THE USE OF INTERVALS OF POSSIBILITIES TO MEASURE AND EVALUATE FINANCIAL RISK AND UNCERTAINTY

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SUMMARY

A major concern in modeling the real world is the expression and treatment of vague, fuzzy, and imprecise information. Common approaches to this concern include probability theory and fuzzy set theory. We introduce instead intervals of possibilities as very general and powerful tools for modeling, analysis and decision making, even with scant amount of information about the uncertain values. To express the imprecision in the available information we replace a real number with an interval of possibilities whose extreme values can be interpreted as the beliefs of a pessimist and an optimist about the possible values of an entity or event. All values in the interval are "possible values," without an associated measure of truth or probability density. This approach has potential for improved decision making, as it enables the decision maker to be cognizant of the impact of imprecision throughout the decision making process.

We present the mathematics of intervals of possibilities and their relation to fuzzy sets and probability theories, and demonstrate their applicability to particular finance and insurance instances.

L'utilisation d'intervalles de possibilités dans la mesure et l'évaluation du risque financier et des incertitudes

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Résumé

L'une des difficultés lors des tentatives de modélisation du monde réel est l'expression et le traitement des informations vagues, floues et imprécises. L'approche habituelle de ce type de problème est l'utilisation de la théorie des probabilités et des ensembles flous. A leur place, nous présentons les intervalles de possibilités en tant qu'instruments d'utilisation généralisée et de grande efficacité pour la modélisation, l'analyse et la prise de décisions, même en présence d'informations fragmentaires concernant les valeurs incertaines. Pour exprimer l'imprécision des informations disponibles, nous remplaçons un nombre réel par un intervalle de possibilités dont les valeurs extrêmes peuvent être interprétées comme le point de vue d'un pessimiste et d'un optimiste sur les valeurs possibles d'une entité ou d'un événement. Toutes les valeurs de l'intervalle sont des « valeurs possibles », sans qu'une mesure de vérité ou de densité de probabilité y soit associée. Cette approche est susceptible d'améliorer la prise de décision, puisqu'elle permet au décisionnaire connaître l'impact de l'imprécision sur l'ensemble du processus de la prise de décision.

Nous présentons les structures mathématiques des intervalles de possibilités et leur relation avec les théories des ensembles flous et des probabilités et nous démontrons leurs possibilités d'application dans certains domaines de la finance et des assurances.

1. INTRODUCTION

Modeling the real world means idealizing it. A model is developed on the basis of hypotheses and assumptions, which may more or less resemble reality. One of the major concerns when modeling real world phenomena is the expression, and treatment, of vague, fuzzy, and imprecise information that abound in real world situations.

The underlying uncertainties of real events and activities can often be well described by probability theory. In ordinary Boolean algebra on which probability theory can be based, an element is either contained in a given set or not, and the transition of an element from the outside into a set is abrupt, rather than gradual. Consequently, probabilities represent uncertainties but not degrees of partial truths which are used for the description of imprecision by degrees of membership [2,7]. If imprecision is the state of nature of a situation and the resulting uncertainty is <u>possibilistic</u> rather than <u>probabilistic</u>, then the situation is said to be fuzzy [3].

Zadeh [11] first described imprecise, possibilistic uncertainties by degrees of membership which require a gradual transition from non-membership to full membership. To this end he enhanced the membership concept of set theory to the grade of membership of fuzzy set theory.

Probability theory and fuzzy set theory are well suited to describe many specific instances of uncertainty. Such a description, through the assumption of a certain distribution function with given parameters, or of a membership function that specifies the grade of membership, requires a vast amount of inherent knowledge. Real situations, however, are often characterized by vagueness and lack of information and knowledge. In such cases it seems in principle contradictory to analyze lack of information by models and assumptions which require considerable amount of reliable knowledge. Rather, the less exact the information in a model, the better it may fit a reality that is vague and actually characterized by lack of knowledge and information.

