

A Numerical Examination of Asset-Liability Management Strategies

Meije Smink¹

University of Groningen,
POBox 800, 9700 AV Groningen,

The Netherlands

Telephone: 31-50-63.45.39

Fax: 31-50-63.72.07

Summary

This article presents a detailed analysis of Asset-Liability Management Strategies. In the paper these strategies are classified according to three categories. These are: [1] static techniques, e.g. gap-analysis, static duration analysis; [2] dynamic value driven strategies, e.g. immunization, key-rate immunization, model-dependent immunization, contingent immunization, portfolio insurance, pay-off distribution optimization; and [3] return driven strategies, e.g. spread management, rate of return optimization.

Each strategy is analyzed using of multi-period framework based on the Vasicek interest-rate model. This model is represented through a trinomial interest rate-tree. The analysis thus obtains a high degree of comparability with regard to the risks and possible rewards associated to each strategy. The article's final section discusses the potential for application of each strategy, as part of the financial institution's ALM policy.

The author gratefully acknowledges helpfull comments from T.K. Dijkstra and R.A.H. van der Meer.

Examen numérique des stratégies de gestion actif-passif

Meije Smink ¹

Université de Groningue,

Pays-Bas

Téléphone : 31-50-63.45.39

Télécopie : 31-50-63.72.07

Résumé

Le présent article propose une analyse détaillée des stratégies de gestion actif-passif. Ces stratégies sont classées en trois catégories : [1] Les techniques statiques, à savoir analyse des écarts, analyse de duration statique ; [2] Les stratégies dynamiques axées sur la valeur, à savoir immunisation, immunisation aux taux clés, immunisation dépendante de modèles, immunisation contingente, assurance de portefeuille, optimisation de la répartition des pay-offs; et [3] Les stratégies axées sur le rendement, à savoir gestion des spreads et optimisation des taux de rendement.

Chaque stratégie est analysée dans un cadre multipériode basé sur le modèle de taux d'intérêt de Vasisek. Ce modèle est représenté par une structure arborescente trinomiale de taux d'intérêt. L'analyse permet ainsi d'établir des comparaisons portant sur les risques et les récompenses éventuelles associés à chaque stratégie. La dernière partie de l'article examine le potentiel d'application de chaque stratégie dans le cadre de la politique d'ALM de l'institution financière.

L'auteur tient à remercier TK. Dijkstra et R.A.H. van der Meer de leurs remarques des plus utiles.

I. Introduction

In this article we will review a number of strategies and techniques that have evolved from the literature for managing asset and liabilities. Throughout the analysis our reference point is the situation of a perfect match between assets and liabilities. Matching will be defined here as strategy that results in full replication of a financial institution's liability portfolio. This is conceptually similar to the hedging strategy as used in the option pricing literature, but includes the obvious cases of cashflow matching and dedication.

Considering the products of a financial institution, the determination of the matching strategy is the core element of the institution's production process. Matching can thus be considered as the normal production situation and clearly may serve as a benchmark for any non-matching strategy.

The strategies and techniques discussed below may be categorized into static techniques, dynamic value driven strategies and dynamic return driven strategies, as in Van der Meer and Smink [1993]. Where the techniques are essentially static, considering the portfolio profile at a particular moment, the strategies considered are explicitly dynamic, describing future acts contingent on the outcome of stochastic variables. In most strategies these variables are related to the term structure of interest rates. Also, in our illustrations we adopt an approach that is based on a term structure model.

We distinguish between value and return driven strategies since the focus of these classes of strategies is different and so are the management control variables. We remark here that to a large extent this difference may be overcome on a conceptual level once we develop these strategies from mutual assumptions. The value driven strategies considered are immunization type of strategies, further classified into preservation and protection strategies. The first correspond to a passive management style, while the latter correspond to active styles seeking outperformance, while guaranteeing a minimum performance.

The article will proceed as follows. First in section 2 we develop our analytical framework. Here we define the stochastic environment. We consider an environment with term structure risk only, according to the Vasicek [1977] term structure model, using a lattice framework developed by Hull and White [1990]. Using the framework of section 2 we consider in section 3 the static ALM techniques. The next sections 4 and 5 consider the passive and active value driven strategies, while section 6 focuses on the return driven strategies. Section 7 generalizes the results and provides the main conclusions.

II. Evaluative Framework

In our illustrations we will use a discretized version of the Vasicek [1977] interest rate model. This model specifies the stochastic evolution of the term structure of interest rates assuming one stochastic variable: the instantaneous short-term interest rate. For the stochastic development of this variable a stochastic process of the Ornstein-Uhlenbeck-type is assumed. In particular, we have that the short-term interest rate process satisfies the stochastic differential equation:

$$dr(t) = a(b - r(t))dt + \sigma dB(t), \quad (1)$$

where $r(t)$ represents the value of the short-term interest rate at time t , a represents the mean-reversion parameter, denoting the rate of reversion of the short-term rate to its long term average Θ , and σ represents the instantaneous volatility of the process. The $dB(t)$ term denotes a standard Wiener process.

Using stochastic calculus and equilibrium arguments Vasicek [1977] obtains expressions for zero-coupon bond prices. He shows that we have for the time t value of a zero-coupon bond with residual maturity T :

$$P(t, T) = A(T)e^{-B(T)r}, \quad (2)$$

where

$$B(T) = \frac{1 - e^{-\alpha T}}{\alpha}, \quad (3)$$

and

$$A(T) = e^{[(B(T) - T)(\theta^* - \frac{1}{2} \frac{\sigma^2}{\alpha^2}) - \frac{\sigma^2 B(T)^2}{4\alpha}]}. \quad (4)$$

In this expression Θ^* denotes the risk-neutral long-term average. With risk-neutral we refer to an adjustment with regard to the expected value of the diffusion process. This change in expectation is made in order to adjust for the interest rate risk that arises from the uncertain future interest rates over longer holding periods. The risk-neutral long-term average is related to the actual long-term average through the following relationship:

$$\theta^* = \theta + \lambda \frac{\sigma}{\alpha}, \quad (5)$$

where λ represents the -- generally positive -- market price of (interest-rate) risk.

Since in this model the short-term interest rate is the only stochastic factor, the evolution of

this rate determines the stochastic behaviour for the entire term structure. We note here that although this model may not be completely adequate for practical purposes since it allows for differences between the observed and the model term-structures. However it can easily be extended into more general models allowing for an increased number of stochastic factors, e.g. Langetieg [1980] and, Beaglehole and Tenney [1991], and a complete match to initial term structures, see e.g. Hull and White [1991]. These extensions do not affect the content of this article.

Here we will not use the continuous-time model as presented but rather use a discretized version. In doing so, we use the Hull and White [1990] methodology of explicit finite differences. Hull and White [1990] show that the current model can be discretized using trinomial trees. That is, assuming that we have estimated values for the parameters in (1), we can generate scenarios as follows. First we determine the time step, Δt , of the scenarios. These time-steps represent the periods for which we assume no change in the short-term interest rate. Obviously, reductions in the time step will improve the discrete approximation of the continuous process. Here, for the sake of illustration we will set the time step at 1 year. Next, we create scenarios by branching on in the short-term interest rate tree. Here we use jumps in the short-term rate of sizes $\Delta r = j \sigma\sqrt{3\Delta t}$. The integer j , denoting the number of upshifts, is allowed to take values between -2 and 2 depending on the branching type, the general type being -1, 0, +1. The type -2, -1, 0, (0, +1, +2) is used only when the value of r is high (low).

