fuzzy payment transforms into the future like a usual payment by compounding interest rates.

In part 4 we shall analyze empirical examinations on geometrically growing annuities. For this purpose we shall restrict ourselves to the most important special case of generalized fuzzy zooming by setting $t_i = i$, $i = 1, \ldots, n$ and derive the following results:

Assertion:

$$S = (1x)^2 + \ldots + (nx^n)^2 =$$

$$= \frac{x^2}{(x^2-1)^2} \cdot \left\{ x^{2n+2} \cdot n \cdot \left[n + 2 - \frac{2x^2}{x^2-1} \right] - x^{2n} \cdot (n+1) \cdot \right.$$

$$\left. \cdot \left[(n+1) - \frac{2x^2}{x^2-1} \right] + 1 - \frac{2x^2}{x^2-1} \right\} .$$

PROOF

$$(k \cdot x^{2k+1})' = k \cdot (2k+1) \cdot x^{2k} = 2k^2 \cdot x^{2k} + k \cdot x^{2k}$$

$$\Rightarrow (k \cdot x^k)^2 = k^2 \cdot x^{2k} = \frac{1}{2} \left((k \cdot x^{2k+1})' - k \cdot x^{2k} \right)$$

$$\Rightarrow S = \sum_{k=1}^{n} (kx^k)^2 = \frac{1}{2} \left[\left(\sum_{k=1}^{n} kx^{2k+1} \right)' - \sum_{k=1}^{n} kx^{2k} \right] = \frac{1}{2} (A' - B) .$$

For $x^2 = w$ we have

$$B = \sum_{k=1}^{n} kw^k = w \sum_{k=1}^{n} \frac{d}{dw} (w^k) = w \frac{d}{dw} \left(\sum_{k=1}^{n} w^k \right) =$$

$$= w \frac{d}{dw} \left(\frac{w^{n+1} - 1}{w - 1} - 1 \right) = w \frac{nw^{n+1} - (n+1)w^n + 1}{(w-1)^2}$$

$$\Rightarrow B = \frac{x^2}{(x^2-1)^2} \left(nx^{2n+2} - (n+1)x^{2n} + 1 \right)$$

$$A = \sum_{k=1}^{n} kx^{2k+1} = x \cdot \sum_{k=1}^{n} kx^{2k} = x \cdot B$$

$$\Rightarrow S = 0,5(x'B + xB' - B) = 0,5 \cdot x \cdot B'$$

$$= \frac{x^2}{(x^2-1)^2} \left\{ x^{2n+2} \cdot n \cdot \left(n + 2 - \frac{2x^2}{x^2-1} \right) - \right.$$

$$\left. - x^{2n} \cdot (n+1) \cdot \left(n + 1 - \frac{2x^2}{x^2-1} \right) + 1 - \frac{2x^2}{x^2-1} \right\} \quad \text{q.e.d.}$$

We now concentrate on *a geometrically increasing cash flow (GIC) which is a geometrically increasing annuity (GIA)*:

time:	0	1	2	...	$n-1$	n
cash flow:		$1+g$	$(1+g)^2$	...	$(1+g)^{n-1}$	$(1+g)^n$

THEOREM. –
The duration of a GIA is

$$D_{\text{GIA}} = \frac{1+j}{j} - \frac{n \cdot w^n}{1-w^n} .$$

The dispersion of a GIA is

$$M^2_{\text{GIA}} = \frac{j}{1-w^n} \cdot \frac{w}{(w-1)^2} \left\{ n \cdot w^{n+1} \left(n+2-\frac{2w}{w-1} \right) - \right.$$
$$\left. -(n+1) \cdot w^n \left(n+1-\frac{2w}{w-1} \right) + 1 - \frac{2w}{w-1} \right\} - D^2_{\text{GIA}} ,$$

where

$$w = \frac{1+g}{1+i}, \quad j = \frac{1-w}{w} \quad \textit{and } 1+i \quad \textit{is a constant inflation factor.}$$

PROOF For geometrically increasing annuities it is practical and common to use an assisting factor $w = (1+g)/(1+i)$ and the corresponding interest rate $j = (1-w)/w$ for the calculation of present values. The assisting terms w and j can be formally used like the corresponding terms for constant annuities:

Constant annuities	$\rightarrow$	geometrically increasing annuities
i	$\rightarrow$	j
$v = 1/1+i$	$\rightarrow$	$w = 1/1+j$