Based on this principle, we represent imprecise information by intervals of possibilities, which replace exact real values numbers by intervals of possibilities on the real line. Each interval represents a set of possible values that a particular entity or variable may assume, without any a-priori assumption about exact value, probability measure or grade of membership. In particular, intervals of possibilities should be used whenever decision variables can assume different values, but a probability measure on these values is not available or justifiable. Circumstances where the introduction of intervals of possibilities to describe real life positions is recommendable can be found in numerous situations in many economical, engineering and natural scientific branches.

We define an interval of possibility by an "infimum" and a "supremum" values that can be interpreted as the beliefs of a pessimist and an optimist about the value that an event may assume, or as the "worst" and "best" that an "average" person may expect. All values in the interval are "possible values", without any associated measure of probability or truth, respectively. The values outside the interval of possibility are assumed to be "impossible." It is convenient, however, but usually not necessary, to denote one of the possible values as "plausible" and associate with such a "plausible value" the belief of an "average person".

For the simplification of problems and their solution, randomness is traditionally ignored, e.g., by replacing random variables with expected values, at an early stage of a decision making process. By using intervals of possibilities, and the arithmetic operations that we define on such intervals, a decision maker can remain cognizant of the impact of uncertainty and imprecision throughout a decision making process, and delay any artificial elimination of uncertainty to the last possible moment.

The mathematics of intervals is well developed [5,8,9], especially in the computer science domain where an inherent imprecision results from the finite representation of real numbers; the inclusion of plausible values, however, have not yet been applied. Furthermore, the use of this approach in decision making and its application to financial and insurance problems are new. Buckley [3] introduced fuzzy numbers whose membership function requires the specification of four points and two functional forms that actually include much, usually non-existent or unknown, information. Berliner and Buehlman [2] introduced a special, simplified fuzzy number called triangular fuzzy number; it is a fuzzy number whose two inner points are combined and linear functional forms are used. If the fuzzy arithmetics used by Buckley [3] is applied to two triangular fuzzy numbers the result may be a non-triangular fuzzy number.

In section 2 we present the mathematics of intervals of possibilities, and extend it to positive intervals of possibilities. We show that these intervals are a natural extension of real numbers, and that they can be regarded as fuzzy sets through the application of an appropriate membership function. In section 3 we present cash flows and their value, and apply the mathematics of intervals of possibilities to these cash flows. In section 4 we specialize our results to particular financial and insurance instances, and demonstrate their usefulness and applicability.

2. INTERVALS OF POSSIBILITIES

An interval of possibilities around a plausible value b is defined as an ordered triplet of real values, $B = (\underline{b}, b, \bar{b})$[1], where $\underline{b} \leq b \leq \bar{b}$. $\underline{b}$ is called the infimum value of the interval of possibility, b is the plausible value of the interval, and $\bar{b}$ is the supremum value of the interval. These correspond to the intuitive notion of an "average" person regarding uncertainty and imprecision: a plausible value is assumed to exist, but the "true" value may differ from the plausible value and be anywhere in the interval of possibilities around this plausible value. This interval is bounded by infimum and supremum values; any value outside the interval is considered to be an impossible, or non-achievable, value.

An interval of possibilities B represents the set of all the possible values for the plausible value b; a possible value for b is any $b' \in B$.[2] A real number b is the limiting case of the interval of possibilities B in which $\underline{b} = b = \bar{b}$, and the value of b is known precisely and without any uncer-

[1] We identify intervals with capital letters, and real values with small letters. Further, an interval of possibilities may be denoted with the triplet $(\underline{b}, b, \bar{b})$ or with the closed real interval $[\underline{b}, \bar{b}]$.