The actual branching type is determined by choosing the transition probabilities corresponding to the transition from the time t value r to time $t+\Delta t$ value $r+j\Delta r$, so as to satisfy the restrictions resulting from the specification of the stochastic process in (1). That is we choose probabilities p_{-1} , p_0 , p_{+1} corresponding to the values of j so as to satisfy:

$$\begin{aligned}
 p_{-1} + p_0 + p_{+1} &= 1, \\
 p_{-1}, p_0, p_{+1} &\geq 0, \\
 +\Delta r p_{+1} - \Delta r p_{-1} &= \alpha(\theta^* - r)\Delta t, \\
 (\Delta r)^2 p_{+1} + (-\Delta r)^2 p_{-1} &= (\sigma \Delta t)^2 + (\alpha(\theta^* - r))^2.
 \end{aligned}
 \tag{6}$$

the first and second constraint restricting the probabilities to sum to 100% and preventing them to become negative, the third constraint restricting the probabilities to satisfy the expected change in (1) and the fourth constraint restricting the probabilities to satisfy the volatility of (1). Clearly when these constraints can not be met by the general branching type we require one of

the alternative branching types.²

In our illustrations we use the initial term structure and corresponding parameter values presented in table 1.

Table 1

Parameter Values:

a: 0.1
 b: 0.1
 σ: 0.0075
 r(0): 0.07

Initial Term Structure:

	Time to Maturity (years):					
	0	1	2	3	4	5
Price (per \$ 1.00):	1.0000	0.9324	0.8668	0.8037	0.7434	0.6863
Yield:	0.0700	0.0715	0.0729	0.0741	0.0753	

This term structure represents a slightly upward sloping term structure starting at an initial instantaneous short-term interest rate of 7%. We have chosen the market price of risk parameter equal to 0, for reasons that will be clear hereafter.

Given the absence of arbitrage opportunities, we can obtain the value, V(t), of any interest rate derivative instrument (e.g. zero coupon bonds and bond-options) paying a cashflow of CF_T T-periods from now (time t) as:³

$$V(t) = E^*[e^{-\sum_{i=t}^T r_i \Delta t} CF_T]. \tag{7}$$

That is: the value of any interest-rate derivative instrument equals the expected discounted value of its cashflows, where the expectation is taken with regard to the risk-neutral interest rate process. This implies that we have one of two situations: either we consider an interest rate environment where the risk parameter λ equals 0, i.e. where investors are risk-neutral, or we have an environment where investors command a risk-premium for interest rate risk through adjusted probability assessments, i.e. with Θ replaced by Θ*. In case of the latter, this can be loosely interpreted as discounting the cashflows on average at an interest rate that is higher as expected, in order to adjust for the interest rate risk resulting from holding the instrument instead of cash (or investing in a money-market account).

In our illustrations we consider five-periods and thus have the possible interest rate scenarios as presented in table 2.

² The typical violation being that the non-negativity constraint is not satisfied.

³ See e.g. Duffie [1992].

Table 2 Interest Rate Tree

Number of upshifts:	Time to Maturity:					
	0	1	2	3	4	5
5						0.1350
4					0.1220	0.1220
3				0.1090	0.1090	0.1090
2			0.0960	0.0960	0.0960	0.0960
1		0.0830	0.0830	0.0830	0.0830	0.0830
0	0.0700	0.0700	0.0700	0.0700	0.0700	0.0700
-1		0.0570	0.0570	0.0570	0.0570	0.0570
-2			0.0440	0.0440	0.0440	0.0440
-3				0.0310	0.0310	0.0310
-4					0.0180	0.0180
-5						0.0050

As a result of the restrictions imposed above on the transition probabilities we obtain the following matrix of transition probabilities corresponding to the trinomial-tree in table 2. Note that we can not reach all possible nodes of the tree in table 2 as a result of the restrictions on the probabilities and their impact on the type of branching process.

Table 3 Transition Probabilities

To number of upshifts:	From number of upshifts:							
	4	3	2	1	0	-1	-2	-3
5	0.0964	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.6381	0.1345	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3	0.2655	0.6619	0.1826	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0000	0.2036	0.6657	0.2407	0.0000	0.0000	0.0000	0.0000
1	0.0000	0.0000	0.1517	0.6495	0.3088	0.0000	0.0000	0.0000
0	0.0000	0.0000	0.0000	0.1098	0.6133	0.3869	0.0000	0.0000
-1	0.0000	0.0000	0.0000	0.0000	0.0779	0.5571	0.4750	0.0421
-2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0560	0.4810	0.5112
-3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0441	0.4466

Using (7) we can calculate the value of the primitive, or Arrow-Debrue, security prices in this economy. These primitive securities, whose prices are denoted by $\pi[i,j]$, are securities that have a pay-off of \$ 1 if and only if the *state of nature* characterized by time to maturity i and number of upshifts j is reached. The values of the primitive security prices for each state are presented in table 4.

Table 4 Values of primitive securities

Number of upshifts	Time to maturity:					
	0	1	2	3	4	5
5						0.0001
4					0.0013	0.0037
3			0.0106	0.0251	0.0386	
2			0.1132	0.1413	0.1539	
1		0.2879	0.3368	0.3203	0.2883	0.2551
0	1.0000	0.5719	0.3827	0.2820	0.2197	0.1770
-1		0.0726	0.0797	0.0715	0.0608	0.0510
-2			0.0038	0.0060	0.0066	0.0064
-3				0.0002	0.0003	0.0004

Using these primitive securities we can value each financial claim, or investment strategy, as the sum of all state contingent pay-offs times their primitive security prices.⁴ With these preliminary results we are able to analyze the strategies and techniques for Asset-Liability Management.

III. Techniques for ALM

Before we turn to dynamic strategies we first discuss some methods that are essentially static in nature. In particular, we will consider the following techniques for ALM:

- * Maturity-Gap Analysis
- * Duration-Gap Analysis
- * Cashflow-Matching

A. Maturity Gap-Analysis

The technique of maturity gap analysis is based on a general interest rate dependency of assets and liabilities. Assets and liabilities are classified according to the term of fixed interest. This is the period until the interest rate on a nominal balance sheet item is reset or adjusted in line with market rates. The gap is usually defined with regard to a maturity segment or *gapping period* (GP). We define it here as in Toevs [1984]:

$$GAP = ISA - ISL,$$

where ISA and ISL are resp. the interest sensitive assets and liabilities with a fixed term equal or less then the gapping period.

⁴ Using continuous-time notation these primitive securities can be expressed by means of the so called Green's function, see e.g. Beaglehole and Tenney [1991] or Duffie [1992].

Example

Consider a bank that has issued a \$ 100000 two year bond paying 6% nominal interest on an annual basis. The value of this liability can be calculated using the zero-coupon bond prices from table 1, and equals \$ 97475 (= 6000 * 0.9324 + 106000 * 0.8668). The bank invests the proceeds equally in a one-year 7.15% coupon bond and a four year 7.5% coupon bond. The value of these bonds per \$ 1000 face value can be calculated similar to that of the liability and equals, resp. \$ 994.37 and \$ 999.07. We assume that \$ 50000 is invested in each. Thus if we consider the gapping period to be given as 1 year then the bank has a (maturity) gap of \$ 50000 - \$ 0 = \$ 50000.