For constant annuities the duration can be calculated directly and easily [2, p. 129-130]. The result is

$$D = \frac{1+i}{i} - \frac{n \cdot \nu^n}{1-\nu^n} ,$$

which leads to

$$D_{\text{GIA}} = \frac{1+j}{j} - \frac{n \cdot w^n}{1-w^n} \quad \text{q.e.d.}$$

By using the assertion just proved, we arrive for $w = x^2$ at:

$$
\begin{aligned}
M^2_{\text{GIA}} &= \frac{1}{PV}\,(1^1 \cdot w + 2^2 w^2 + \ldots + n^2 w^n) - D^2_{\text{GIA}} \\
&= \left(w \cdot \frac{1-w^n}{1-w}\right)^{-1} \left((1\cdot\sqrt{w})^2 + (2\cdot\sqrt{w^2})^2 + \ldots + (n\cdot\sqrt{w^n})^2\right) - D^2_{\text{GIA}} \\
&= \frac{j}{1-w^n}\,\frac{w}{(w-1)^2} \cdot \left\{ w^{n+1} \cdot n \cdot \left(n + 2 - \frac{2w}{w-1}\right) + \right. \\
&\quad - w^n \cdot (n+1) \cdot \left(n + 1 - \frac{2w}{w-1}\right) + \\
&\quad \left. + 1 - \frac{2w}{w-1} \right\} - D^2_{\text{GIA}} \quad \text{q.e.d.}
\end{aligned}
$$

With the help of *equivalent two-payments cash flows (ETPC)* — which we shall introduce in the following — this theorem and the transformation theorem will allow us to analyze empirically in part 4 matching qualities of fuzzy zoomings of geometrically increasing cash flows.

Before we turn to the introduction of ETPC we feel it in place to note an interesting property of M^2_{GIA}:

Assertion:

The dispersion of a geometrically increasing annuity adopts an extremum for $g = i$, i.e. for $w = (1+g)/(1+i) = 1$ and $j = (1-w)/w = 0$.

PROOF

$$
\begin{aligned}
M^2(w) &= \frac{\sum k^2 w^k}{\sum w^k} - \frac{\left(\sum k \cdot w^k\right)^2}{\left(\sum w^k\right)^2} \\
(M^2(w))' &= \frac{\left(\sum k^3 w^{k-1}\right) \cdot \left(\sum w^k\right) - \left(\sum k^2 w^k\right) \cdot \left(\sum k w^{k-1}\right)}{\left(\sum w^k\right)^2} + \\
&\quad - \frac{2 \cdot \left(\sum k \cdot w^k\right) \cdot \left(\sum k^2 w^{k-1}\right) \cdot \left(\sum w^k\right)^2}{\left(\sum w^k\right)^4} + \\
&\quad + \frac{\left(\sum k \cdot w^k\right)^2 \cdot 2\left(\sum w^k\right) \cdot \left(\sum k w^{k-1}\right)}{\left(\sum w^k\right)^4} \\
(M^2(w=1))' &= \frac{n \cdot \left(\sum k^3\right) - \left(\sum k^2\right) \cdot \left(\sum k\right)}{n^2} + \\
&\quad - \frac{2 \cdot \left(\sum k\right) \cdot \left(\sum k^2\right) \cdot n^2 - \left(\sum k\right)^3 \cdot 2n}{n^4} \,.
\end{aligned}
$$

By inserting

$$\sum_{k=1}^{n} k = \frac{n \cdot (n+1)}{2}, \ \sum_{k=1}^{n} k^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$$

and

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

we arrive at

$$\left(M^2(w=1)\right)' = (n-1)^2 \cdot 0 = 0 \ . \quad \text{q.e.d.}$$

$$\underline{M^2(w=1)} = \frac{1}{PV} \cdot \left(\sum_{k=1}^{n} k^2\right) - \left[\frac{1}{PV} \cdot \left(\sum_{k=1}^{n} k\right)\right]^2$$

$$= \frac{1}{n} \cdot \frac{n \cdot (n+1) \cdot (2n+1)}{6} - \frac{1}{n^2} \cdot \frac{n^2 \cdot (n+1)^2}{4} =$$

$$= \underline{\frac{(n+1)(n-1)}{12}} \ .$$

Direct calculations and computer simulations for varying n show that for $h > 0$

$$M^2(1-h) < M^2(w=1) \quad \text{and}$$

$$M^2(1+h) < M^2(w=1) \ .$$

The extremum that the dispersion of a GIA adopts for $g = i$ is thus a maximum. We can therefore conclude:

$$M^2_{\text{GIA}} \leq \frac{(n+1)(n-1)}{12} \ .$$

We have moreover good reasons to assume that for $w < 1$, i.e. for $g < i$, $M^2_{\text{GIA}}(g)$ is monotonically increasing with increasing g and for $w > 1$, i.e. for $g > i$, M^2_{GIA} is monotonically decreasing with increasing g.