[2] For simplicity of presentation we define an interval of possibilities as a continuous interval. We could have instead defined a discrete "interval" of possibilities, e.g., $B = \{\underline{b}=b_1, .., b_n=\bar{b}\}$, where the plausible value b is one of the b_is for i=1, .., n. With this definition, though, one must be careful to assure that the result of arithmetical operators remain valid, i.e., within the set of feasible values.

tainty; we will denote the related redundant interval of possibilities as $\bar{\underline{B}}$.

Two intervals of possibilities are equal, if all the related components (infimum, plausible and supremum) are identical. While a complete order cannot be defined on the set of intervals of possibilities, we can define the partial order relation "≤": $A \le B$ if $\underline{a} \le \underline{b}$, $a \le b$, and $\bar{a} \le \bar{b}$; further, we say that $A < B$ if $A \le B$ and at least one of the three defining inequalities is strict. In particular, $\bar{\underline{A}} \le B$ whenever $a = \bar{a} \le \underline{b}$, and $B \le \bar{\underline{A}}$ whenever $\bar{b} \le a = \underline{a}$. Clearly, $A \neq B$ whenever $\underline{a} \neq \underline{b}$, $a \neq b$ or $\bar{a} \neq \bar{b}$.

The <u>width</u> of an interval of possibilities B is defined as $\bar{b} - \underline{b}$; the smaller the width, the narrower the interval, and the more precise, or less uncertain, is the knowledge of the plausible value b. Two intervals of possibilities A and B intersect, i.e., $A \cap B \neq \emptyset$, whenever $\underline{a} \le \underline{b} \le \bar{a}$ or $\underline{b} \le \underline{a} \le \bar{b}$ or $\underline{a} \le \bar{b} \le \bar{a}$ or $\underline{b} \le \bar{a} \le \bar{b}$. An interval of possibility B is contained in an interval of possibility A, denoted as $B \subsetneq A$, if and only if $\underline{a} \le \underline{b} \le \bar{b} \le \bar{a}$.

2.1 <u>Arithmetical Operations on Intervals of Possibilities</u>

If A and B are intervals of possibilities, and $\odot$ is any arithmetical operator, we define

$$A \odot B = (\min_{a' \in A, b' \in B}\{a' \odot b'\},\ a \odot b,\ \max_{a' \in A, b' \in B}\{a' \odot b'\}) \quad (1)$$

This definition assures that the resulting interval of possibilities contains all possible combinations of pairs of values from the intervals of possibilities that are related by the arithmetical operator. When this definition is applied to particular operators, the result is often simplified. In particular, we have for addition and substraction:

$$\left.\begin{aligned}
&A + B = (\underline{a}+\underline{b},\ a+b,\ \bar{a}+\bar{b}) && (2a)\\
&\bar{\underline{0}} - A = -A = (-\bar{a},\ -a,\ -\underline{a}) && (2b)\\
&A - B = A + (-B) = (\underline{a}-\bar{b},\ a-b,\ \bar{a}-\underline{b}) && (2c)
\end{aligned}\right\} \quad (2)$$

Addition is associative and commutative, and $\bar{\underline{0}} = (0, 0, 0)$ is the unit element for addition and substraction, as $\bar{\underline{0}} + A = A + \bar{\underline{0}} = A - \bar{\underline{0}} = A$ and $\bar{\underline{0}} - A = -A$. However, $A - A \neq \bar{\underline{0}}$ whenever $\underline{a} \neq \bar{a}$.

Multiplication is well defined for any two intervals of possibilities. In the general case, (1) leads to expressions which take into account the end-points of the two intervals of possibilities; but then one has to consider nine separate cases relating to the signs of the various end-points (see, e.g., [9] page 12.) Of these cases, the following are of particular interest to us:

$$\left.\begin{aligned}
&A \cdot B = (\underline{a}\cdot\underline{b},\ a\cdot b,\ \bar{a}\cdot\bar{b}) && \text{if } 0 \le \underline{a} \text{ and } 0 \le \underline{b} && (3a)\\
&A \cdot B = (\bar{a}\cdot\bar{b},\ a\cdot b,\ \underline{a}\cdot\underline{b}) && \text{if } \bar{a} \le 0 \text{ and } \bar{b} \le 0 && (3b)\\
&A \cdot B = (\underline{a}\cdot\bar{b},\ a\cdot b,\ \bar{a}\cdot\underline{b}) && \text{if } \bar{a} \le 0 \text{ and } 0 \le \underline{b} && (3c)
\end{aligned}\right\} \quad (3)$$