The rationale is based on an accounting treatment of the bank’s interest income. Clearly the bank’s income as measured over the gapping period is subject to change when interest rates move. In particular, the net interest margin with regard to the gapping period, NIM(GP), of the bank changes according to:

$$NIM(GP) = GAP \Delta r.$$

The maturity gap-analysis thus presents an income approach to ALM. By managing the maturity gap the bank is able to modify the sensitivity of the net-interest margin to interest rate changes over the gapping period. However, by focusing on the bank’s gapping period income the maturity gap analysis ignores income to be received in periods extending the gapping period. This capitalized income is reflected in the asset and liability values and thereby in the bank’s solvency.

Example continued

Given the interest rate tree of table 1 we consider how changes in interest rates affect the banks income (net interest margin) and solvency. If at time 1 we are at state [1,1] then \$ 50000 * (0.083 - 0.07) = \$ 650.00 additional interest is earned since the one-year bond can now be reinvested at higher rates (see table 2, and note that we have ignored the interest income). The expected change in interest rate income can be calculated using the transition probabilities from table 3 and equals: 0.3088 * \$ 650 + 0.0779 * (-\$ 650) = \$ 150.09.

Table 5 Maturity Gap Analysis

Number of Upshifts:	ΔNIW(1 year):		Solvency:	
	Deviations from: No change	Expected	Surplus	Deviation: from Expected
1	\$ +650	\$ +400	\$ 2437	\$ -273
0	\$ 0	\$ -150	\$ 2789	\$ + 79
-1	\$ -650	\$ -800	\$ 3178	\$ +466
Expected value:		\$ +150		\$ 2711

With regard to solvency we calculate the values of the bonds and the liability at time 1 for each possible state. Given the \$ 50000 investments in both bonds we obtain asset portfolio value of \$ 106000 in [1,1], \$ 107620 in [1,0], and \$ 109304 in [1,-1] (see the state-contingent bond prices from the appendix). The liability value in these states equals resp. \$ 103562, \$ 104834, and \$ 106128. The surplus in each state is presented above, its expected value being $0.3088 * \$ 2437.43 + 0.6133 * \$ 2789.46 + 0.0779 * \$ 3176.57 = \$ 2710.90$. From table 5, gapping period income and future period income (as measured by the market value difference of assets and liabilities at time 1 compared to the scenario of no change in interest rates), are apparently not always compatible.

B. Duration-Gap Analysis

Where the maturity-gap analysis is directed to interest rate income, the duration-gap analysis focuses on the present value of interest rate payments. The duration of an asset or liability provides a measure for the sensitivity of its present value to changes in interest rates. This concept was first defined by Macaulay [1938]. Macaulay defined the duration of a future stream of N fixed cashflows with known payment times, $C_1(T_1), \dots, C_N(T_N)$, as the present value weighted average maturity of these cashflows. Defining the present value of these cashflows by:

$$P_N \equiv P_N(C(T_1), \dots, C(T_N)) = \sum_{i=1}^N C(T_i) e^{-y T_i}, \quad (10)$$

then the duration, $D(P_N(T))$, is defined as:

$$D(P_N) = \frac{\sum_{i=1}^N T_i C(T_i) e^{-y T_i}}{P_N} \quad (11)$$

Analogy with the economic elasticity concept, as in Hicks [1938], leads to the conclusion that this duration measures the instantaneous percentage change in present value P_N with regard to changes in the discount rate, y , assumed. Moreover, duration can be interpreted as the length of time required before a change in present value due to an instantaneous interest rate movement, is off-set by a corresponding change in reinvestment opportunities.⁵ This is particularly obvious in case of a zero-coupon bond. From (11) it is clear that the duration of a zero-coupon bond equals its time to maturity and changes in present value are fully amortized over the residual maturity of the bond.

Thus by analyzing the duration of assets and liabilities it is possible to determine the impact of interest rate changes on the net present value (surplus) of the portfolio.

⁵ This interpretation is particularly clear from the so-called duration windows, see e.g. Bierwag [1987].

Example

Considering the previous example we can calculate the duration of assets and liabilities. The liability duration equals 1.94 (= $[6000 * 1 * 0.9324 + 106000 * 2 * 0.8668]/97475$). Likewise the assetportfolio duration may be calculated as the equal weighted average of 1 and 3.60 (the duration of the four-year bond). Thus the asset portfolio duration equals 2.30. A rise in interest rates will thus decrease the value of the asset portfolio more then it will decrease the value of the liability. This explains the solvency effect of table 5. Moreover, given the previous interpretation of duration it is clear that even though a rise in interest rates will increase current income, it will take a longer period before the decline in value is off-set.

Table 6 Duration Gap Analysis

Security:	Value: (\$ 1000 face value)	Duration:
Liability:	974.74	1.94
One-year zero:	990.07	1.00
Four-year zero:	994.37	3.60
Asset-Portfolio:	1000.00	2.30

C. Cashflow Matching

The cashflow matching approach may be regarded as an optimized maturity gap analysis. The approach aims at minimizing maturity differences between asset and liability cashflows. As such it is a normative approach, where the gap analysis is only descriptive. However, there are differences in the degree of matching implied by this approach. It has become more or less customary to distinguish between:

- * exact cashflow matching, and
- * dedication.

Under exact cashflow matching complete matching of asset and liabilities is pursued for a generally large number of (small) maturity segments. Of all possible portfolios that provide for such a position the portfolio with minimum cost is purchased. Once such a position is obtained, minimum portfolio risk (as measured by either asset and liability cashflows or asset and liability values) results. The only possible risks arising from timing differences within the maturity segments.

Example

Considering the example above, the cashflow matched position is off-course equivalent to the purchase of two year zero-coupon bond, with face value of \$ 106000, and \$ 6000 one year zero-

coupon bonds. The purchase price of these bonds equals $\$ 106000 * 0.8668 (= \$ 91.880) + \$ 6000 * 0.9324 (= \$ 5594) = \$ 97474$. Here we have multiplied the face amounts times their corresponding term structure discount rates (table 1). Thus a profit of $\$ 100000 - \$ 97474 = \$ 2526$ is locked in.

Since a cashflow position may prove to be too restrictive, i.e. can not be purchased, or is too costly. If it can only be purchased at substantial premia above current market rates, a more general approach may be desirable. This is provided by dedication where expected accrued interest payments are taken into account. Hereby we introduce some risk with regard to the actual profit realized (locked in), but are still cashflow matched with regard to possible interest rate changes.

Example continued

From the interest rate tree in table 2 and the corresponding probabilities it is clear that the one-year interest rate at time 1 equals at least 5.7%. Given this consideration we may decide to invest not all of the $\$ 106000$ face amount in the two-year treasuries but invest an additional $\$ 106000 * 0.9446 = \$ 100128$ in one-year treasuries. This amounts to a maximum of $\$ 93359 + \$ 5594 = \$ 98953$ invested in one-year treasuries. Note that now we have locked in a $\$ 1047$ profit, which will increase if time 1 interest rates are above 5.7%. It may be verified that the difference between the $\$ 1047$ and $\$ 2526$ locked in profits under the two cashflow matching strategies is equal to the value of put options on the two-year treasury (face value $\$ 100128$) which are exercised under time 1 scenario's [1,1] and [1,0] if the dedication strategy is used. The pay off from these put options are $\$ 2568$ in state [1,1] and $\$ 1298$ in state [1,0], with a resulting value of $\$ 1480 (= \$ 2568 * 0.2879 + \$ 1298 * 0.5719)$, where we have multiplied by the primitive security prices from table 4).