Comment

When introducing the dispersion in paragraph 2.2, we mentioned that it characterizes the deviation of a cash flow from a zero coupon bond of the same present value and duration. This very restrictive property made it appear reasonable to use the dispersion's square root as a deviation measure when defining the equivalent fuzzy payments.

The fact that $M^2_{\mathrm{GIA}}(w)$ adopts a maximum for $w = 1$, i.e. when the annuity increases geometrically exactly with inflation ($g = i$), is rather unexpected. $g = i$ means a static rather than a dynamic situation since the present values of all payments are identical. In such a case we would expect a measure of deviation that characterizes the cash flow to assume a minimum rather than a maximum value. A dispersion in financial mathematics is not like the variance — its analogous term in the theory of probability — a measure of deviation that we can understand and interpret in every situation. When mentioning in section 2.2 the present value principle, the duration principle and the convexity principle for immunization, we left out on purpose the dispersion principle because dispersion is a term that can only be comprehended as a term of deviation in very special and restrictive cash flow situations. As we shall illustrate in part 4, the dispersion is, however, very useful in defining equivalent fuzzy payments and using them for generalized fuzzy zoomings of higher orders.

3.3. The equivalent two-payments cash flow (ETPC)

Till now the starting point has always been a cash flow to be characterized by an equivalent fuzzy payment or by a generalized fuzzy zooming. In this section our starting point is a given equivalent fuzzy payment. Infinitely many cash flows with this specific equivalent fuzzy payment exist. For our empirical examination it is practical to pick a specially simple representative out of the infinite number of cash flows at disposal. We call this representative the equivalent two-payment cash flow, ETPC. ETPC, like every other cash flow in the array of cash flows with the same equivalent fuzzy payment, includes an inherent imprecision, at least in time, due to lack of information or for other reasons, since it can be represented by an equivalent fuzzy payment.

ETPC consists of the following two payments:

$$P_1 = \frac{1}{2} PVr^{D-\Delta t} \quad \text{at time point } D - \Delta t \quad \text{and}$$
$$P_2 = \frac{1}{2} PVr^{D+\Delta t} \quad \text{at time point } D + \Delta t \ ,$$

whereby the present value of the original cash flow PV and the interest rate factor r can be fuzzy numbers. Also the powers of r can be defined to be fuzzy numbers [6, p. 52]. For the sake of simplicity, we limit ourselves here to the case of real amounts of payment, i.e., to real present value PV and a real interest rate factor r.

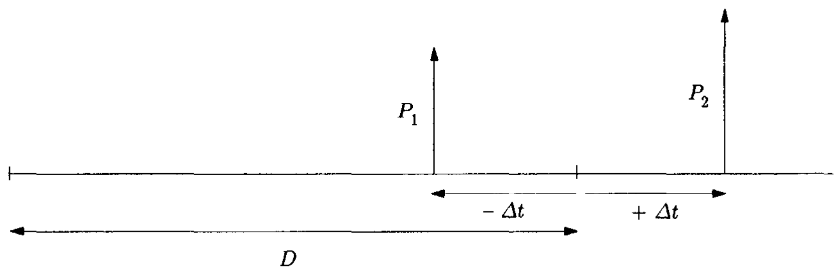


Fig. 3

THEOREM. –
a) $D_{\text{ETPC}} = D$
b) $M^2_{\text{ETPC}} = (\Delta t)^2$.

