

On the Asset Models

as a part of All-Company Insurance Analysis
by Finnish Insurance Modelling Group (FIM-GROUP)

Summary. Our work is aimed at testing and further developing the asset models, the Wilkie model and its variants, which are presented in the Practical Risk Theory study book by Daykin, Pentikäinen and Pesonen (1993). A number of empirical data collected from 12 countries are studied. The observations confirm the appropriateness of the basic structure of the above models where inflation is used as a background factor, driving the flow of asset values as well as income gained from different sorts of investments. Furthermore, the feature is also confirmed that the behaviour of the capital market is subject to significant changes in its character, period by period, and that abrupt crashes in equity and property values appear so frequently and in such large dimensions that they should be taken into account in one way or another. This seems to suggest that the models can still be improved by also engaging other background variables than merely inflation.

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Les modèles d'actifs
dans le cadre d'une analyse d'une assurance de toutes les compagnies
par le groupe de modélisation d'assurance finlandais
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Résumé

Nos travaux ont pour objectif de tester et de poursuivre l'élaboration des modèles d'actifs, modèle de Wilkie et ses variantes, qui ont été présentés dans le manuel de théorie pratique du risque de Daykin, Pentikäinen et Pesonen (1993). Il porte sur des données empiriques recueillies dans douze pays. Les observations confirment la validité de la structure fondamentale des modèles ci-dessus lorsque l'inflation est utilisée comme facteur d'arrière-plan, qui affecte l'évolution de la valeur des avoirs ainsi que le revenu produit par différents types d'investissements. En outre, il est également confirmé que le comportement du marché des capitaux est sujet à des changements significatifs de nature, par période, et que des chutes abruptes de valeurs des titres et des biens surviennent si fréquemment et avec une amplitude telle qu'elles doivent être prises en compte d'une manière ou d'une autre. Ceci semble indiquer que les modèles peuvent encore être améliorés en faisant intervenir d'autres variables d'arrière-plan que la seule inflation.

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1. Introduction

This paper was prepared by an informal study group, for which the name 'Finnish Insurance Modelling Group', briefly FIM-group is assumed. It is a continuing series of research and reporting projects, which were initiated by solvency studies on general insurance (Pentikäinen and Rantala 1982) and continued by similar studies on pension insurance (Tuomikoski 1987) and life insurance (Rantala 1992). Originally the analyses were focused on the stochastic behaviour of the underwriting business and methods, in particular simulations were developed for its special needs. In the context of the co-operation on the solvency issues with the British working parties, chaired by C Daykin since 1982, the idea to extend the stochastic treatment to the asset side of the insurance business spread also to Finland and has since then been gaining increasing attention, also inspired by the AFIR colloquiums. Accordingly, our group has been studying the present state-of-the-art of theories and practices of investment performances and, in particular, looking for whether the methods, among others simulation, which are developed for the analysis of other parts of the insurance business, could find a useful application also in this environment.

The work is still continuing and this paper is only an intermediate report on those items which we have, at least preliminarily, discussed so far. Even though our studies concern only some rather limited items, the whole procedure of investment performances is briefly described in order to show how the various modules are to be placed in the total schedule which, for its part, is a submodel in the all-company analysis comprising both assets and liabilities. In those numerous items where we have no new development to report or which are still open awaiting future research efforts we have made only references to other sources or expressed our regret for the incompleteness which we hope future studies will rectify.

The considerations and notations closely follow the study book *Practical Risk Theory* by Daykin, Pentikäinen and Pesonen, which is frequently referred to briefly herein as DPP. The modelling of inflation, which is a primary issue also in our considerations, is dealt with by Pukkila, Ranne and Sarvamaa (1994) in a separate paper submitted to this colloquium as well as the Schwarz-Rissanen test method which also is referred to in several contexts.

(a) Purpose of the model is to be a tool in exploring, in quite general terms, how the risks involved with investments can be evaluated, in particular, the

influence of capital markets. The consequences of various adverse events, whenever they may occur, are tested. For this purpose it is sufficient to find models which generate high and low periods in similar patterns to those which can be expected to appear in the future, having regard for past experience and the expected future development in future.

It is necessary to emphasize that these kinds of models are not intended to be used for forecasting actual asset movements, or the times when the return on investments is high or low. This is a crucial difference to many of the conventional econometric models.

(b) Basic definitions. The assets $A(t)$ are divided into categories, indexed by k , for instance, handling as separate groups bonds, loans, equities, properties, etc. If appropriate, these can still be further subdivided into their own classes or, on the other hand, combined together with other items which are of similar character having regard for the level of expected return, variability of values and income, sensitivity to inflation, term of maturity, etc. so far as these features are relevant in the analysis of concern. Hence, we have

$$(1.1) \quad A(t) = \sum_k A_k(t) = \sum_k w_k(t) \cdot A(t), \quad \sum_k w_k = 1$$

where the distribution coefficient w_k indicates the proportional share of the assets of the category k in the portfolio.

The return on investments J is decomposed into the cash income J' and to the changes in asset values ΔA

$$(1.2) \quad J(t) = \Sigma J'_k(t) + \Sigma \Delta A_k(t)$$

The movements of values are often convenient to measure by indices $I_k(t)$ which give the relative value with reference to a fixed initial value. Then

$$(1.3) \quad \Delta A_k(t) = i_k(t) \cdot A_k(t-1)$$

where

$$(1.4) \quad i_k(t) = I_k(t) / I_k(t-1) - 1 \approx \ln[I_k(t) / I_k(t-1)]$$

is called a growth rate.

2. Interest rates

The model includes both long and short term interest rates. The long term rates are generated first and the model uses them to produce the short term interest rates.

In our data the long term interest rates are represented by long term government bond yields and the short term rates by money market rates. Fig. 2.1 shows the inflation rates and government bond yields in different countries. It is seen that the interest rate level follows the changes in the rate of inflation. However, the interest rate doesn't generally react immediately to the changing inflation rate but adjusts only after some years (Germany seems to be an exception where the adjustment occurs almost immediately). This can be explained by the fact that the investors demand a real rate of interest added to the expected value of inflation in the future. The expected inflation doesn't react immediately to every rise and fall in the actual inflation rate because these are usually thought at first to be only temporary.

Following an idea of Wilkie (in his submodel for the consols yield) we divide the long term interest rate $j_{\text{long}}(t)$ into two components:

$$(2.1) \quad j_{\text{long}}(t) = j_1(t) + j_2(t)$$

where $j_1(t)$ represents the expected level of inflation and $j_2(t)$ represents the real rate. Because the expected level of inflation cannot be measured directly, it is modelled by smoothing the actual values by the formula

$$(2.2) \quad j_1(t) = \delta i(t) + (1 - \delta) j_1(t - 1)$$

where the rate of inflation $i(t)$ is calculated from the price index $I(t)$ by the formula

$$(2.3) \quad i(t) = \ln(I(t)/I(t - 1))$$

The real interest rate $j_2(t)$ is then modelled by an autoregressive formula:

$$(2.4) \quad j_2(t) - \mu = \sum_{k=1}^p \phi_k [j_2(t - k) - \mu] + \varepsilon_{\text{long}}(t)$$