Thereby we have introduced risk (with regard to the actual profit realized) but are still protected (cashflow matched) against the interest rate changes assumed. Note that here we assumed that interest rates move according to the model where under general dedication strategies more conservative rates may be assumed.

In both cases, cashflow matching and dedication, the portfolio selection problem may be formulated as a linear programming problem. Examples of this formulation may be found in Fabozzi, Tong and Zhu [1989] and Dert and Rinnooy Kan [1991].

IV. Passive Value Driven Strategies: Surplus Preservation

We distinguish between value driven strategies that are directed to the preservation of surplus and those that aim at an outperformance. The first type of strategies are of the immunization type. These strategies are essentially passive since no bet on future interest rates is made. They differ according to the assumptions made regarding the stochastic development of interest rates. We will discuss here:

- * Redington Immunization
- * Key Rate Immunization
- * Model Contingent Immunization

A. Redington Immunization

All immunization strategies are essentially extensions to the concept of interest rate duration defined in the previous section. From this Samuelson [1945] and Redington [1952] formulated a strategy for maintaining the net present value, surplus, of a bank or life insurance company. By continuously managing asset and liability durations so as to equate asset and liability durations, the surplus is immune to changes in the interest rates. Thus the value of surplus is preserved. There are two necessary criteria for Redington immunization. The first consists of the assumption that all changes in interest rates are parallel or -- with regard to the term structure -- shape preserving⁶. The second consists of an asset and liability portfolio constraint, requiring that the liability portfolio convexity does not exceed the asset portfolio convexity. Convexity may be regarded as a measure of cashflow dispersion, and is defined by:

$$C(P_N) \equiv \frac{\sum_{i=1}^{i=N} T_i^2 C(T_i) e^{-y T_i}}{2 P_N} \tag{12}$$

Example

We consider the immunization of a \$ 1000 face value, three-year zero-coupon liability. For the asset portfolio we use both one-year and five-year zero-coupon bonds. In the initial state, state [0,0] of table 2, an amount of \$ 803.7 is available, i.e. the term structure discounted value (see table 1). From (10) it is clear that in case of zero-coupon bonds the duration equals the residual time to maturity.

⁶ Note that in our present value formula we have assumed that the discount rate is equal for all maturities. In general, this is not required. The corresponding duration measure for arbitrary term structure is referred to as the Fisher-Weill duration, see Fisher and Weill [1971]. However, the validity of Fisher-Weill duration as an approximation of percentage changes in present value does depend on the shape-preserving shift assumption.

Thus in order to obtain a duration matched asset and liability portfolio initially 50% is invested in both the one and five-year zero-coupon bonds. Since the dispersion of the asset portfolio exceeds that of the liability portfolio the convexity of the asset portfolio will exceed the convexity of the liability portfolio and the convexity constraint is met (using (12), the reader may verify this).⁷

The 50% investment thus has a value of $\$ 803.7 / 2 = \$ 401.85$. This amount is used to invest in 0.5855 five-year zeros and 0.4319 one-year zeros, both with face value \$ 1000. At time 1 three alternative situations may occur. In state [1,1] the 0.5855, then four-year zeros, have value \$ 416.25 ($= 0.5855 * 1000 * \$ 0.7109$, appendix 1) and we receive \$ 431.9 in cash from the one-year zeros. This amounts to \$ 847.25. Likewise, we find in states [1,0] a total value of \$ 866.28 and in state [1,-1] a total value of \$ 886.25.

The results of the immunization strategy are presented in table 7. Table 7 is organized as follows. The first row presents the state under consideration. The second row lists the previous state. The next three rows list the state value of the liability and the asset portfolio values under the Redington Immunization program for moving to the next state. The next three rows list the values which result as outcomes from the transition from the previous state to the current state in terms of the portfolio composition and the net result. Considering the time 1 value distribution of the then two-year liability, state [1,1] \$ 845.6, state [1,0] \$ 866.8 and state [1,-1] \$ 888.4, we note that when rates move up small gains are made, when rates move down losses are incurred.

⁷ This type of duration matched portfolio is generally denoted as a barbell portfolio, in contrast to a bullet portfolio where asset cashflows are located near the liability duration. For an exposition and empirical results on the use of these strategies, see Ingersoll [1981].

Table 7 Redington Immunization

From State:	To State:						
	[0,0]	[1,1] [0,0]	[1,0] [0,0]	[1,-1] [0,0]	[2,2] [1,1]	[2,1] [1,1]	[2,1] [1,0]
Value Liability:	803.70	845.60	866.80	888.40	908.50	920.40	920.40
Initial Values							
- five-year zero:	401.85	281.87	288.93	296.13	0.00	0.00	0.00
- one-year zero:	401.85	563.73	577.87	592.27	908.50	920.50	920.40
Ending Values							
- five-year zero:		416.25	435.28	455.25	296.98	307.64	301.56
- one-year zero:		431.00	431.00	431.00	612.48	612.48	619.76
Net Result:	0.00	+2.15	-0.52	-2.15	+0.94	-0.28	+0.98

From State:	To State:						
	[2,0] [1,1]	[2,0] [1,0]	[2,0] [1,-1]	[2,-1] [1,0]	[2,-1] [1,-1]	[2,-2] [1,-1]	[2,-2] [1,-1]
Value liability:	932.40	932.40	932.40	944.60	944.60	956.90	956.90
Initial Value							
- five year zero:	0.00	0.00	0.00	0.00	0.00	0.00	0.00
- one-year zero:	932.40	932.40	932.40	944.60	944.60	956.90	956.90
Ending Value							
- five year zero:	315.48	312.37	306.11	323.56	317.08	329.38	329.38
- one-year zero:	612.48	619.76	627.01	619.76	627.01	627.01	627.01
Net Result:	-1.26	-0.27	+0.72	-1.28	-0.51	-0.51	-0.51

As is apparent from the examples the immunization is not complete. This is due to the fact that in our example, and in general, the term structure shifts are not shape preserving. This phenomenon is known as the stochastic process risk of immunization. Here, short term rates move stronger than do long term rates. In general stochastic process risk increases with the convexity difference between assets and liabilities. Fong and Vasicek [1986] have suggested a measure for this type of immunization risk, called the M^2 . The concept of M^2 is similar to that convexity, it measures the dispersion of the cashflows of a portfolio, except it does so with regard to a particular investment horizon, H . It is defined by:

$$M^2(P_N, H) \equiv \frac{\sum_{i=1}^{i=N} (T_i - H)^2 C(T_i) e^{-y T_i}}{P_N} \tag{13}$$

In case of an asset-liability portfolio the appropriate horizon H is given as the liability duration.⁸ Thus for implementation of Redington immunization strategy some bound on M^2 must be imposed, although no clear level for this bound can be presented.

Example continued

We consider states [0,0] and [1,0]. In state [0,0] asset portfolio duration, convexity and M^2 are resp. 3, 6.5, and 4. In state [1,0] they are resp. 2, 3, and 2. In both cases assets and liabilities are duration matched. Liability convexities are resp. $3^2/2 = 4.5$ and $2^2/2 = 2$. Thus the convexity difference is larger in state [0,0] and so is the M^2 . This explains that the immunization errors by moving from

⁸ It can easily be verified that for a duration-matched asset-liability portfolio with H equal to the liability duration, the asset portfolio M^2 equals $2C - H^2$, i.e. the difference between asset-portfolio convexity and squared duration. Thus M^2 increases with asset portfolio convexity.

state [0,0] or larger than those by moving from state [1,0].