PROOF

$$D_{\text{ETPC}} = \frac{\sum t_k G_k \nu^{t_k}}{\sum G_k \nu^{t_k}} =$$

$$= \frac{(D-\Delta t)\frac{1}{2}\, PV r^{D-\Delta t}\cdot \nu^{D-\Delta t} + (D+\Delta t)\frac{1}{2}\, PV r^{D+\Delta t}\cdot \nu^{D+\Delta t}}{\frac{1}{2}\, PV r^{D-\Delta t}\nu^{D-\Delta t} + \frac{1}{2}\, PV r^{D+\Delta t}\nu^{D+\Delta t}}$$

$$= \frac{\frac{1}{2}(D-\Delta t) + \frac{1}{2}(D+\Delta t)}{\frac{1}{2}+\frac{1}{2}} = D$$

since $r = 1 + i = 1/v$

$$M^2_{\text{ETPC}} = \frac{\sum t_k^2 G_k \nu^{t_k}}{\sum G_k \nu_k^t} - D^2_{\text{ETPC}}$$

$$= \frac{(D-\Delta t)^2\cdot\frac{1}{2} + (D+\Delta t)^2\cdot\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}} - D^2$$

$$= \frac{1}{2}\left(D^2 - 2d\cdot(\Delta t) + (\Delta t)^2 + D^2 + 2D\cdot(\Delta t) + (\Delta t)^2 - 2D^2\right) =$$

$$= (\Delta t)^2 \quad \text{q.e.d.}$$

We proceed now to the empirical examinations of the general fuzzy zooming's matching qualities.

4. Empirical examinations on geometrically increasing annuities

We shall examine on GIA the matching quality of generalized fuzzy zoomings of the 1st order and 5th order from the following starting position:

a) Level of yield: 10% for the whole observed long–term period of 100 years (flat yield curve).

b) Growth rate: 5% p.a.

c) Kind of payment: At year's end.

We shall now expose this convenient GIA cash flow to four different kinds of "interest rate shocks":

First kind of interest rate shock:
Parallel shifting of the interest rate curve from 10% to 15%.

Second kind of interest rate shock:
Parallel shifting of the interest rate curve from 10% to 5%.

Third kind of interest rate shock:
Payments due before the 51st year are subject to an interest rate of 5%; those afterwards to an interest rate of 15%.

Fourth kind of interest rate shock:
Payments due before the 51st year are subject to an interest rate of 15%; those afterwards to an interest rate of 5%.

These examples were chosen according to the following criteria:

- Very long–term cash flow (100 years).
- Very large fluctuations of the level of yield.
- Very large deviations from a parallel shifting of the flat yield curve under the third and fourth kind of interest rate shocks.

For our analysis we choose the ETPCs as specially simple cash flows representing the required generalized zooming characteristics.

The generalized fuzzy zooming of the 1st order is the equivalent fuzzy payment of the following characteristics:

$$\begin{aligned} i &= 0,1 \\ g &= 0,05 \\ w &= 0,954545 \\ j &= 0,047619 \\ D &= 21,0366 \\ M^2 &= 364,7276 \\ PV &= 20,800 \ . \end{aligned}$$

The time points of the corresponding ETCP are

$$\begin{aligned} t_1 &= D - \sqrt{M^2} = 21,0366 - 19,0978 = 1,9388 \\ t_2 &= D + \sqrt{M^2} = 21,0366 + 19,0978 = 40,1344 \end{aligned}$$

and the corresponding payments

$$\begin{aligned} P_1 &= 10,4 \cdot r^{t_1} = 12,510 \\ P_2 &= 10,4 \cdot r^{t_2} = 476,765 \ . \end{aligned}$$

By using the decomposition functions described in section 3.2 we can easily proceed to a generalized fuzzy zooming of the 2nd order. We move, however, directly to a generalized fuzzy zooming of the 5th order and choose a division of the total period of 100 years into 5 equal, disjunctive partial periods of 20 years each, the first including years 1–20, the second 21–40 and so on. For the generalized fuzzy zooming characteristics of the first partial interval, we arrive at:

$$\begin{aligned} n &= 20 \\ D &= 8,9751 \\ M^2 &= 31,8554 \\ PV &= 12,7177 \ . \end{aligned}$$

The values for the corresponding ETCP are:

$$\begin{aligned} t_1 &= D - \sqrt{M^2} = 3,3311 \\ t_2 &= D + \sqrt{M^2} = 14,6191 \\ P_1 &= 6,3588 \cdot r^{t_1} = 8,7349 \\ P_2 &= 6,3588 \cdot r^{t_2} = 25,6157 \ . \end{aligned}$$