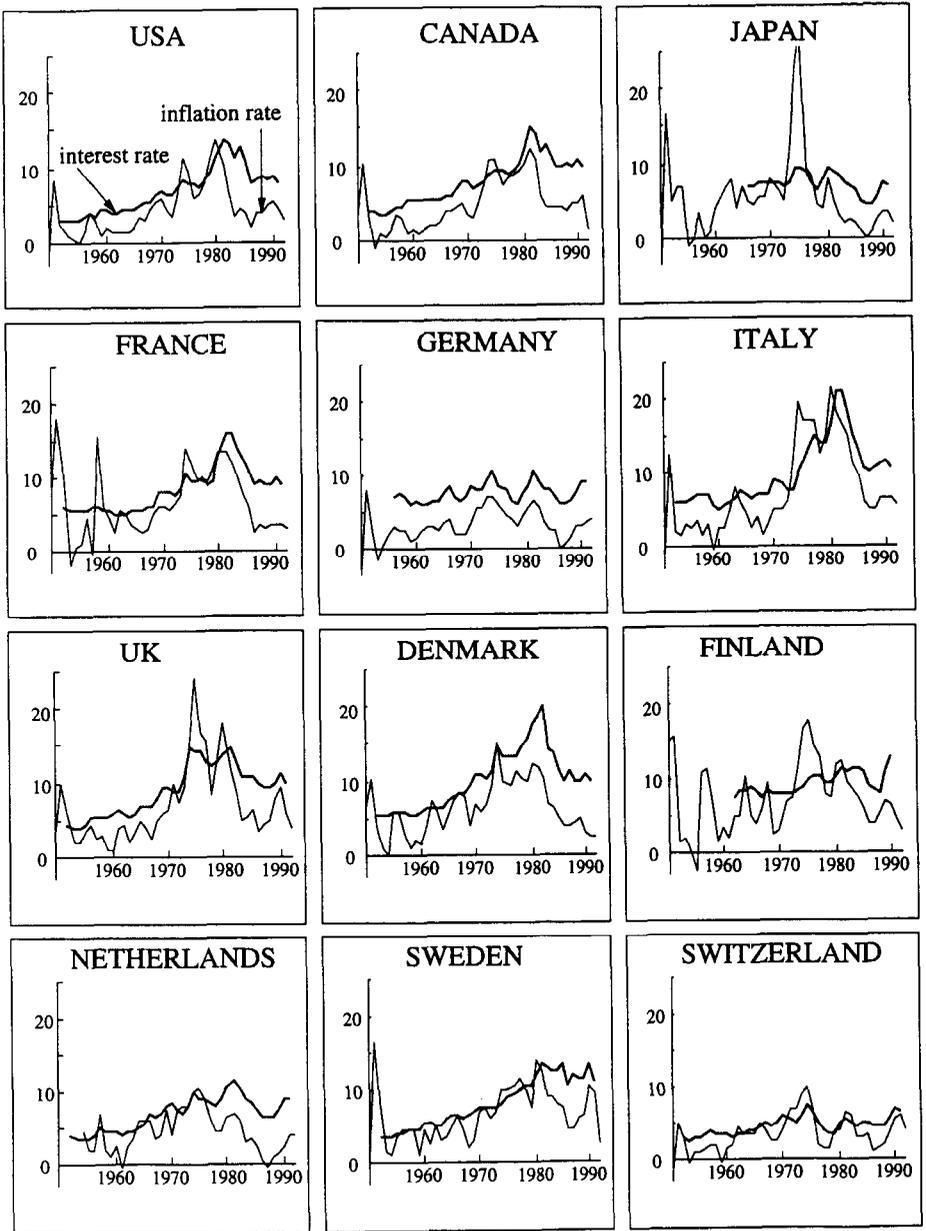


Fig. 2.1. The rates of interest and inflation

where the order of the process, p , is selected by the Schwarz-Rissanen test (Pukkila et al. 1994, section 3). By this test it is seen that $j_2(t)$ is AR(2) in France, Italy and the Netherlands and AR(1) in the other nine countries. The estimated parameters with the standard deviation and skewness of the residuals are shown in Table 2.1.

Table 2.1. Parameters of the long term interest rate model

	δ	μ	ϕ_1	ϕ_2	σ_ϵ	γ_ϵ
USA	0.146	0.031	0.706		0.008	0.717
CANADA	0.152	0.038	0.692		0.008	0.451
JAPAN	0.065	0.021	0.783		0.009	0.502
FRANCE	0.155	0.025	0.951	-0.352	0.009	0.411
GERMANY	0.670	0.043	0.692		0.007	-0.318
ITALY	0.235	0.030	1.075	-0.519	0.009	1.307
UK	0.220	0.027	0.729		0.009	-0.295
DENMARK	0.225	0.043	0.771		0.012	-0.412
FINLAND	0.093	0.029	0.612		0.009	0.682
NETHERLANDS	0.111	0.034	0.863	-0.468	0.007	0.352
SWEDEN	0.099	0.033	0.756		0.008	-0.318
SWITZERLAND	0.299	0.013	0.786		0.005	-0.353

The short term rates $j_{\text{short}}(t)$ are produced using the long term rates. This is done according to the observed dependence between the real interest rates, which are defined according to the formulae

$$(2.5) \quad j_{\text{long,real}}(t) = j_{\text{long}}(t) - i(t)$$

$$j_{\text{short,real}}(t) = j_{\text{short}}(t) - i(t)$$

Fig. 2.2 shows the dependence between the long and short term real interest rates in the data. It is seen that there are strong positive correlations and that the dependence is linear. Therefore, it seems natural to model the short term real interest rates by:

$$(2.6) \quad j_{\text{short,real}}(t) = a j_{\text{long,real}}(t) + b + \epsilon_{\text{short}}(t)$$

where $\epsilon_{\text{short}}(t)$ are normally distributed noise terms. The estimated parameters and the standard deviation of the noise are shown in Table 2.2.

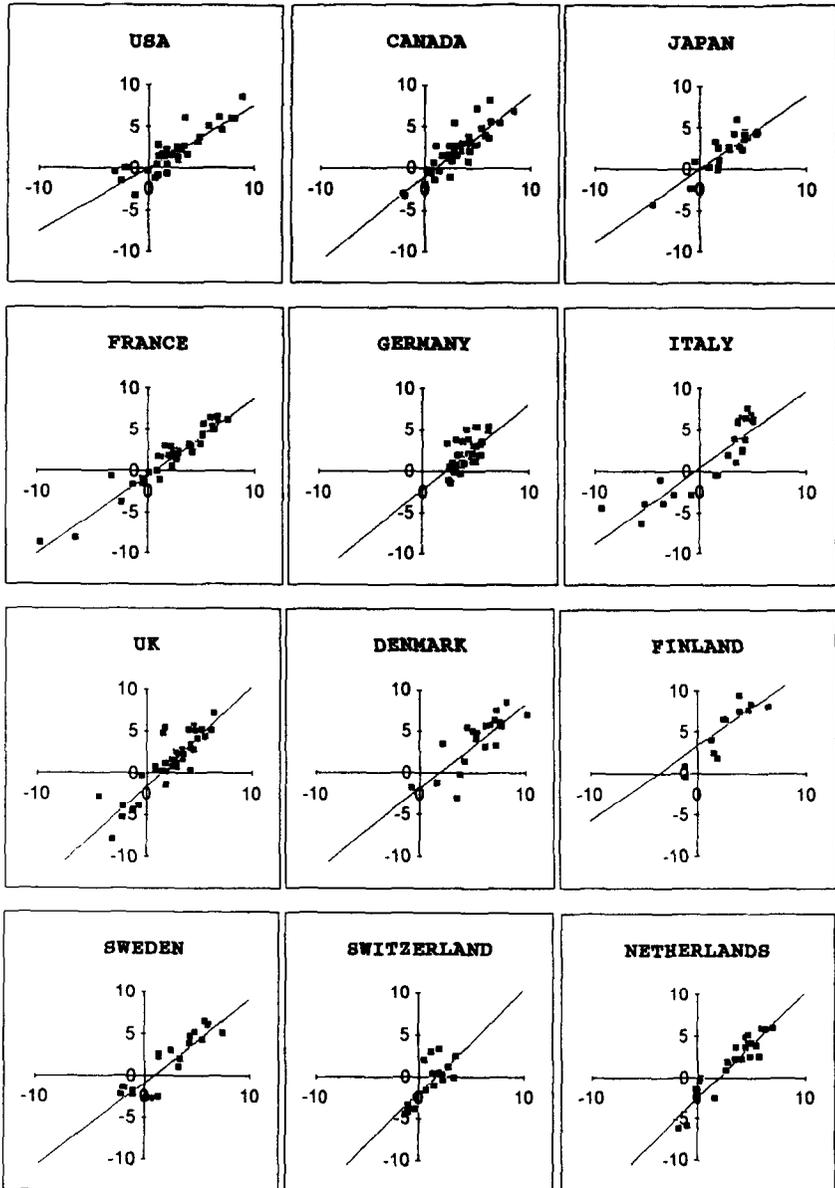


Fig. 2.2. Observed real rates of short-term (horizontal axis) and long-term interest (vertical axis)

Table 2.2. Parameters and the coefficient of determination (r^2) of the short term interest rate model

	a	100b	100 σ_e	r^2
USA	0.751	0.0	1.21	0.76
CANADA	1.002	-1.0	1.31	0.74
JAPAN	0.885	0.0	1.18	0.95
FRANCE	0.949	-0.6	0.97	0.92
GERMANY	1.034	-2.2	1.40	0.38
ITALY	0.928	0.4	1.96	0.79
UK	1.193	-1.6	1.62	0.84
DENMARK	1.009	-1.8	1.90	0.63
FINLAND	0.907	3.3	2.16	0.81
NETHERLANDS	1.273	-2.5	1.25	0.87
SWEDEN	1.005	-0.9	1.49	0.79
SWITZERLAND	1.237	-2.0	1.60	0.52

The value of a bond is obtained in a straightforward way from the interest rates (see DPP, section 8.5(e)).

3. Equities

3.1 Equity values, empirical data

(a) **Equity value statistics.** We have collected information on the flow of equity price indices $I_e(t)$ as is exhibited in Fig. 3.1a for the USA and in Fig. 3.1b also for 11 other countries. It is plotted, together with the inflation index $I(t)$ (consumer prices), in the topmost boxes of the country diagrams (the shading of the variation range around $I_e(t)$ will be explained in section 3.2(f) below). Time is denoted by t counting in years from some specified initial point, which is conveniently $t = 0$ for the formulae and a calendar year (1950) in the figures. The scaling of the indices is defined so that

$$(3.1.1) \quad I_e(0) = I(0) = 1$$

(b) **Real values.** A well-known feature is that in the long run the equity prices compensate inflation and, in addition, as a rule give real proceeds. This is seen as a stronger growth of the equity indices than the inflation index.