B. Key-Rate Immunization

The concern for non-shape preserving shifts of the term structure has encouraged the development of more general immunization approaches. A straightforward approach is provided by key-rate immunization as developed by Reitano [1991] and Ho [1990]. Here the term structure is partitioned in maturity segments. The separating maturities correspond to key- (interest-) rates, e.g. 3 month, 1 year, 5 year, 10 year, etc. It is assumed that the term structure may be (linearly) interpolated from these key rates.

Given a number of key-rates, key-rate durations and convexities may be defined as in (10) and (11). With regard to these key-rates an immunization strategy as in Redington immunization is feasible.

Example

We consider the same situation as in the case of Redington immunization. We intend to immunize a three-year zero-coupon liability. Assume that there are two relevant key rates: the two- and four-year interest rates. We assume here, for sake of illustration only, that these key rates determine the shape of the term structure (at least for maturities up to 5 years). The other term structure rates are assumed to depend on these rates by two corresponding directional vectors: N_2 and N_4 . For the 1, 2, 3, 4, and 5 year to maturity yields, these vectors are chosen as resp. [1, 1, 0.5, 0, 0] and [0, 0, 0.5, 1, 1]. That is: the one year rate experiences the change in the two-year rate multiplied by 1 plus the change in the four-year rate times 0, the three-year rate experiences 0.5 times the change in either rate, etc..

The key-rate durations with regard to the two key-rates for zero-coupon bonds with maturities up to

Table 8 Key-Rate Durations

Key-Rate:	Years to Maturity:				
	1	2	3	4	5
2-year	1	2	1.5	0	0
4-year	0	0	1.5	4	5

five year are presented in table 8. They are simply calculated as the multiple of their time to maturity times the value of the corresponding element of the directional vectors.

Starting from state [0,0] with portfolio value \$ 803.70, we now search a portfolio with identical key-rate durations as the three year liability. The portfolio so obtained has portfolio weights for the one-,

two- and five-year zero coupon bonds of resp. -10%, 80%, and 30%.⁹ Thus we invest resp. - \$ 80.37, \$ 642.96, and \$ 241.11 in these bonds. From the zero-coupon prices in table 2 it is clear that this amounts to face values of resp. - \$ 86.20, \$ 741.76, and \$ 351.32. The time 1 value of the portfolio under the different states is presented in table 9 and can be calculated as the state contingent zero-coupon price from appendix 1 times the face value of the bond. At time 1 the portfolio is rebalance in all states into two-year zero coupon bonds.

Table 9 Key-Rate Immunization

	To State:			
From State:	[0,0]	[1,1]	[1,0]	[1,-1]
	[0,0]	[0,0]	[0,0]	[0,0]
Value Liability:	803.70	845.60	866.80	888.40
Initial Values				
- five-year zero:	241.11	0.00	0.00	0.00
- two-year zero:	642.96	845.60	866.80	888.40
- one-year zero:	-80.37	0.00	0.00	0.00
Ending Values				
- five-year zero:		249.75	261.17	273.15
- two-year zero:		682.72	691.62	700.67
- one-year zero:		-86.20	-86.20	-86.20
Net Result:	0.00	+0.67	-0.21	-0.78

As is clear from the example, imposing additional constraints on the term structure exposure of the portfolio results in less immunization risk. Depending on the number of key-rates considered (and thus on the number of key-rates that are immunized) the stochastic process risk can almost entirely be eliminated.

C. Model Contingent Immunization

Where the previous immunization models make little explicit assumptions on the stochastic process underlying the evolution of the term structure of interest rates, we now turn to models that do. Model contingent immunization utilizes an explicit formulation of the stochastic term structure process. First a stochastic process is been specified and identified by estimating model paramaters from market data. Second, the values of interest rate sensitive instruments can be determined as well as their sensitivities with regard to the stochastic factors determining the term structure. Once these values and sensitivities have been found, an immunization strategy, similar to that in the previous examples, with regard to these sensitivities can be implemented.

⁹ Off course the simplest portfolio consists of three-year zero-coupon bonds only. Since this is trivial we have not done so. It is clear however that the requirement to match two-durations imposes additional constraints -- as compared to Redington immunization -- that cannot always be met.

Example

We consider the immunization problem from the previous examples. However, we now utilize the fact that we know the underlying stochastic process of the term structure. That is we utilize the sensitivities of the zero-coupon bond prices with regard to the short-term rate -- the stochastic factor in this Vasicek [1977] model. Note that from (2) follows that the sensitivity of the zero-coupon bond prices with regard to the short-term rate is presented through $B(T)$. This is the *model-contingent duration*, MCD , and it only applies in case of the Vasicek model assumed. For the zero-coupon bonds with maturities from one year to five years, the MCD 's are resp. 0.9516, 1.8127, 2.5918, 3.2968, and 3.9347.¹⁰ From this it is clear that, in order to be immunized at state $[0,0]$, we do not allocate 50%/50% over one-year and five-year zero coupon bonds as in Redington immunization. The proper allocation amounts to 54.98% in the five-year bond (= \$ 441.90) and 45.02% in the one-year bond (= \$ 361.80).

We thus purchase \$ 643.90 (= 441.90/0.6863, table 1) face value of the five-year and \$ 388.03 (= 361.80/0.9324, table 1) face value of the one-year zero-coupon bonds. Thus at time 1 we receive \$ 388.03 in all states. This amount is added to the value of the then four-year zero coupon bonds, which have values of resp. \$ 457.74 in $[1,1]$, \$ 478.67 in $[1,0]$, and \$ 500.62 in $[1,-1]$. Thus immunization errors of resp. \$ 0.17, - \$ 0.10, and \$ 0.25. The full immunization program is presented in table 10.

Table 10 Model-Contingent Immunization

	To state:						
From State:	[0,0]	[1,1]	[1,0]	[1,-1]	[2,2]	[2,1]	[2,1]
	[0,0]	[0,0]	[0,0]	[0,0]	[1,1]	[1,1]	[1,0]
Value Liability:	803.70	845.60	866.80	888.40	908.50	920.40	920.40
Initial Values							
- five-year zero:	441.90	310.48	318.27	326.20	0.00	0.00	0.00
- one-year zero:	361.80	535.12	548.53	562.20	908.50	920.50	920.40
Ending Values							
- five-year zero:		457.74	478.67	500.62	327.12	338.87	332.18
- one-year zero:		388.03	388.03	388.03	581.40	581.40	588.30
Net result:	0.00	+0.17	-0.10	+0.25	+0.02	-0.13	+0.08
	To State:						
From State:	[2,0]	[2,0]	[2,0]	[2,-1]	[2,-1]	[2,-1]	[2,-2]
	[1,1]	[1,0]	[1,-1]	[1,0]	[2,-1]	[2,-1]	[1,-1]
Value Liability:	932.40	932.40	932.40	944.60	944.60	956.90	
Initial Values							
- five year zero:	0.00	0.00	0.00	0.00	0.00	0.00	0.00
- one-year zero:	932.40	932.40	932.40	944.60	944.60	956.90	
Ending Values							
- five-year zero:		350.88	344.08	337.19	356.41	349.27	362.83
- one-year zero:		581.88	588.30	595.17	588.30	595.17	595.17
Net result:		-0.12	-0.01	-0.03	+0.12	-0.11	+1.10

From the example we conclude that model-contingent immunization allows for the best results.