For the calculation of the characteristics of the subsequent 4 partial intervals we make use of the transformation theorem for equivalent fuzzy payments and of the results for duration and dispersion for geometrically increasing annuities. We then arrive easily at the following results for the time points and amounts of payment that determine the 5 respective ETPCs:

Table 1

Time Interval	ETPC-time points	ETPC-payments
1–20	3.3311	8.7349
	14.6191	25.6155
21–40	23.3311	23.1765
	34.6191	67.9656
41–60	43.3311	61.4941
	54.6191	180.3330
61–80	63.3311	163.1621
	74.6191	478.4771
81–100	83.3311	432.9176
	94.6191	1269.5420

We now calculate the present values of the cash flows subject to the four kinds of interest rate shocks by zoomings of 1st order and of 5th order and compare them to the corresponding present values of the original cash flows. This comparison will indicate the matching qualities of zoomings of different orders.

Table 2 – Present values

Kind of interest rate shock	Original cash flow	1st order zooming	5th order zooming
1st kind	10.49	11.29	10.51
2nd kind	100.00	78.66	99.89
3rd kind	50.11	78.66	47.51
4th kind	60.39	11.29	62.88

An exact calculation of these results and charts of the original and ETPC cash flows of the 1st respectively 5th order zoomings are given in the appendix.

As can be seen, the present values calculated by the 1st order generalized zooming are not good approximations of the exact values under the hard conditions of these shocks, even in the case of parallel shifting in the second kind of interest rate shock, whereas the 5th order generalized zooming results in good approximations even under the very rough conditions it is exposed to in the third and fourth kind of interest rate shocks.

We can assume that 5th order zoomings produce good approximations and — if easily computable — can well be used for all kinds of characteristics of long-term cash flows like, for example, pension funds.

Comment

In part 5 of (1) we calculated special fuzzy zoomings of different orders in a special life annuity example. Now we not only extend our considerations to general fuzzy zoomings of different orders and to geometrically increasing annuities but we chose a special and simple representative in the class of generalized fuzzy zoomings of a given order by introducing in section 3.3 via equivalent fuzzy payments the equivalent two-payment cash flows which we inserted into the generalized zooming procedure. Tools for using a general fuzzy approach in a generalized fuzzy zooming of the k-th order are developed in this paper and in (1).

5. Modeling of insecure assets

In (1) it was shown how insecure future interest rates can be modeled by triangular fuzzy numbers and how they can be used for the calculation of present values.

Often, however, the amounts of payment as well as the times of payment and, of course, the interest rates can be defined as TFNs. In such cases the corresponding cash flow terms are inherently imprecise due to fuzziness of information or feelings on future developments, as discussed in part 1. Examples are market indexes of shares, long-term insurance covers or cash flows resulting from complicated projects. In all these cases the generalized fuzzy zooming of cash flows based on triangular fuzzy numbers provides a comprehensive frame of modeling.

We shall restrict ourselves here to indicating a proposal on how to model on the assets' side the future development of a stock market index that takes into account the reinvestment of dividends (e.g. the DAX index in Germany). Since an investment in such an index has no real cash flow, an artificial cash flow must be constructed for the generalized fuzzy zooming. We assume for the artificial cash flow that the market index at the beginning of a period of observation is sold during that period. Starting from a given generalized fuzzy zooming of the liabilities of a portfolio, we are able to define a partition of the period of observation into time intervals for the assets and calculate for them the generalized fuzzy zooming of the same order as for the liabilities.

We propose that the relative amount of nominal value of the stock market index which is sold during each time interval of the zooming is proportional to the length of this interval. For example, we assume for a generalized fuzzy zooming of the 4th order, where the period of observation is divided into four time intervals of equal length, that during each time interval of 5 years 25% of the original stock market index is sold.

By this method, with the help of fuzzy arithmetics, we can calculate the generalized fuzzy zooming of mixed portfolios which contain both stock market indexes and bonds. The generalized fuzzy zooming supplies a valuable tool to match such mixed assets with vaguely known future liabilities (e.g. long–term annuity liabilities) of the same generalized fuzzy zooming order and division of the period of observation into partial time intervals. A problem is to develop a matching algorithm that selects securities matching the vaguely known liabilities in a way that minimizes costs.