For further analyses it is useful to consider the ratio of these two indices, termed as *equity real price index*

$$(3.1.2) \quad I_{\text{real}}(t) = I_e(t)/I(t)$$

This index is displayed in Fig.3.2. Furthermore, its growth rate is also employed:

$$(3.1.3) \quad i_{\text{real}}(t) = I_{\text{real}}(t)/I_{\text{real}}(t-1) - 1.$$

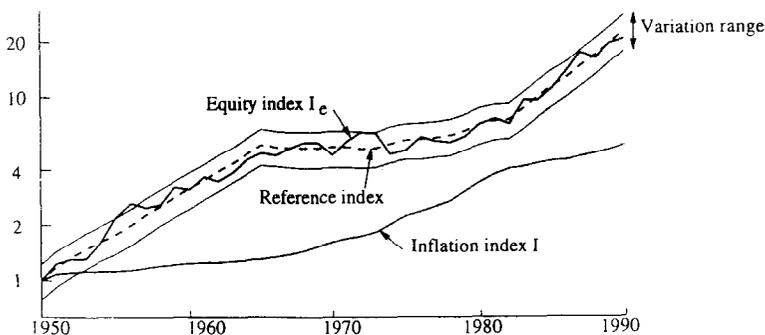


Fig.3.1a. The equity price index and the inflation index (consumer prices) in the USA. The dotted line represents the reference index on which the modelling will be based (see 3.2(e)) and the shaded area the variation range around it ($\pm 2 \times \text{st. dev.}$ of the variable δ to be introduced in section 3.2(f)). Semi-log scale.

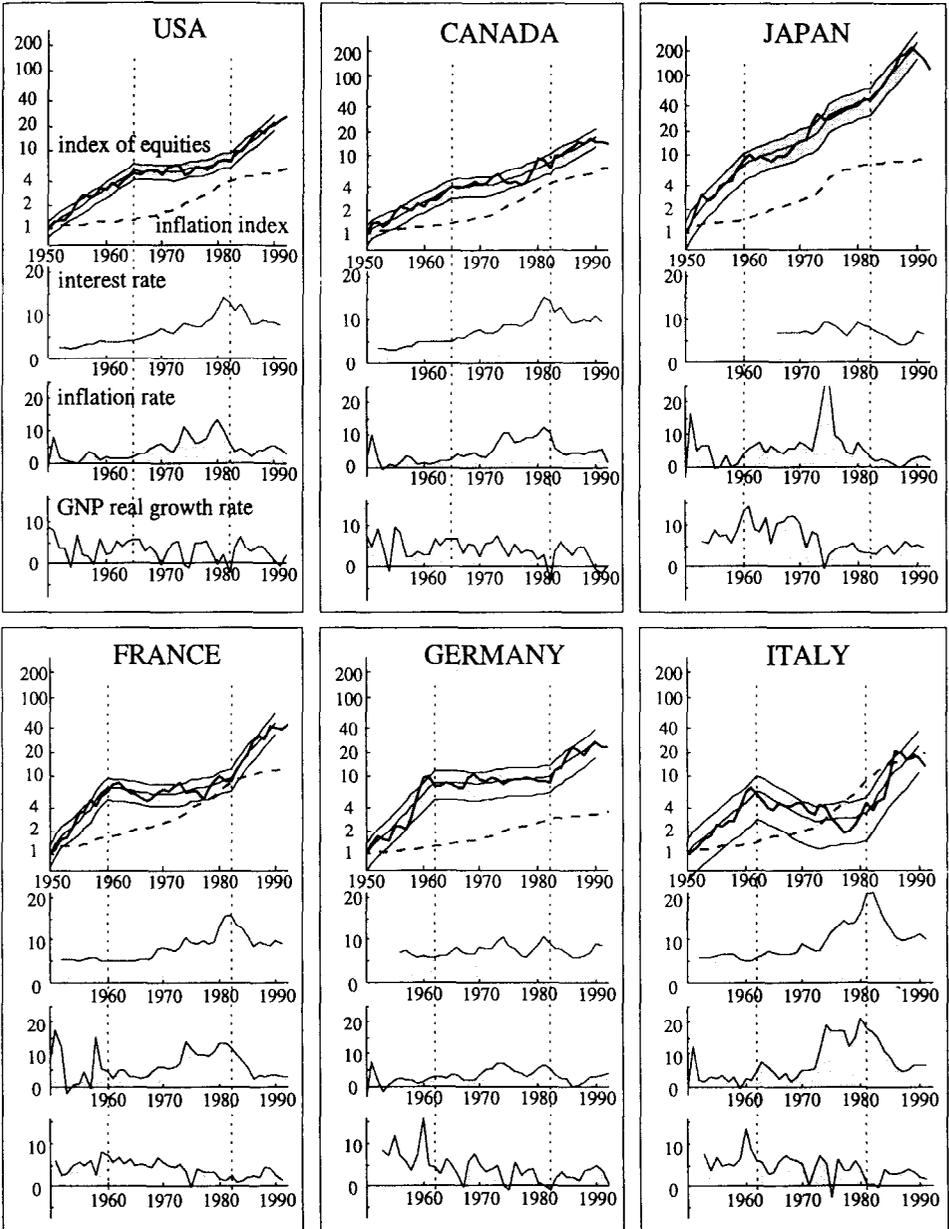


Fig. 3.1b. Equity price indices compared against the inflation index and the rates of interest (long-term bonds) and the growth rates of GNP. Semi-log scale.