¹⁰ Note that these durations do not proportionally increase with maturity. This effect is caused by the mean-reversion property of the short-term rate model.

This is even more true when we realize that the immunization errors in the example are only result of the discretization of the continuous process. Conditional on the interest-rate tree we could have matched asset and liability values at every node. Since the tree is trinomial this would be comparable to imposing three restrictions -- one for each possible transition -- on the portfolio. We thereby immunize the changes in asset and liability values for each transition. This type of model contingent immunization is referred to as *stochastic dedication*, or *pathwise immunization*, e.g. Ho [1992].

A significant advantage of the model-contingent immunization approach is that it is attached to a term structure model that can be used both for valuation as well as for hedging purposes. Where the previously defined (Macaulay) duration can not straightforwardly be used in case of floating-rate assets and assets with embedded (prepayment) options, the approach here can.

However, some care must be taken. Clearly model-contingent immunization depends on the accuracy of the term structure model in describing actual bond price movements. Had we incorrectly estimated the model's parameters, or, assumed a model not matching actual behaviour, significant immunization errors could result. In particular the number and stability over time of the parameters is a crucial element in the succes of this technique. As a practical solution to this Hull and White [1991] recommend a combination of this approach and the key-rate duration analysis.

V. Active Value Driven Strategies: Surplus Protection

The second category of value driven strategies are of the portfolio insurance. These strategies allow for active management, combined with a floor protective constraint. From a theoretical viewpoint these strategies involve the stochastic dedication in combination with synthetically created call- and/or put-options on the active portfolio management strategy.¹¹ The actual exercise price of these options is determined by the stochastic behaviour of the liability portfolio and the relationship between assets and liabilities.

Here we discuss:

- * Contingent Immunization
- * Portfolio Insurance
- * Pay-Off Distribution Optimization

¹¹ This may be regarded as put-call parity at the strategy level. Each active strategy plus protective puts may be regarded as a stochastic dedication strategy with upward potential providing calls. This will be clear from the examples provided in the text.

A. *Contingent Immunization*

Contingent Immunization as formulated by Leibowitz and Weinberger [1982], [1983], starts with the determination of a criterion that triggers the passive investment mode. This criterion may consist of an asset portfolio floor or a trigger yield. Now at first an active strategy is implemented. However, as soon as -- due to adverse developments -- the portfolio floor or the trigger yield is reached, the active strategy is terminated and a passive, immunization strategy is implemented in order to protect the asset value from ending below the required portfolio value at the strategies final date.

As such contingent immunization implies immunization from the time that surplus is not sufficient as to protect against shortfall risks. Essentially, no restrictions are necessary with regard to the actual active strategy implemented. The only assumption stemming from the required liquidity for termination of the active mode and pursuing an immunization strategy.

Example

We reconsider the immunization problem as in the previous examples. Thus the value of the three-year liability equals \$ 803.70. However, we assume that now a \$ 13.00 surplus, S , is present. Thus asset portfolio value equals \$ 816.70. Moreover, assume that we expect a decline in interest rates. It may then appear to be desirable not to match the portfolio, either by means of cashflow matching or from pursuing one of the immunization strategies above, but to maintain a maturity mismatch so as to speculate on interest rate movements. However, at all times we wish to be able to meet the liability value. That is, the probability of ending with negative surplus, $\text{Prob}(S < 0)$, must be equal to zero.

The \$ 816.70 allows for the purchase of 1.19 ($= 816.70/0.6863$, table 2) five-year zero-coupon bonds. At time 1 we may end up in the following situations. In state [1,1] the active asset portfolio value equals \$ 846.00 ($= \$ 1000.00 * 1.19 * 0.7109$, appendix). In states [1,0] and [1,-1] this value equals resp. \$ 884.65, and \$ 925.22. The liability value in these states equals \$ 845.60, \$ 866.80, and \$ 888.40. Thus, at state [1,1] the value of surplus has almost evaporated. However, at states [1,0] and [1,1] a profit is made. Since at state [1,1] surplus is insufficient to continue with the five-year active strategy, the immunization mode is triggered. At the other two states the active strategy is continued. Table 11 presents the further results of the contingent immunization strategy.

We make the following observations. First, we note that the succes of contingent immunization depends on the timely change of mode. Therefore, sufficient liquidity in both the active as well as the passive strategy is required. Second, when considered on an asset-only basis, surplus is invested on a highly leveraged basis, resulting in a substantial risks. Third, the contingent immunization may be regarded, or can be implemented, as a *stochastic dedication plus calls*

Table 11 Contingent Immunization

	To State:						
From State:	[0,0]	[1,1]	[1,0]	[1,-1]	[2,2]	[2,1]	[2,1]
	[0,0]	[0,0]	[0,0]	[0,0]	[1,1]	[1,1]	[1,0]
Value Liability:	803.70	845.60	866.80	888.40	908.50	920.40	920.40
Initial Value:							
- active asset:	816.70	846.00	884.65	925.22	-	-	923.32
- surplus (S):	13.00	0.40	17.85	36.60	-	-	2.92
Prob(S<0) > 0:	No	Yes	No	No	-	-	Yes
Mode:	Active	Passive	Active	Active	Passive	Passive	Passive

	To State:						
From State:	[2,0]	[2,0]	[2,0]	[2,-1]	[2,-1]	[2,-1]	[2,-1]
	[1,1]	[1,0]	[1,-1]	[1,0]	[1,-1]	[1,-1]	[1,-1]
Value Liability:	932.40	932.40	932.40	944.60	944.60	956.90	
Initial Value:							
- active asset:	-	956.40	956.40	990.68	990.68	1029.11	
- surplus (S):	-	24.00	24.00	46.08	46.08	72.21	
Prob(S<0) > 0:	-	No	No	No	No	No	No
Mode:	Passive	Active	Active	Active	Active	Active	Active

strategy. The calls created synthetically through the active strategy. This will be clarified by the continued example.

Example continued

At state [0,0] the value of surplus equals the sum of all the state-contingent future surplus values. As such, the contingent immunization strategy may be regarded as a stochastic dedication strategy plus calls that pay-off a particular surplus value given that a particular state occurs. Utilizing table 4 we can calculate the present value of these calls as: $0.40 * \$ 0.2879 (= \pi[1,1]) + 17.85 * \$ 0.5719 (= \pi[1,0]) + 36.60 * \$ 0.0726 (= \pi[1,-1])$. This equals: $\$ 0.12 + \$ 10.20 + \$ 2.66 = \$ 13.00$, the present value of surplus.

B. Portfolio Insurance

Where contingent immunization may be regarded as a stochastic dedication plus calls strategy, portfolio insurance explicitly accounts for an *active plus puts strategy*. Here the focus is on the synthetic provision of a put option for an active strategy, so as to protect the value of surplus from becoming negative. For the actual implementation of the portfolio insurance strategy there are basically three alternatives:

- * obtain puts on the active portfolio in the financial markets;
- * create the puts on a synthetic basis by:
 - pursuing a dynamic portfolio insurance replication strategy;
 - pursuing a dynamic constant proportion portfolio insurance strategy.