Returning to the stock market index, we register that the prices P are often well modeled as *a Geometric Brownian Motion* (9) that is described by the following stochastic differential equation:

$$\frac{dP}{P} = k \cdot dt + s \cdot dz \ ,$$

where k and s are constants, t is the time and z a zero drift and unit variance Wiener process. Let P_0 and P_t be the "prices" of the stock market index at the beginning of the period of observation and at time t in that period. From the theorem of Itô follows that $\ln(P_t/P_0)$ is normally distributed with expected value $(k - s^2/2) \cdot t$ and variance $s^2 \cdot t$. s and k must be estimated.

For modeling the duration and dispersion of the equivalent fuzzy payment in a partial time interval, we assume that one third of the stock market index sales during that partial time interval is sold at the beginning, one third at the expected value and one third at the end of the respective partial time interval. A TFN $(a_1, a_2 = a_3, a_4)$ can then be used where $a_2 = a_3$ is the expected sales value in time, a_1 a lower quantile and a_4 the respective upper quantile of the log–normal distribution function. The fact that the variance $s^2 \cdot t$ of $\ln(P_t/P_0)$ is proportional to time t results in increasing wideness of TFN with increasing t. The definition of the TFN depends strongly, moreover, on the matching goals that are to be achieved. If, for example, a rather riskless matching strategy is desired, the maximum of the membership function in each general fuzzy zooming interval should be assumed to be close to the respective lower quantile.

APPENDIX

TIME	PAYMENT		PV 15%		PV 5%	5%↔15%	15%↔5%
0	0	0	0	0	0	0	0
1	0	1.05	0	0.913043	1	1	0.913043
2	0	1.1025	12.5	0.833648	1	1	0.833648
3	8.7349	1.157625	0	0.761157	1	1	0.761157
4	0	1.215506	0	0.69497	1	1	0.69497
5	0	1.276282	0	0.634538	1	1	0.634538
6	0	1.340096	0	0.57936	1	1	0.57936
7	0	1.4071	0	0.528981	1	1	0.528981
8	0	1.477455	0	0.482983	1	1	0.482983
9	0	1.551328	0	0.440984	1	1	0.440984
10	0	1.628895	0	0.402638	1	1	0.402638
11	0	1.710339	0	0.367626	1	1	0.367626
12	0	1.795856	0	0.335658	1	1	0.335658
13	0	1.885649	0	0.306471	1	1	0.306471
14	0	1.979932	0	0.279821	1	1	0.279821
15	25.6155	2.078928	0	0.255489	1	1	0.255489
16	0	2.182875	0	0.233272	1	1	0.233272
17	0	2.292018	0	0.212988	1	1	0.212988
18	0	2.406619	0	0.194467	1	1	0.194467
19	0	2.52695	0	0.177557	1	1	0.177557
20	0	2.653298	0	0.162117	1	1	0.162117
21	0	2.785963	0	0.14802	1	1	0.14802
22	0	2.925261	0	0.135149	1	1	0.135149
23	23.1765	3.071524	0	0.123397	1	1	0.123397
24	0	3.2251	0	0.112667	1	1	0.11266
25	0	3.386355	0	0.102869	1	1	0.102869
26	0	3.555673	0	0.093924	1	1	0.093924
27	0	3.733456	0	0.085757	1	1	0.085757
28	0	3.920126	0	0.0783	1	1	0.0783
29	0	4.116136	0	0.071491	1	1	0.071491
30	0	4.321942	0	0.065275	1	1	0.065275
31	0	4.538039	0	0.059598	1	1	0.059598
32	0	4.764941	0	0.054416	1	1	0.054416
33	0	5.003189	0	0.049684	1	1	0.049684
34	0	5.253348	0	0.045364	1	1	0.045364
35	67.9656	5.516015	0	0.041419	1	1	0.041419
36	0	5.791816	0	0.037817	1	1	0.037817
37	0	6.081407	0	0.034529	1	1	0.034529