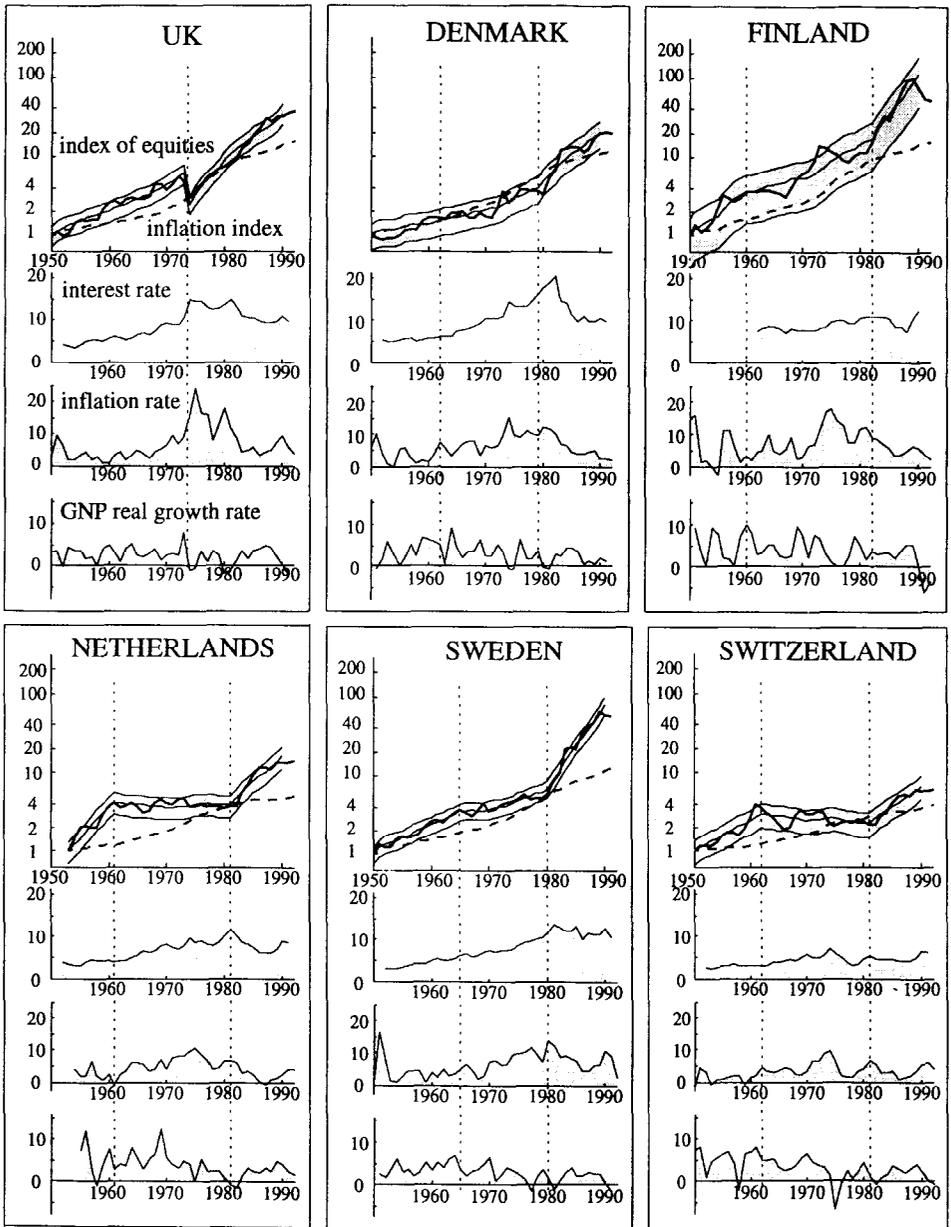


Fig. 3.1b continued

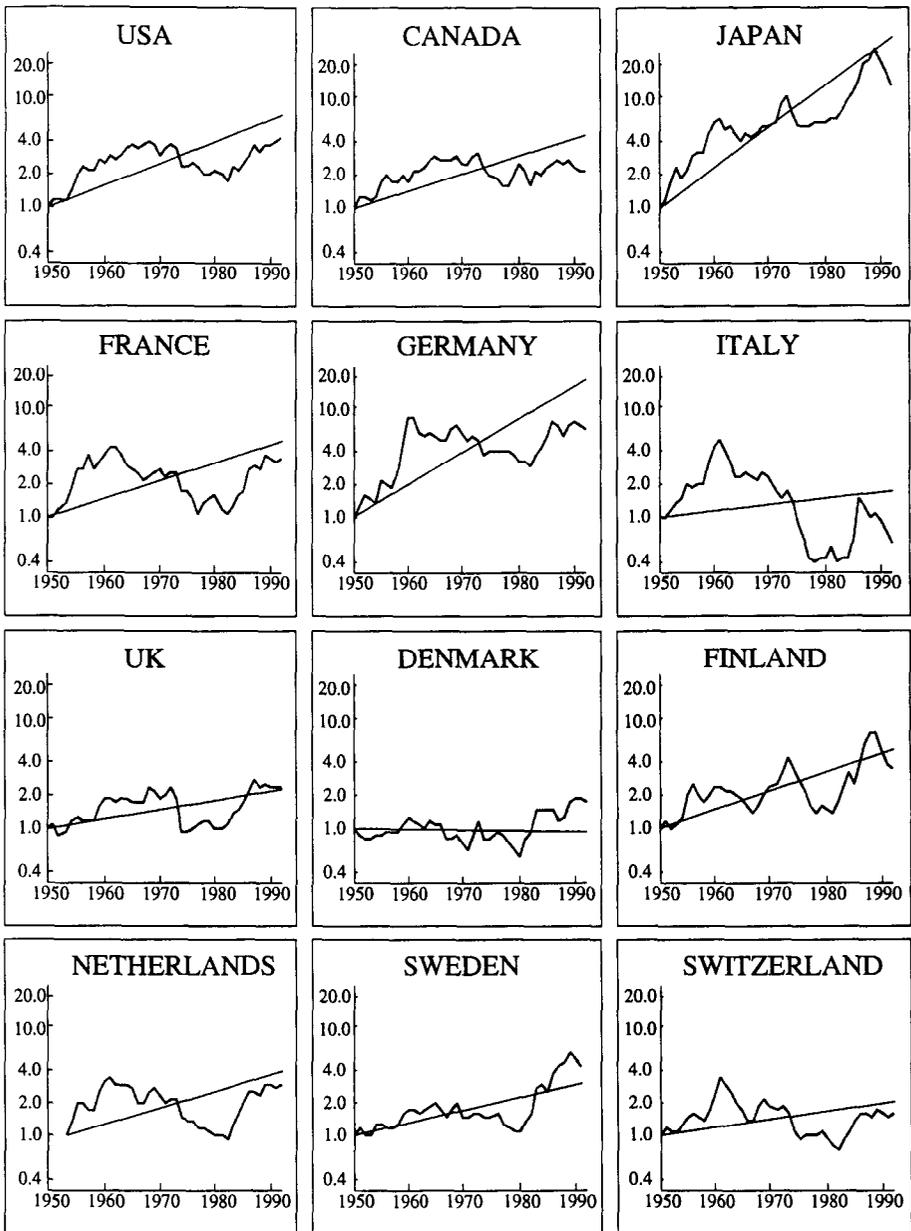


Fig. 3.2. Equity real price indices $I_{real}(t)$, derived from the data of Fig. 3.1b. Semi-log scale.

3.2. Fitting statistical models for the equity index

(a) Structuring the behaviour of the equity price index. We are now going to look for a model for the equity price index $I_e(t)$ which would fit as well as possible the observations detectable from the diagrams of the above figures. Unfortunately the time span from which the relevant empirical data are inherent is rather short and the shapes of the curves in different countries are so different that it obviously is not possible, within the framework of that information which was available, to design any model which could be shown to be universally valid with convincing reliability. What can be done is only to propose variants to be experimented with in various applications.

Clearly inflation is the most relevant factor affecting the flow of the equity prices. It was described by the inflation index $I(t)$ in section 3.1(a).

The equity real values measured by the index $I_{real}(t)$ defined by (3.1.2) is clearly another relevant background factor. It seems advisable to decompose it into two factors, letting the first one $\bar{I}_{real}(t)$ describe consolidated (smoothed) real growth and the second one $s(t)$ residual fluctuation,

$$\text{i.e. } I_{real}(t) = \bar{I}_{real}(t) \cdot s(t).$$

Accordingly, the following considerations are based on the fundamental assumption that the model should allow for superimposition of 1) inflation, 2) smoothed real growth and 3) random fluctuation (noise):

$$(3.2.1) \quad I_e(t) = I(t) \cdot \bar{I}_{real}(t) \cdot s(t)$$

The residual variation $s(t)$ can be modelled, for example, by using the well-known autoregressive time series AR(1), AR(2) or higher as will be considered in section (f) below. It will be briefly called noise. Note that it may depend on the definition of the growth $\bar{I}_{real}(t)$.

(b) Constant real growth. Following the proposal by Bonsdorff (1991) a simple variant is to present the smoothed real growth by an exponential function (a straight line in our semi-logarithmic scale in Fig. 3.2)):

$$(3.2.2) \quad \bar{I}_{real}(t) = (1 + \bar{i}_{real})^t.$$

Here \bar{i}_{real} represents the average real growth rate of the equity values during the observation period (years 1950-90 in our figures).

The country-related noises $s(t)$ were derived from the relevant data by using the Schwarz-Rissanen test method (see Pukkila et al. 1994, section 3, paragraph 3). However, even though autoregressive solutions were found which

passed the test, tentative experiments resulted in realisations which did not well correspond to such types as were desired (see Fig.5.1). Therefore, other variants were also studied.

(c) **Varyingly progressing real growth.** The diagrams of Fig.3.2 show that the unsmoothed real growth index considerably deviates (note the semi-logarithmic scale!) from the line (3.2.2). Therefore, this is often a rather rough describer of the real growth. Clearly there are rather long 'carrier waves' superimposed by a short-term fluctuation. Their combined effect was not sufficiently regarded by the simple model of section (b). This observation suggests a separation of these two by introducing a random variable $z(t)$ to describe the carrier waves and limiting the above $s(t)$ to indicate the (short-term) residual fluctuation which still remains. Hence, we have

$$(3.2.3) \quad \bar{I}_{\text{real}}(t) = (1 + \bar{i}_{\text{real}})^t \cdot z(t).$$

It is to some degree a matter of discretion which part of the total random fluctuation is assigned to $z(t)$ and which part is left to $s(t)$.

The variable $z(t)$ can be modelled, for example, by using a stationary first or second order autoregressive time series as will be dealt with in section (f) below.

(d) **Broken line version of the smoothed real growth** was suggested in DPP, section 8.5(i). It aims to reduce the variance of the noise term and is based on the observation that the real index $I_{\text{real}}(t)$ in Fig.3.2 has periods of rapid growth and periods of zero growth and all the time is subject to random fluctuation. In particular, the slope of the trend, i_{real} (see (3.1.3)), seems to be approximately fairly *constant* during subperiods but then is abruptly changed being after that again constant for many years until a new change occurs. This feature suggests a version for (3.2.3) which allows the coefficient to be defined as time-dependent, for instance, as a most simple case, to be piecewise constant. Then

$$(3.2.4) \quad \bar{I}_{\text{real}}(t) = \prod_{\tau=1}^t (1 + \bar{i}_{\text{real}}(\tau)).$$

Table 3.1 exhibits an evaluation of the period splits and the values which are constant inside the subperiods but different for subsequent periods. Three subperiods were taken into the table and the constant slope of i_{real} inside of each

of them estimated. The positions of the winding points were determined to give an optimal fit with the observed data.

A common trait in quite many countries seems to have been that $i_{\text{real}}(t)$ was clearly positive in the 1950's, but then sometimes in the 1960's turned to be zero or even slightly negative, and in the 1980's again resumed a positive value.