To avoid the complexities of the general case, we define a <u>positive interval of possibilities around a plausible value</u> $\underline{b}$ as an interval of possibilities around b where $0 \le \underline{b}$. Consequently, for two positive intervals of possibilities multiplication is defined by (3a).

When division is considered, one has to worry about the inclusion of the real value 0 in the interval of possibilities. Indeed, if 0 is not contained in the interval of possibilities A, then

$$1 / A = (1/\bar{a},\ 1/a,\ 1/\underline{a}) = \{1/a' : a' \in A\} \qquad (4)$$

and the width of $1 / A$ is $1/\underline{a} - 1/\bar{a}$, i.e., it is bounded. However, when $0 \in A$, the set given in (4) is unbounded and cannot be described by an interval of possibilities whose endpoints are finite real numbers such as $1/\bar{a}$ or $1/\underline{a}$. The value of $1/a$ will approach $+\infty$ or $-\infty$, as a approaches 0 from above or below; consequently, the set definition in (4) will result in $1 / A$ being the whole real line. In the sequel, we shall assume that 0 is not included in the intervals of possibilities we consider. Further, if B is positive (i.e., $0 < \underline{b}$), we have from (4) for the division

$$A / B = A \cdot (1 / B) = (\underline{a}/\bar{b},\ a/b,\ \bar{a}/\underline{b}) \tag{5}$$

Multiplication of intervals of possibilities, like addition, is associative and commutative. It has $\bar{\underline{1}}$ as the unit element, as $\bar{\underline{1}} \cdot A = A \cdot \bar{\underline{1}} = A$, and obeys $A \cdot \bar{\underline{0}} = \bar{\underline{0}} \cdot A = \bar{\underline{0}}$. In analogy to substraction, we have $A / \bar{\underline{1}} = A$, but $A / A \neq \bar{\underline{1}}$ whenever $\underline{a} \neq \bar{a}$. Furthermore, the distributive law does not always hold, e.g., $A \cdot (B - B) \neq A \cdot B - A \cdot B$ whenever A and B are non-redundant intervals of possibilities (i.e., $A \neq \bar{\underline{A}}$ and $B \neq \bar{\underline{B}}$.) Rather, intervals of possibilities are sub-distributive, as $A \cdot (B + C)$ is included in $A \cdot B + A \cdot C$ (see, e.g., [9] page 13.) The distributive law holds, however, when A is real (i.e., $A = \bar{\underline{A}}$) or $\bar{\underline{0}} < B \cdot C$.

Powers of intervals can be defined using (1). We, however, will be interested only in the power operation for positive intervals of possibilities. In this case we have for the positive intervals of possibilities A and B

$$A^B = \begin{cases} (\underline{a}^{\underline{b}},\ a^b,\ \bar{a}^{\bar{b}}) & \text{if } \bar{\underline{1}} \leq A \quad \text{(6a)} \\ (\underline{a}^{\bar{b}},\ a^b,\ \bar{a}^{\underline{b}}) & \text{if } A < \bar{\underline{1}} \quad \text{(6b)} \\ \text{computed by (1)} & \text{if } 1 \in A \quad \text{(6c)} \end{cases} \tag{6}$$

Note that whenever B is real, i.e., $B = \bar{\underline{B}}$, the definitions in (6a) and (6b) lead to the same result, since then $\underline{b} = \bar{b}$.