continued

Continued

TIME	PAYMENT		PV 15%		PV 5%	5%↔15%	15%↔5%
38	0	6.385477	0	0.031526	1	1	0.031526
39	0	6.704751	0	0.028785	1	1	0.028785
40	0	7.039989	476.8	0.026282	1	1	0.026282
41	0	7.391988	0	0.023997	1	1	0.023997
42	0	7.761588	0	0.02191	1	1	0.02191
43	61.4941	8.149667	0	0.020005	1	1	0.020005
44	0	8.55715	0	0.018265	1	1	0.018265
45	0	8.985008	0	0.016677	1	1	0.016677
46	0	9.434258	0	0.015227	1	1	0.015227
47	0	9.905971	0	0.013903	1	1	0.013903
48	0	10.40127	0	0.012694	1	1	0.012694
49	0	10.92133	0	0.01159	1	1	0.01159
50	0	11.4674	0	0.010582	1	1	0.010582
51	0	12.04077	0	0.009662	1	0.009662	1
52	0	12.64281	0	0.008222	1	0.008222	1
53	0	13.27495	0	0.008055	1	0.008055	1
54	0	13.9387	0	0.007354	1	0.007354	1
55	180.333	14.63563	0	0.006715	1	0.006715	1
56	0	15.36741	0	0.006131	1	0.006131	1
57	0	16.13578	0	0.005598	1	0.005598	1
58	0	16.94257	0	0.005111	1	0.005111	1
59	0	17.7897	0	0.004667	1	0.004667	1
60	0	18.67919	0	0.004261	1	0.004261	1
61	0	19.61315	0	0.00389	1	0.00389	1
62	0	20.5938	0	0.003552	1	0.003552	1
63	163.1621	21.62349	0	0.003243	1	0.003243	1
64	0	22.70467	0	0.002961	1	0.002961	1
65	0	23.8399	0	0.002704	1	0.002704	1
66	0	25.0319	0	0.002469	1	0.002469	1
67	0	26.28349	0	0.002254	1	0.002254	1
68	0	27.59766	0	0.002058	1	0.002058	1
69	0	28.97755	0	0.001879	1	0.001879	1
70	0	30.42643	0	0.001716	1	0.001716	1
71	0	31.94775	0	0.001566	1	0.001566	1
72	0	33.54513	0	0.00143	1	0.00143	1
73	0	35.22239	0	0.001306	1	0.001306	1
74	0	36.98351	0	0.001192	1	0.001192	1
75	478.4771	38.83269	0	0.001089	1	0.001089	1
76	0	40.77432	0	0.000994	1	0.000994	1
77	0	42.81304	0	0.000907	1	0.000907	1
78	0	44.95369	0	0.000829	1	0.000829	1
79	0	47.20137	0	0.000757	1	0.000757	1
80	0	49.56144	0	0.000691	1	0.000691	1
81	0	52.03951	0	0.000631	1	0.000631	1
82	0	54.64149	0	0.000576	1	0.000576	1
83	432.9176	57.37356	0	0.000526	1	0.000526	1
84	0	60.24224	0	0.00048	1	0.00048	1
85	0	63.25435	0	0.000438	1	0.000438	1
86	0	66.41707	0	0.0004	1	0.0004	1
87	0	69.73792	0	0.000365	1	0.000365	1
88	0	73.22482	0	0.000334	1	0.000334	1