Table 3.1. Rates of the real growth of equity values and the standard deviation σ_ϵ and skewness γ_ϵ of the residual variation

	$\bar{i}_{\text{real}}(t)$			\bar{i}_{real}	σ_ϵ	γ_ϵ
	1951-1965	1966-1982	1983-1990	1950-90		
USA	0.099	-0.046	0.106	0.0465	0.111	0.884
CANADA	0.073	-0.025	0.049	0.0366	0.134	0.151
JAPAN	0.177	0.011	0.211	0.0887	0.189	1.533
FRANCE	0.154	-0.057	0.172	0.0389	0.153	0.521
GERMANY	0.170	-0.032	0.119	0.0715	0.194	0.947
ITALY	0.132	-0.120	0.146	0.0133	0.278	0.946
UK	0.037	-0.654	0.072	0.0193	0.141	0.207
DENMARK	0.000	-0.020	0.090	-0.0023	0.185	0.707
FINLAND	0.085	-0.008	0.194	0.0393	0.306	1.465
NETHERLANDS	0.171	-0.061	0.147	0.0353	0.152	0.569
SWEDEN	0.043	-0.028	0.187	0.0275	0.123	0.622
SWITZERLAND	0.078	-0.055	0.095	0.0175	0.171	0.525

(e) **Reference index.** It is useful to introduce as an auxiliary variable the product of the inflation index and the *smoothed* real growth index as a particular *reference index* $I_{\text{ref}}(t)$ as follows

$$(3.2.5) \quad I_{\text{ref}}(t) = I(t) \cdot \bar{I}_{\text{real}}(t)$$

It indicates the combined effect of inflation and modelled (smoothed) real growth and is plotted in Figs. 3.1 as a central line of the shaded areas. These areas describe the variation range around $I_e(t)$.

The above construction can be illustrated imagining that there is an *attracting force* towards the reference index $I_{\text{ref}}(t)$, as if the equity index $I_e(t)$ would be tied it by a rubberband. If its course deviates from the reference curve, the market reacts by a tendency to restore a balance. A force (likely) proportional to the deviation will drag it back. This force is to be introduced by a proper construction of the noise factor $s(t)$. It remains still to find modelling for it.

(f) **The noise factor.** It is appropriate to write the noise variable $s(t)$ into form

$$(3.2.6) \quad s(t) = 1 + \delta(t).$$

where $\delta(t)$ is the above mentioned attraction force. The empirical data suggests an autoregressive form for it. Owing to the above definitions its mean is convenient to be chosen to be equal to zero. This can be achieved by a proper determination of $\bar{i}_{\text{real}}(t)$. Another approach is to determine the latter first e.g. by the method of the least squares. Then the mean of $\delta(t)$ will, as a rule, slightly deviate from zero.

Having regard to the multiplicative model of the indices (note the half-logarithmic scales in our figures), it is advisable to deal with the logarithm

$$(3.2.7) \quad d(t) = \ln(1 + \delta(t))$$

The Schwarz-Rissanen test, when applied to the data of Table 3.1, suggested in some countries a second order autoregressive time series, AR(n) with $n=2$,

$$(3.2.8) \quad d(t) = \alpha_1 \cdot d(t-1) + \alpha_2 \cdot d(t-2) + \sigma_\varepsilon \cdot \varepsilon(t)$$

whereas in some other countries a first order series ($n=1$, $\alpha_2 = 0$) turned out to be sufficient. $\varepsilon(t)$ is a gamma-distributed variable with mean zero, standard deviation unity and skewness according to the application of concern (DPP, Appendix D, section G.2(a)). The results are given in Table 3.2.

The variable z in (3.2.3) can be modelled similarly as s finding proper coefficients for (3.2.8)

Table 3.2. Estimates for the parameters of (3.2.8).

	n	α_1	α_2	σ_ε	γ_ε
USA	1	0.381		0.101	0.354
		0.498	-0.308	0.098	0.195
CANADA	1	0.393		0.124	0.150
		0.492	-0.252	0.122	0.064
JAPAN	2	0.513		0.152	0.292
		0.622	-0.213	0.144	0.234
FRANCE	1	0.361		0.139	-0.081
		0.411	-0.141	0.139	-0.156
GERMANY	2	0.529		0.163	0.477
		0.800	-0.511	0.143	0.070
ITALY	2	0.598		0.210	0.648
		0.822	-0.375	0.196	0.322
UK	2	0.600		0.124	0.123
		0.685	-0.142	0.118	0.184
DENMARK	1	0.599		0.159	0.034
		0.743	-0.241	0.155	-0.186
FINLAND	2	0.656		0.197	0.011
		0.876	-0.334	0.174	-0.056
SWEDEN	0	0.090		0.122	0.039
		0.111	-0.240	0.121	-0.018
SWITZERLAND	2	0.596		0.135	0.407
		0.920	-0.543	0.111	0.204
NETHERLANDS	1	0.674		0.138	-0.500
		0.767	-0.138	0.136	-0.436

The upper lines at each country indicate the case where the first order series AR(1) is applied and the lower lines the case AR(2).

It remains for the future studies to deliberate which of the above versions or possibly some quite different one turns out to be most appropriate. Fig.5.1. exhibits examples on simulated realizations experimenting with them.

3.3. Background factors

For an understanding of the behaviour of the equity price index potential determinants for the observed phenomena such as are seen e.g. in Figs 3.1 are now sought. In particular, explanations for the periodical changes in the slope of the real growth trend would be useful because it could help in potential further sharpening the modelling of the auxiliary real growth variables $z(t)$ in (3.2.3) and $\bar{i}_{\text{real}}(t)$ in (3.2.4). As candidates the long-term variability of a) the rate of interest, b) excessive inflation and c) GNP are tried. Their flow is plotted in the

lower boxes of the country diagrams in Fig. 3.1 and will be commented on in what follows.

(a) Rate of interest. A well-known feature is the cross-effect between the rate of interest and equity values. If the level of interest, or rather expectations of its future, increases, bonds and other interest-related instruments gain favour in capital market and investments have a tendency to move away from equities. As a consequence the equity prices go down. Hence, the equity curve and the interest curve can be expected to be mirrors of each other, ie when one is going up, the other is falling. Indeed, this is seen in some of the diagrams. For example, when the rate of interest (and obviously the expectations of its future growth) in the USA was trending up during the period about 1966-82, the real growth of the equity prices was slightly negative. When the interest curve turned downwards since 1982, the equity prices resumed a rapid real growth. A similar behaviour can be seen clearly in France, Germany, Italy, the Netherlands, Sweden, Switzerland and in the UK as well.

(b) The rate of inflation was experimented with as another potential determinant as is shown in the respective subboxes of Fig. 3.1b. When the rate of inflation is increasing, the growth of the real equity values is reduced. Likely the need to compensate for excessive inflation absorbs the capacity to give proceeds still over the inflation rate. In fact, the movements in the rate of inflation seem to be an equally good predictor for equity prices as the rate of interest and mainly in the same countries where the interest also is a satisfactory background variable. This is not surprising, because the inflation expectations may be provoking the movements of the level of interest, i.e. the inflation impacts are also coming through this way.

The oil crisis and the resultant soaring oil prices also affected world economies and naturally reflected on the stock exchange values as well. The UK temporary downturn in equity prices in the year 1974 (see Fig.3.1b) might be explainable by it. This kind of impulse is convenient to take into account by providing the inflation model with a submodule which allows to give a shock top in the rate of inflation (DPP, 7.3(g), Pukkila et al 1994, section 5).

(c) Gross national product can also be expected to have a significant influence on the real growth of the equity prices, because these are eventually backed by the increasing national wealth. This is confirmed in Fig. 3.3., but only for quite a long run, in fact mostly for the whole observation period.