With these operators, arithmetics on intervals of possibilities can be extended to any real function. Indeed, as Petkovic shows ([9], section 3.3), if f is a rational function of n real variables, and F is the interval extension of f, i.e., an interval valued function of n interval variables which coincides with f when the intervals degenerate to real values, then the following holds:

> The interval value of F contains the range of values of the corresponding real function f, when the real arguments of f lie in the intervals used in the evaluation of F.

Symbolically, if the variables are $x_1, .., x_n$, and the intervals are $X_1, .., X_n$, then $f(x_1, .., x_n) \epsilon F(X_1, .., X_n)$ whenever $x_i \epsilon X_i$ for $1 \leq i \leq n$.

Often many different expressions can be presented for the same real-valued function f. While all the interval extensions of these forms contain the range of values of f, they are not necessarily all of the same width. Indeed, certain functional forms yield narrower intervals, and thus a higher precision and lower uncertainty for the results; considerable work in the interval mathematics literature is devoted to this subject (see, e.g., [5, 8, 9].)

2.2 Intervals of Possibilities, Fuzzy Sets and Fuzzy Numbers

While intervals of possibilities present a very different approach to the representation and treatment of uncertainty, as compared with the approaches of probability and fuzzy sets theories, they are also related to these methodologies. In this section we demonstrate how a membership function, as required by fuzzy sets theory, can be defined for an interval of possibilities. For additional discussion of these issues see Babad and Berliner [1].

As noted by Lemaire [7], "a fuzzy set is a class of objects in which there is no sharp boundary between the objects that belong to the class and those that do not. More specifically, a fuzzy set A in a collection of objects X is a set of ordered pairs

$A = \{(x, U_A(x)), \text{ where } x \in X\}$

where U_A: $X \mapsto M$ is the membership function that determines the grade of membership of $x \in X$ in A. M is the membership space, and it can be assumed that M is the interval [0, 1], with 0 and 1 representing, respectively, the lowest and highest grades of membership. In general, the degree of membership of x in A corresponds to the truth value of the statement: x is a member of A." For additional discussion and illustrations see, e.g., Lemaire [7].

An interval of possibilities is not a fuzzy set since we do not associate any measure of membership with particular points within the interval. However, one may construct for each interval of possibilities a function which corresponds to a fuzzy set's membership function. To this end, let A be an interval of possibilities. As a first step, define for each $u \in [0, 1]$ a <u>certainty interval of level u</u>

$I_A(u) = [a - u\cdot(a-\underline{a}), a + u\cdot(\bar{a}-a)]$

Then $I_A(0) = [a, a] = \underline{\bar{A}}$, while $I_A(1) = [\underline{a}, \bar{a}]$ which coincides with the interval of possibilities A. Now define an interval function from the real line to [0, 1], which corresponds to a membership function of a fuzzy set:

$$U_A(x) = \begin{cases} 0 & \text{if } x \notin I_A(1) = [\underline{a}, \bar{a}] \\ 1-u & \text{if } x \in I_A(v) \text{ but } x \notin I_A(w) \text{ for } w < u \leq v \leq 1 \end{cases}$$

Then $\{(x, U_A(x))$, where x is a real value$\}$ is a fuzzy set corresponding to an interval of possibilities A. Note that $U_A(a) = 1$, while $U_A(\underline{a}) = U_A(\bar{a}) = 0$.