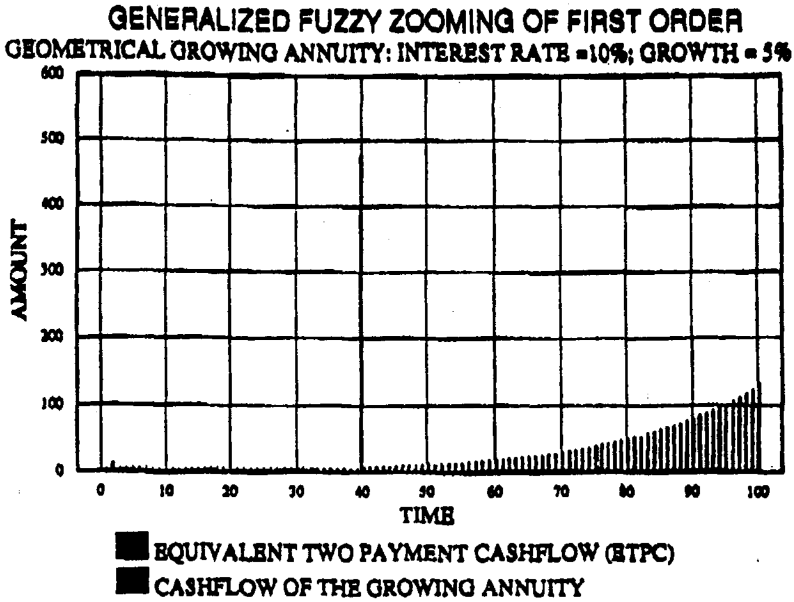
continued

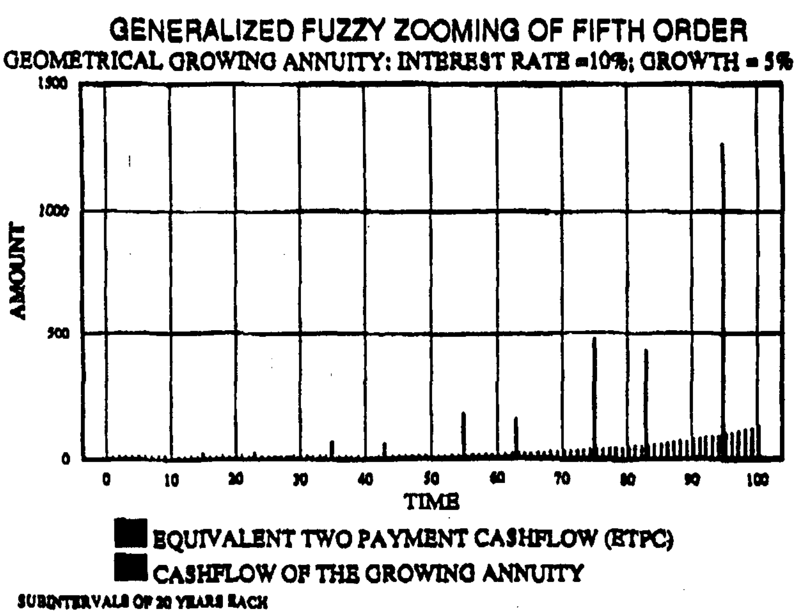
Continued

TIME	PAYMENT		PV 15%		PV 5%	5%↔15%	15%↔5%
89	0	76.88606	0	0.000305	1	0.000305	1
90	0	80.73037	0	0.000278	1	0.000278	1
91	0	84.76688	0	0.000254	1	0.000254	1
92	0	89.00523	0	0.000232	1	0.000232	1
93	0	93.45549	0	0.000212	1	0.000212	1
94	0	98.12826	0	0.000193	1	0.000193	1
95	1269.542	103.0347	0	0.000176	1	0.000176	1
96	0	108.1864	0	0.000161	1	0.000161	1
97	0	113.5957	0	0.000147	1	0.000147	1
98	0	119.2755	0	0.000134	1	0.000134	1
99	0	125.2393	0	0.000123	1	0.000123	1
100	0	131.5013	0	0.000112	1	0.000112	1
			PV:	10.49882	100	50.10994	60.38889

FIFTH ORDER TIME	PAYMENT	PV 15%	PV 5%	1-50:5% 51- . . . :15%	1-50:15% 51- . . . :5%
3.3311	8.734969	5.483663	7.424679	7.424679	5.483663
14.61913	25.61552	3.320119	12.55261	12.55261	3.320119
23.3311	23.17647	0.888996	7.424679	7.424679	0.888996
34.61913	67.96561	0.538248	12.55261	12.55261	0.538248
43.3311	61.49408	0.144122	7.424679	7.424679	0.144122
54.61913	180.333	0.087259	12.55261	0.087259	12.55261
63.3311	163.1621	0.023365	7.424679	0.023365	7.424679
74.61913	478.4771	0.014146	12.55261	0.014146	12.55261
83.3311	432.9176	0.003788	7.424679	0.003788	7.424679
94.61913	1269.542	0.002293	12.55261	0.002293	12.55261
	PV:	10.506	99.88646	47.51011	62.88234

FIRST ORDER TIME	PAYMENT	PV 15%	PV 5%
1.9388	12.5105	9.540996	11.381133
40.1345	476.7574	1.746707	67.27838
	PV:	11.2877	78.6597


GENERALIZED FUZZY ZOOMING OF FIRST ORDER
GEOMETRICAL GROWING ANNUITY: INTEREST RATE =10%; GROWTH = 5%
AMOUNT
TIME
EQUIVALENT TWO PAYMENT CASHFLOW (ETPC)
CASHFLOW OF THE GROWING ANNUITY


GENERALIZED FUZZY ZOOMING OF FIFTH ORDER
GEOMETRICAL GROWING ANNUITY: INTEREST RATE =10%; GROWTH = 5%
AMOUNT
TIME
EQUIVALENT TWO PAYMENT CASHFLOW (ETPC)
CASHFLOW OF THE GROWING ANNUITY
SUBINTERVALS OF 20 YEARS EACH

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