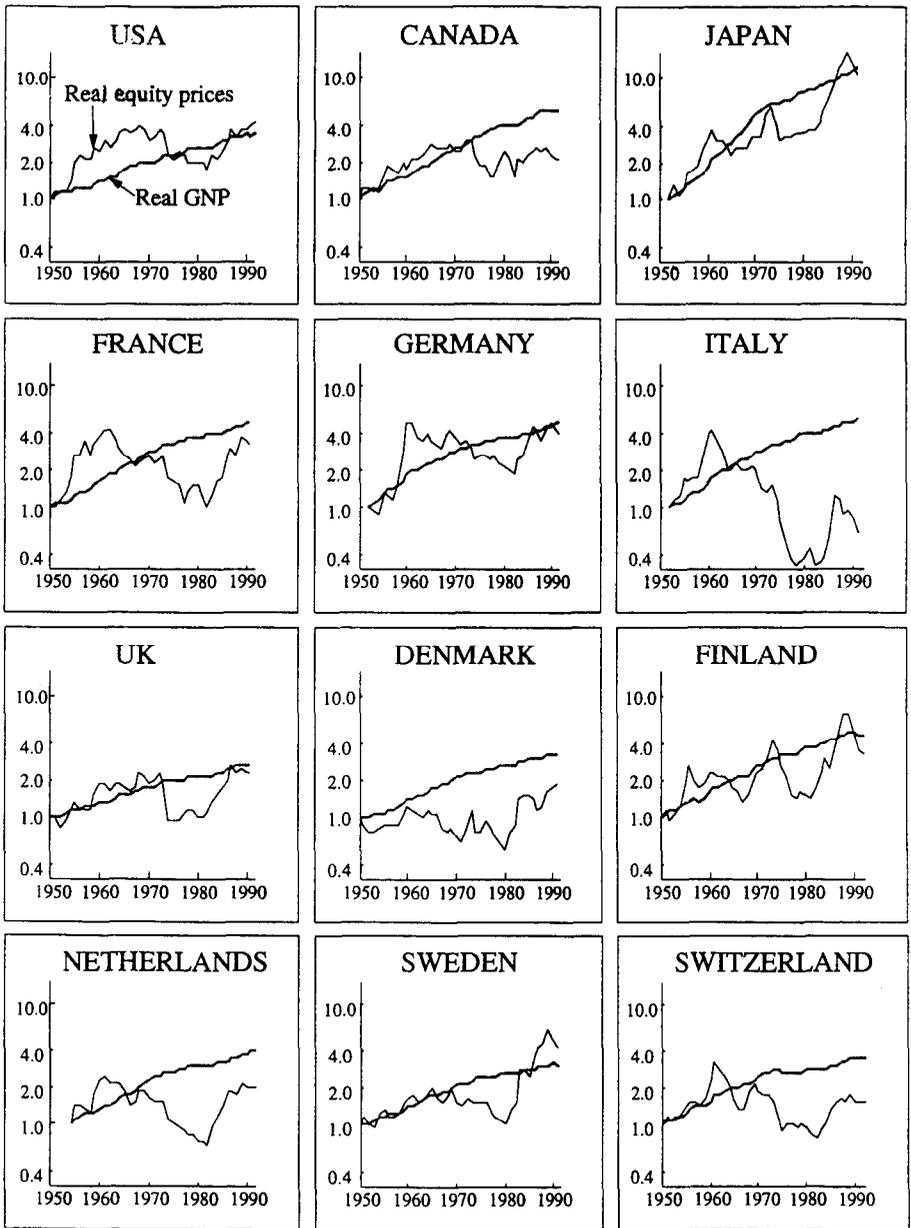


Fig. 3.3. The indices of real GNP and equity prices. Semi-log scale.

The order of the magnitude of the average growths of the equity prices and the GNP are approximately equal in most countries. However, during shorter periods the growth rates diverge so far that no useful correlation can be found. Though, it is justified to assume that if the equity prices deviate considerably and for a lengthy time from the trajectory determined by the long-term trend such as (3.2.2), the market forces drag them back just as was described as the attraction force effect for the short-term deviations $s(t)$ in the second paragraph of section 3.2(e) above. This can well be one of the contributing factors which appear in windings of the slope of the real value index seen in Fig. 3.2.

(d) Other factors. Furthermore, we must keep in mind that, alike inflation, there are also for capital markets numerous other relevant influencing factors and impulses, for instance, provoked by the actions of governments, changes in taxation, possibly parliamentary and presidential elections, the policy of central banks, labour market agreements, extreme weather conditions influencing agriculture, international impulses also other than oil crises etc. A strikingly similar behaviour of the markets has appeared in several countries as can be seen in Fig. 3.1.b. This indicates close international relationships and reflect possibly the USA-market's leading influence on other markets.

All in all, the behaviour of the capital markets is rather unpredictable at least in the short term. However, the uncertainties do not prevent us from building models, because, as was stated in the introduction section 1(a), we are not aiming at forecasting but in finding models which can generate outcomes of fluctuation structures which have similar features as what according to experience and sound actuarial adjustment can be expected to occur in the more or less remote future whatever are the reasons and their timing. The differences in environments and the uncertainty of the future can be explored by using different scenarios for basic assumptions. Hence, the model should be flexible enough to allow for various background factors as driving forces to be chosen according to local conditions of the application.

3.4. Outlines for the simulation of the equity prices

(a). There are already in existence well tried models for the simulation of inflation (DPP, Chapter 7, Pukkila et al 1994). Nor will the handling of the noise variable $s(t)$ in (3.2.1) give rise to serious difficulties. So the problem remains how to cope with the real growth factor $\bar{i}_{\text{real}}(t)$ in sections 3.2(b-d).

(b) **The real growth factor** $\bar{i}_{\text{real}}(t)$ was defined by suggesting three alternative versions in section 3.2 above. In all these cases standard statistical methods can be used to derive the necessary model parameters from the available past data and experience.

A particular special problem is to support and hone the model by trying to utilize the variables such as which were dealt with in section 3.3. This may be specially useful when an investment model is used as a part of an all-company asset-liability consideration. However, at the present stage of the work we have had to postpone these kinds of studies for the future. The comments in the subsequent sections are limited to the simple piecewise constant case of section 3.2(d) which is amenable to straightforward programming. A simple simulation of the versions in parallel will be exemplified in Fig.5.1.

(c) **Semi-deterministic approach.** A simple, but still useful way is to give the increment rates $\bar{i}_{\text{real}}(t)$ deterministically for equation (3.2.4). The idea might be to test the reaction of the portfolio (and the solvency of the insurer) if a period of high inflation, high rate of interest and low growth of the real equity values occur, for instance, about in the way and dimensions which can be seen in the diagrams of Fig.3.1b and Table 3.1 as the middle subperiod. The growth parameters may be set to be about normal for a certain time. Then they are changed as was drafted and after some suitably assessed time the normal values are again resumed, possibly providing the growth with some degree of acceleration. This kind of constellation can well simulate that kind of processes which are observed in the diagrams of Fig.3.1. The stochasticity is introduced by inflation and the term $s(t)$ (see (3.2.1)).

Furthermore, of particular importance is to provide the model with the ability to generate a drop, if not to speak about a crash, in the equity values, because the chance of such occurrences may be the most dangerous risk which is fatally threatening an insurer (or e.g. a pension fund) in cases where a considerable share of the asset portfolio is invested in equities. The past history has experienced so many such crashes that the possibility of a reoccurrence should not be ignored.

The reaction of the investment portfolio and, more generally, the solvency of the insurer can then be tested against various sorts of changes in the capital markets which are reflected via the rates $\bar{i}_{\text{real}}(t)$.

(c) **A fully stochastic approach.** A more sophisticated approach, which might be worth trying, is also to *randomize* the behaviour of the real index via its increment rates $\bar{i}_{\text{real}}(t)$ and the occurrence of a crash. A particular trigger indicator is drafted in DPP, section 8.5(i) to give a signal for a change of $\bar{i}_{\text{real}}(t)$ and for a potential downturn in equity values. Then further assumptions are needed for distributions of the changed value of $\bar{i}_{\text{real}}(t)$ and of the depth of a potential crash. They can be linked to the flow of inflation, the rate of interest or other relevant variables. The development of this approach was left to a future date.

3.5. Dividend income gained from equities.

The cash income term in (1.2) is obtained from

$$(3.5.1) \quad \mathcal{J}'_e(t) = j_e(t) \cdot \frac{A_e(t) + A_e(t-1)}{2}$$

where A_e is the relevant value of the equities of concern and the coefficient $j_e(t)$ indicates the dividend rates. The impact of inflation and real growth of equity values is coming via the A_e values as was dealt with in the previous section. Then the yield coefficient $j_e(t)$ is fairly stable. Therefore, a first approximation would be to model it as a constant around which it randomly fluctuates as is suggested by Ibbotson and Sinquefield (DPP, section 8.5(f)).

We tested the yield against various potential background phenomena similarly as was presented in section 3.1 above. The well-known feature (DPP, section 8.5(f)) was again found that the yield has a tendency to react to the changes in equity values with a time lag. If, owing to inflation or other reasons, the market values of the equities suddenly increase, $j_e(t)$ may lag behind due to inertia. This is seen in Fig. 3.4. (we regret that we could acquire for this edition long data series only from the UK and Finland).

The above observations suggest an amendment to the Ibbotson-Sinquefield formula linking it to variation of the equity prices. Following an idea originally proposed by Bonsdorff (1991) (see DPP, equations (8.5.7) and (8.5.8)) the noise term $(1 + \delta(t))$ (see (3.2.6)) is damped and its influence delayed. This can be

achieved defining the coefficient in (3.5.1) as follows

$$(3.5.2) \quad j_e(t) = \bar{j}_e \cdot \frac{1 + \zeta(t)}{1 + \delta(t)}$$

where \bar{j}_e is the average dividend yield rate and the damping variable is generated from

$$(3.5.3) \quad 1 + \zeta(t) = \beta \cdot (1 + \delta(t)) + (1 - \beta) \cdot (1 + \zeta(t-1))$$

Here β is a smoothing factor subject to the constraint $0 < \beta < 1$.