The first alternative is the most straightforward. The second two alternatives are based on option pricing theory. The concept of portfolio insurance was first developed for stock options by Leland and Rubinstein [1981]. Constant proportional portfolio insurance is a modification to this strategy that is somewhat more simple in its application: it suggests continuous rebalancing to

constant proportion portfolio weights. This strategy was developed in Black [1988] and Black and Perold [1992].

Example

We assume that a two-year zero-coupon liability with \$ 1000.00 face value has been issued at \$ 866.80. As an active investment strategy an investment in a five-year zero-coupon bond is chosen. We intend to provide a protection against a surplus shortfall at time 2. From the time 0 prices of the two- and five-year bonds, it is clear that we can purchase 1.263 (= 0.8668/0.6863, table 1) \$ 1000.00 face value five-year zero-coupon bonds. We may protect these bonds from a surplus shortfall by either:

- [1] buying put-options with exercise prices so as to provide for pay-offs at time 2 when surplus is negative. Note that from the three-year zero-coupon bond prices at time 2 (appendix) we should have pay-offs equal to \$ 54.00 in state [2,2], \$ 20.00 in state [2,1], and \$ 0.00 in the other states. We thus require puts on states [2,2] and [2,1]. The value of these puts can be calculated using the state-prices of table 4. The put-value equals: \$ 10.18 (= 54.00 * π [2,2] + 20.00 * π [2,1]).
- [2] create the puts synthetically by a stochastic dedication program for the puts with pay-offs as in [1]. The actual portfolio weights can be calculated by moving backward through the trinomial tree-structure starting from the ultimate put pay-offs. As already noting when discussing the model-contingent immunization strategy, we have to meet three constraints. Assuming that we implement the synthetic creation of the puts by utilizing three- and five-year zero-coupon bonds and cash, at each node of the tree we should have:

$$\begin{aligned}
 (1 + r(j)) C(i) + A(5-i) P(5-i,j+1) + A(4-i) P(4-i,j+1) &= P_{put}(j+1) \\
 (1 + r(j)) C(i) + A(5-i) P(5-i,j) + A(4-i) P(4-i,j) &= P_{put}(j) \tag{14} \\
 (1 + r(j)) C(i) + A(5-i) P(5-i,j-1) + A(4-i) P(4-i,j-1) &= P_{put}(j-1).
 \end{aligned}$$

Here we have defined resp. $C(i)$, $A(5-i)$ and $A(4-i)$ as the number of \$ 1000 face value cash, five- and three-year bonds. The symbols $P(5-i,j)$, $P(4-i,j)$ and $P_{put}(j)$ denote the prices, or pay-offs, of the five- and three-year zero-coupon bonds and the put-options. The results of this synthetic put replication strategy are presented in table 12.

Note that the portfolio insurance strategy may be considered as a special type of stochastic dedication strategy: here the put-option is replicated using stochastic dedication. Conceptually equivalent but in practical applications possibly different would be not to purchase the (synthetic) put, but to implement a passive, immunization strategy with calls instead. Of course,

Table 12 Portfolio Insurance

Initial State:	[1,1]			[1,0]			[1,-1]		[0,0]		
Ending State:	[2,2]	[2,1]	[2,0]	[2,1]	[2,0]	[2,-1]	[2,0]-[2,-2]	[1,1]	[1,0]	[1,-1]	
Ending Values:											
C(i):	1086.5	1086.5	1086.5	1072.5	1072.5	1072.5	n.a.	1072.5	1072.5	1072.5	
A(5-i):	749.0	775.9	803.7	775.9	803.7	832.5	n.a.	710.9	743.4	777.5	
A(3-i):	908.5	920.4	932.4	920.4	932.4	944.6	n.a.	775.9	803.7	832.5	
Put:	54.0	20.0	0.0	20.0	0.0	0.0		0.0	23.9	5.8	0.0
Initial State:	[1,1]			[1,0]			[1,-1]		[0,0]		
Number of:											
C(i):	30.7			41.8			0.0		5.8		
A(5-i):	23.5			32.1			0.0		28.9		
A(3-i):	-56.0			-75.7			0.0		-34.4		
Value put:	23.7			5.8			0.0		10.2		

a similar observation yields the contingent immunization program since in both cases these strategies can be reformulated using an argument similar to put-call-parity.

As in stochastic dedication the success of the synthetic portfolio insurance strategy depends on the accuracy of the term structure model employed. As with contingent immunization and stochastic dedication, liquidity will be a crucial concern.

C. Pay-Off Distribution Optimization

The Pay-Off Distribution model extends the technique of stochastic dedication into situations where investors may have arbitrary utility functions. Moreover, it is possible to incorporate probability assessments for the scenarios that may differ from the market implied probabilities, in a consistent way. Theoretically it is based on the dynamic programming formulation of the consumption-investment problem as formulated in Cox and Huang [1989]. It has been further developed in Dybvig [1988a] and [1988b], and Smink [1993].

From Cox and Huang [1989] it is clear that the multiperiod problem of finding an optimal strategy for a given set of preferences can be obtained from an equivalent static programming problem. In order to do so, utility must be optimized across different state pay-offs. It follows that when we consider a particular utility function specified over all possible future states, we can select a strategy by first optimizing state dependent utility over all states and then selecting an investment strategy that replicates the corresponding optimal pay-off distribution. In case of stochastic dedication, the utility function may be defined as having value 1 for all states with pay-offs of the liability function, until the liability pay-off is met, and utility 0 in those states where the liability has no pay-offs or for pay-offs in states in excess of the liability. Given sufficient capitalization, utility is maximized for an investment program that exactly replicates the liability.

However, the Pay-Off Distribution model is more general in that it allows for selective risk taking and for situations when capitalization of the asset portfolio differs from that of the liability portfolio. As an example we consider a situation analyzed in Beaglehole [1992].

Example

We consider a situation where a two-year liability has been issued. The (actuarial) value of this liability equals \$ 866.80. We assume however that the liability is not fully funded, for instance as a consequence of a too low issuing price or adverse developments in the asset portfolio. We assume the issuer has available wealth \$ *W*. If the issuer can not meet the liability at time 2 substantial problems arise.

This problem can be formulated in terms of the Pay-Off Distribution Optimization as follows. First we choose a utility function accurately describing the decision-makers utility. Consider a utility function *U(.)* assuming the value 1 when in those states where the liability is met, and 0 in those states where it is not. Now the decision makers objective function is given as:

$$\begin{aligned} \max \quad & \sum_{j=-2}^2 U(j) \text{prob}[2,j] \\ \text{s.t.} \quad & \sum_{j=-2}^2 U(j)\pi[2,j] = \frac{W}{866.80} \end{aligned} \tag{15}$$

This objective function is specified over all time 2 states, the actual state denoted by the parameter *j*. The probability of reaching state [2,*j*] is denoted by *prob*[2,*j*], this probability can be calculated using table 3. As before, the prices of the primitive securities, i.e. state-contingent pay-offs, are noted by $\pi[2,j]$ (table 4). Using the method of Lagrange-multipliers¹² the solution to this problem can be determined. This solution is formally presented by

$$\frac{\text{prob}[2,j]}{\pi[2,j]} \geq \lambda. \tag{16}$$

Table 13 Pay-Off Distribution Optimization

State:	Ratio:	Prob(success) :	Cumulative Wealth Invested:
[2, 2]	1.165	7.40 %	\$ 63.80
[2, 1]	1.158	46.21 %	\$ 400.60
[2, 0]	1.150	90.36 %	\$ 783.30
[2, -1]	1.145	99.56 %	\$ 863.00
[2, -2]	1.144	100.00 %	\$ 866.80

This amounts to stating that first pay-offs are purchased in those states where the price per unit of probability is at a minimum. The left hand side ratios in (16) and the resulting probabilities of succes are presented in table 13. It is optimal to start buying pay-offs in states determined by the maximum number of upshifts in the short-term rate. Note that given the interest rate tree other values for *W* do not alter the probability of meeting the liability.