Remark. The reason to choose $1 + \delta(t)$ as the linking variable in (3.5.2) instead of e.g. $i_e(t)$ is the fact that the latter is influenced also by inflation and (smoothed) real growth trend in addition to the short-term fluctuation which is presented by $\delta(t)$. But the two first mentioned factors are sufficiently effectuated via the A_k in (3.4.1) and, therefore, are not taken into account still another time.

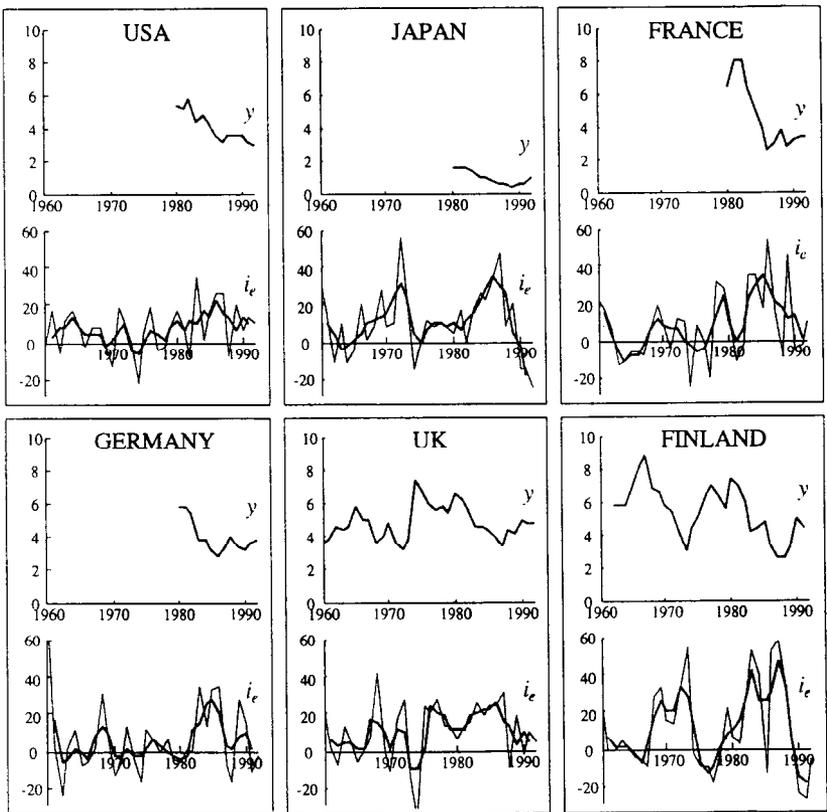


Fig 3.4. Dividend yield y and the changes of the equity prices i_e as annual rates and as moving averages.

4. Property values

DPP suggests that for property values and the yield therefrom the same model structure could be used as was proposed for the equities (DPP section 8.4(e,f) dealing with the Wilkie-model and 8.5(d) with its modification). Being unable as yet to extend our studies to this investment category we have followed the same line when constructing the property component for the portfolio models.

5. Total return on investments, examples

(a) **The total return $J(t)$** on the investment portfolio is obtained according to (1.1) and (1.2) first generating both the values $A_k(t)$ and the cash income $J'_k(t)$ for each of the asset categories by using the methods which were dealt with in the previous sections. The consideration is based on simulation which progresses year by year over a period $t=0,1,2,\dots,T$. Repeating the simulation numerous times (Monte Carlo method) a conception of the general character of the process can be obtained.

Because we are not making any concrete forecasting but rather investigating the properties of various asset categories and their mixes in general terms, the time span T in our example diagrams is chosen as long as 100 years so that different variation patterns would have a chance to appear in the relevant random process.

Only some scattered examples on some of the asset items will be given in the subsequent figures. The dealing with mixed portfolios and other further analysis is postponed for future works as will be outlined in section 6.

(b) **Real growth variants**, which were considered in sections 3.2(b,c and d) are simulated in Fig.5.1 The specifications are given in the caption of the figure.

Inflation is generated by using the AR(1) formulae presented by Pukkila (1994, section 4) fitting the parameters to approximately correspond to the USA data

$$(5.1) \quad \mu = 0.04, \phi_1 = 0.873, \delta_1 = 0.635, \delta_2 = -0.474, \omega_0 = 0.042, \\ \sigma_\varepsilon = 0.01 \text{ and } \gamma_\varepsilon = 0.6, \text{ no outside shocks.}$$

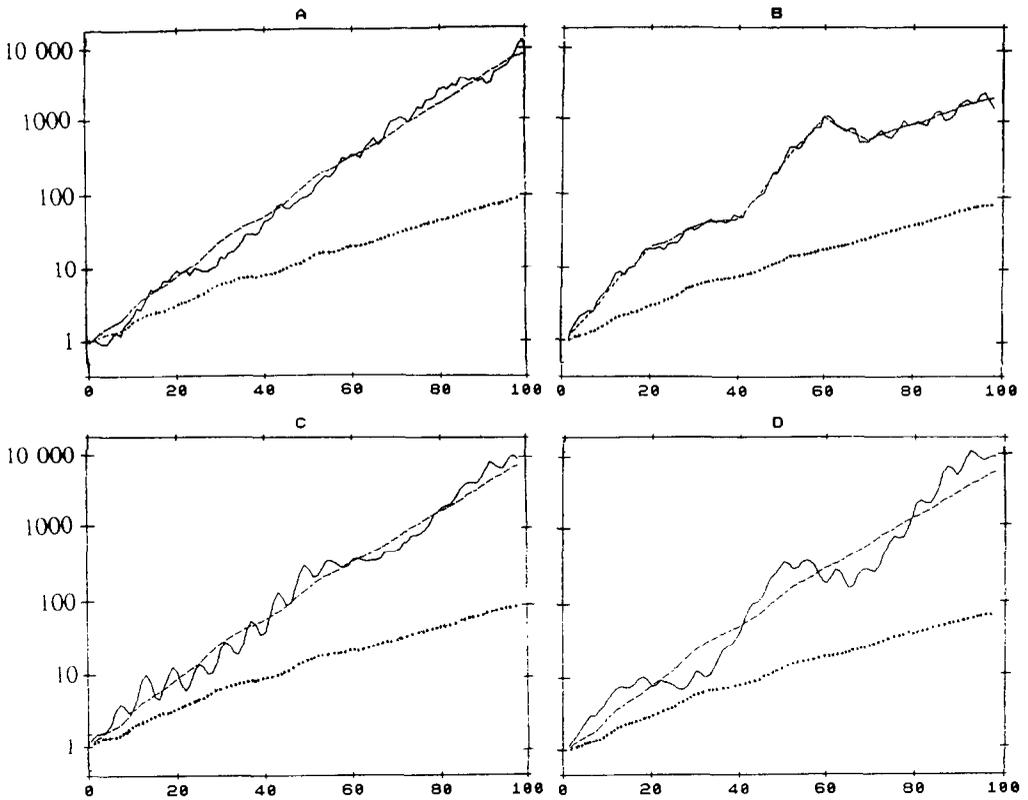


Fig. 5.1. Alternative choices of the real growth definition $\bar{I}_{\text{real}}(t)$. Dotted line displays the simulated inflation and the dash dotted the reference index (3.2.5). A. Constant real growth (3.2.2). B. Piecewise constant real growth (3.2.4). C. The growth function $z(t)$ in (3.2.3) is deterministically given as a sine function. D. Function $z(t)$ is generated as an autoregressive time series AR(2). The noise variable $s(t)$ is generated as an AR(2) series fitting it to the respective cases.

(c) **Single realization diagrams.** Fig. 5.2 displays simulated outcomes of the rates of inflation, interest and dividends as well as the equity prices both as growth rates and index.

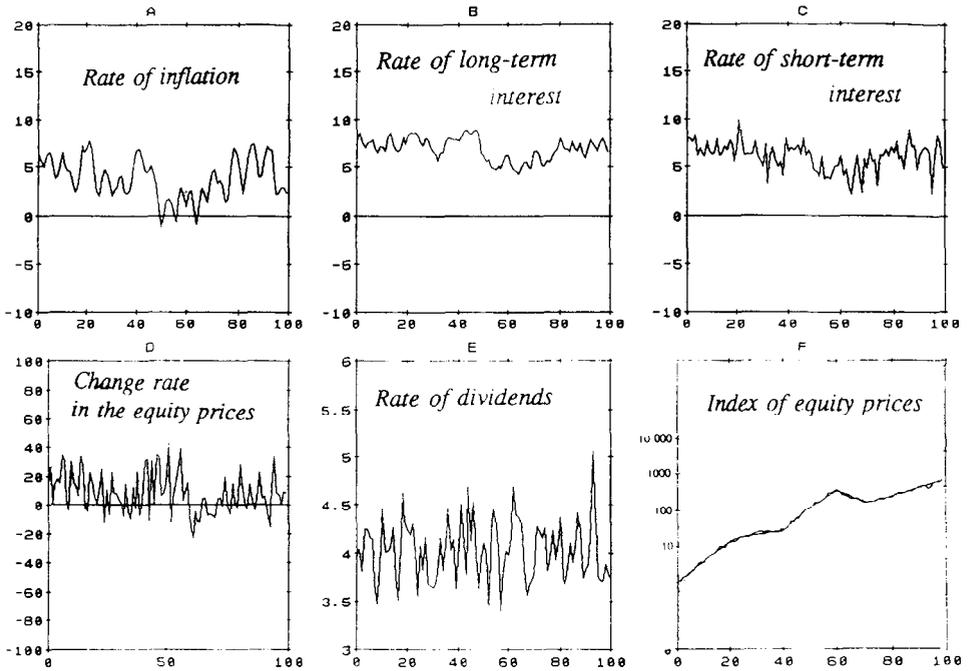


Fig. 5.2. Simulated realizations. The rates are given in percentages and the index in semi-logarithmic scale with the initial level = 1.

The rate of the long-term interest is generated by the formulae of section 2 above applying the parameter values

$$(5.2) \quad \delta = 0.146, \mu = 0.031, \phi_1 = 0.031, \phi_2 = -0.100, \\ \sigma_{\text{clong}} = 0.007 \text{ and } \gamma_{\text{clong}} = 0.$$

and for the short-term formula (2.6)

$$(5.3) \quad a = 0.751, b = 0, \sigma_{\text{cshort}} = 0.012 \text{ and } \gamma_{\text{cshort}} = 0.$$

The equity prices were simulated using the piecewise constant real growth version 3.2(d) with parameters

$$(5.4) \quad \alpha_1 = 0.5, \alpha_2 = -0.3, \sigma_{ed} = 0.1 \text{ and } \gamma_{ed} = 0 \text{ and}$$

for the subsequent periods with end points 20, 40, 60, 70 and 100 respectively $\bar{i}_{real}(\tau) = 1.100, 0.995, 1.120, 0.900$ and 1.010 .

For the dividend formulae of section 3.5 the following parameter values were used

$$(5.5) \quad \bar{j}_e = 0.04 \text{ and } \beta = 0.3.$$

d) Monte Carlo diagrams. Fig. 5.3 deals with the same variables as Fig.5.2. Now inflation is first simulated and then kept fixed. The other variables are generated 20 times in order to demonstrate the influence of their inflation-independent stochasticity separately.

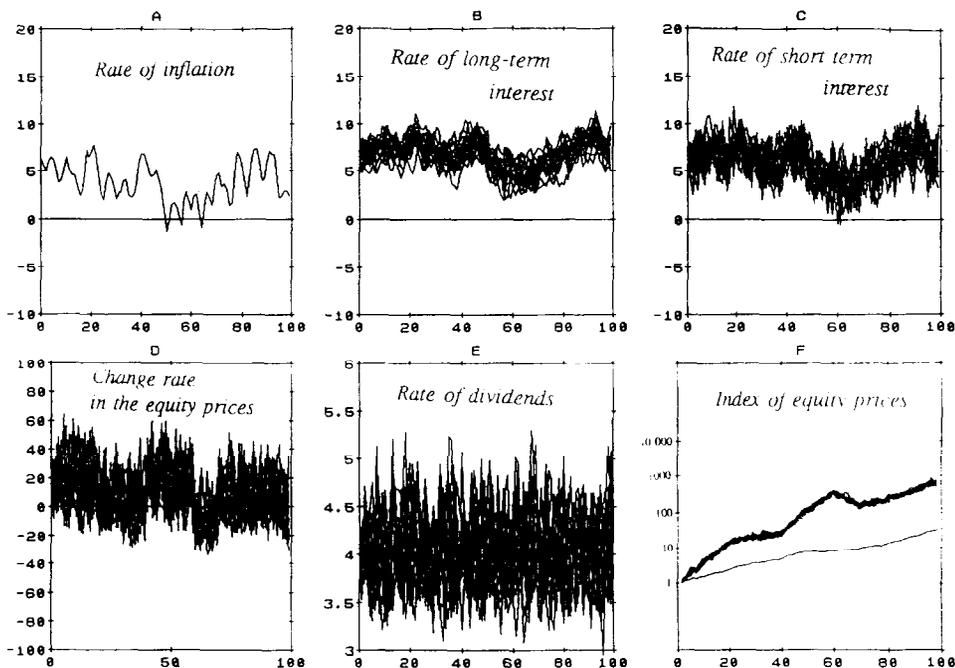


Fig.5.3. Inflation generated into box (A) after which 20 realizations are generated for each of the same variables as depicted in Fig. 5.2.

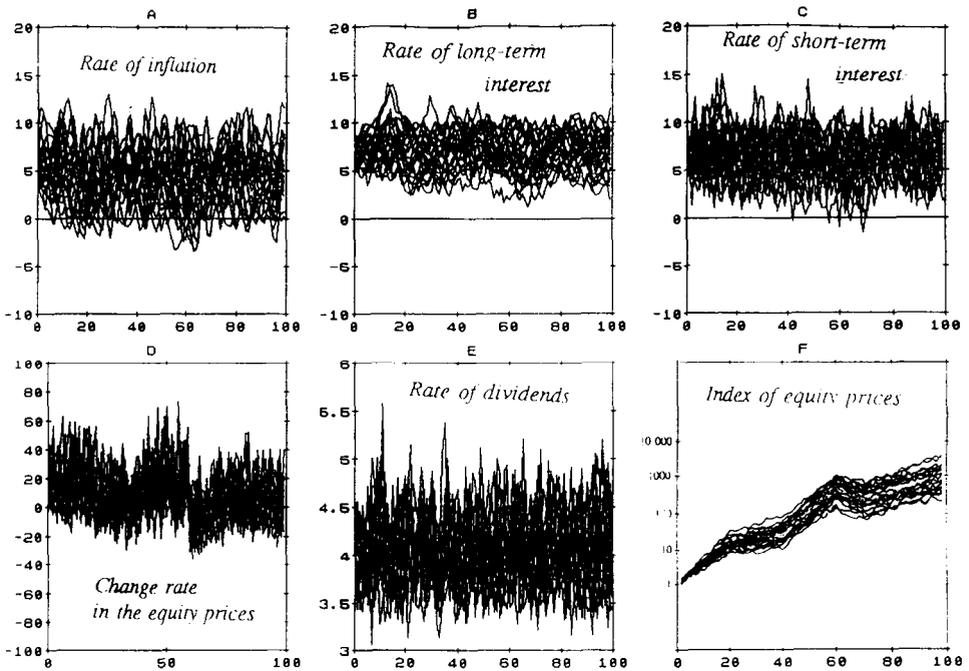


Fig. 5.4. The same variables as in Fig. 5.3. but now for each simulation also inflation is generated anew.

6. Future avenues

We plan to complete the work extending the simulations to concern mixes of various sorts of investments corresponding to suitably specified distribution parameters w_k in the basic equations (1.1) and (1.2).

(a) Various investments strategies can be tested, for instance optimal mixes of investment objects can be sought and sensitivity analyses performed.

Portfolio theories commonly use as a boundary condition the maximal risk which is constructed by means of standard deviations and covariance matrices (DPP, 8.6(d)). In the simulation approach this is replaced by a more general condition that the simulated bundle of the outcomes should not spread below

some acceptable safety level (see e.g. Fig. 5.4).

A major benefit of the simulation method is that it is not limited to any one-period analyses which are typical for many econometric portfolio theories (with few exceptions such as by Coutts). Furthermore, without overwhelming technical complications, background variables and their correlations can be engaged in the considerations such as inflation, movements of interest rates and GNP. Dynamical rules for the continual control of the portfolio can be incorporated into the model, for instance, disinvestments, reinvestments, and other active tactical investment policies.

The immunization might be of interest in asset-liability models (DPP, section 8.1(c))

(b) A simple, but for practical purposes quite expedient **procedure** to perform the optimization of the portfolio mix is trial and error, i.e. trying different mixes until a satisfactory one is found.

(c) A more sophisticated approach is first to simulate each of the relevant investment categories separately (as e.g. in Fig. 5.4) and to deduce therefrom the mean return for any desired period and the variation range (in terms of standard deviation and skewness, see DPP Appendix G) and the correlations between the investment categories. Then we have the basic data which are needed for the conventional portfolio optimization using the quadratic programming (DPP, 8.6(d)). An advantage, compared with the conventional portfolio theories, is that numerous background phenomena and dynamic control is regarded for and the situation can be analyzed at any future time point.

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Numerous further references can be found in DPP among others to the important works of Wilkie, Daykin and Hey, FIMAG, etc.