¹² See e.g. Huang and Litzenberger [1988], pp. 121.

We conclude with the observation that the Pay-Off Distribution Optimization allows for a general collection of ALM strategies. It provides the opportunity to integrate active policies with stochastic dedication. Note that in the example we have used the risk-neutral probability but we are not restricted to do so. This allows of determination of pay-offs in states that are perceived to be priced relatively inexpensive (have a low price per unit of probability). The success of the Pay-Off Distribution Optimization as that of the stochastic dedication strategy depends of course on the accuracy of the model employed.

VI. Return Driven Strategies

We now turn to strategies that are based on the stochastic returns of assets and liabilities. Where the value-driven strategies may be regarded as the dynamic analog of duration-gap-analysis, the return-driven strategies may be considered as extensions to maturity-gap analysis. Here we will consider Realized Return Optimization.

A. *Realized Return Optimization*

The Realized Return Optimization approach, as developed in Miller, Rajan and Shimpi [1989], is founded on four basic components. First a number of scenario's is determined. Second, for each scenario liability returns are calculated. If felt necessary a margin for error may be imposed on the calculated liability return. The so obtained scenario contingent liability returns are the required returns an asset portfolio should meet. Third, for different asset portfolios scenario contingent returns are calculated. In combination with a decision criterion, the fourth component of Realized Return Optimization, a decision with regard to the asset portfolios can be made.

Example

We consider the situation from the previous example. We assume however that the \$ 1000.00 face value liability is fully funded. Thus both asset and liability values equal \$ 866.80. We assume that the decision maker choses our scenario's and we calculate the required liability returns under each scenario over a two-year horizon. Moreover, we consider two alternative asset portfolios, the first consisting of a continued investment in one-year zero-coupon bonds -- at time 1 we reinvest in one-year zero coupon bonds -- and an investment in four year zero-coupon bonds. The bond prices at the time 1 and time 2 states can be obtained from the appendix. The scenario contingent returns on the liability and the two-asset portfolios are presented in table 14. Note that in case of the one-year zero-coupon bond strategy the returns depend on the value of interest rates at time 1 and are path-dependent.

Table 14 Realized Return Optimization

Scenario:	Probability:	Realized Returns:		
		Liability	1 Yr Zeros	5 Yr Zeros
[1, 1] - [2, 2]	7.43 %	7.42 %	7.95 %	4.44 %
[1, 1] - [2, 1]	20.06 %	7.42 %	7.95 %	6.30 %
[1, 1] - [2, 0]	3.39 %	7.42 %	7.95 %	8.19 %
[1, 0] - [2, 1]	18.94 %	7.42 %	7.25 %	6.30 %
[1, 0] - [2, 0]	37.61 %	7.42 %	7.25 %	8.19 %
[1, 0] - [2, -1]	4.78 %	7.42 %	7.25 %	10.11 %
[1, -1] - [2, 0]	3.01 %	7.42 %	6.56 %	8.19 %
[1, -1] - [2, -1]	4.34 %	7.42 %	6.56 %	10.11 %
[1, -1] - [2, -2]	0.44 %	7.42 %	6.56 %	12.23 %

From these returns a choice between the two portfolios can be made after an appropriate criterion has been defined, e.g. below-target probability, which equals 69.12% for the one-year and 46.43% for the 4 year portfolio.

The Realized Return Optimization approach is comparable to the Pay-Off Distribution Optimization strategy. This approach has the advantage of being formulated in terms of returns, which may be attractive from both a practical perspective and as a means of integrating this model in efficient frontier analysis, i.e. expected return versus risk trade-offs, see e.g. Markowitz [1987]. However, a precautionary remark must be made. The integration of valuation and investment strategy selection may be lost. If a mispricing in the liability exists then this may incur unrealistic return requirements leading to unrecognized risk-seeking behaviour in the asset portfolio. When formulated in a general valuation setting it is conceptually equivalent to the pay-off distribution model.

VII. Evaluation and Conclusions

The strategies and techniques discussed may be considered as representative for the methodologies that have evolved in the literature on investment management in the presence of liabilities. Looking at these methods it is clear that the scenario approach based on market valuation has become the unifying factor. Where simple liabilities can be matched by cashflow matching techniques a substantial number of financial products has evolved in the financial intermediary's product markets that can not. As noted above, this creates a situation where asset management may be regarded in terms of put-call parity at a strategy level. Any active portfolio equals the sum of a matching, or stochastic dedication, strategy and call-type and put-type exposures with regard to the financial value-determining variables. The crucial question to be addressed in Asset-Liability Management are thus threefold:

- * how can the matching strategy be determined;
- * to what extent should risks be eliminated;
- * if there is an exposure to financial risks, how can this exposure be structured efficiently.

The first question addresses the determination of the stochastic dedication strategy. Here the focus is on scenario analysis using scenarios that are both consistent with market values and representative for management's expectations. The actual number of scenarios employed may differ for different purposes: where top-management may use a small number with little detail, actual portfolio management will require more. Here consistency is required in the scenarios used, both for the determination of asset and liability values and the values of traded financial instruments, as well as for their risk exposures and expected returns.

The second question addresses the implementation of the stochastic dedication strategy. Given market traded financial instruments, transaction costs, asset and liability payment uncertainty, e.g. from expected prepayments or insurance risks, and the uncertainty inherent on the perception and modeling of risks, it may not be desirable to pursue a minimum risk strategy. Instead within clearly defined boundaries risk exposures may exist, in order to minimize costs. This results in a risk attitude for active element of the portfolio.

This leads to the final question, given a level of risk exposure, is it possible to generate a more efficient risk profile. This is where the Pay-Off Distribution and Realized Return Optimization strategies may become successful. If clear preferences with regard to risk exposure exist then these preferences can be exploited by structuring risk exposure efficiently. Note that for both the valuation and matching strategy formulation as well as for the active portfolio strategy decisions the same scenarios are used: only different probability assessments are made. With the passive strategies the probabilities used are the risk-neutral "market implied" probabilities, while in active strategies expected, subjective, user-determined probabilities are used.

We finally note that although we have restricted our attention to interest rate risk only, both the analytical framework as most of the strategies themselves can be readily implemented in more general settings.

Appendix State Bond Prices

Number of Upshifts:	Time to Maturity:					
	0	1	2	3	4	5
5	1.0000					
4	1.0000	0.8852				
3	1.0000	0.8968	0.8049			
2	1.0000	0.9085	0.8250	0.7490		
1	1.0000	0.9204	0.8456	0.7759	0.7109	
0	1.0000	0.9324	0.8668	0.8037	0.7434	0.6863
-1	1.0000	0.9446	0.8884	0.8325	0.7775	
-2	1.0000	0.9569	0.9106	0.8648		
-3	1.0000	0.9694	0.9933			

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