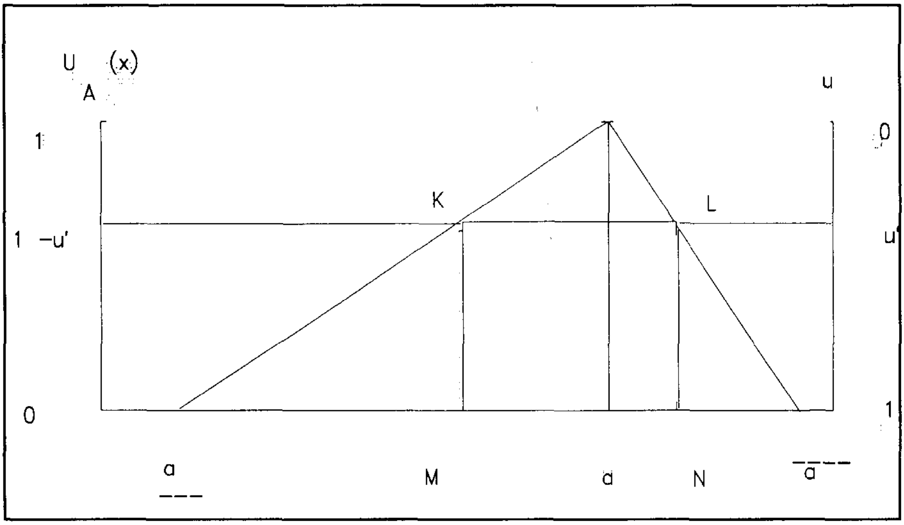

Figure 1: Interval of Uncertainty and its Certainty Function

These concepts are illustrated in Figure 1, which demonstrates an interval of possibilities ($\underline{a}$, a, $\bar{a}$) on the x-axis; above it we display a certainty interval [K, L] of level u', whose interval function has the level 1-u'. [K, L] (or its projection [M, N] on the x-axis) is narrower, and thus more precise and less uncertain than the original interval of possibilities. The triangle with base [$\underline{a}$, $\bar{a}$] and top a = $I_A(0)$ represents all the possible certainty intervals for ($\underline{a}$, a, $\bar{a}$); the nearer one is to the top, the narrower the certainty interval, the more precise the information, and the higher the value of the interval function. The top of the triangle corresponds to the precise real number a, while the base [$\underline{a}$, $\bar{a}$] contains all its possible values.

Fuzzy numbers, as defined by Buckley [3], and their simplification into triangular fuzzy numbers as done by Berliner and Buehlmann [2], are instances of fuzzy sets that are particularly appropriate for arithmetical manipulations. In fuzzy number arithmetics, however, the inverse of the membership functions is used to determine the new interval of membership. Intervals of possibilities, in contrast, require less information than fuzzy numbers, and perform all arithmetical operations on the intervals themselves; then, if it is desired to provide the related membership function, the interval function is computed. Consequently, in our approach the membership relationships between the operands and resulting intervals of possibilities may not be immediately apparent, but the critically important concept of "possible values" remains intact; in fuzzy numbers theory the opposite is true.

3. INTERVALS OF POSSIBILITIES AND CASH FLOWS

The valuation of a cash flow is a major subject of study in finance and insurance. Our objective is to develop expressions for the financial measures of uncertain and/or imprecise cash flows, i.e., cash flows in which some - or all - of the amounts, times, interest rates and/or the number of payments are defined by intervals of possibilities. In particular, we would like to express the resulting intervals of possibilities as functions of the original intervals, and in this way assess the level of uncertainty and imprecision of the financial measures that are associated with cash flows.

3.1 Cash Flows - Definitions and Measures

A Cash Flow is a sequence of amounts which are either received or paid out at certain points of time. Formally, a cash flow can be represented as a sequence of triplets,

$\{(a_n, t_n, i_n), n = 1, .., N\}$,

where each triplet corresponds to an amount a_n being paid/received at time t_n under the applicable interest rate i_n. The sign of a_n determines whether the amount is being paid or received; we are not concerned with the actual assignment of positive or negative values to a payment or receivable. We assume that $t_1 < .. < t_N$; t_1 is the initial time of the cash flow, and t_N is the terminal time. By changing the current time t_0 (which can be anywhere on the time axis), the cash flow can be shifted in time; this shift inevitably will affect the valuation of the cash flow. For the sake of simplicity we assume that $t_0 = t_1$. The span of the cash flow is the time difference $t_N - t_1$, where N is the number of payments. With each interest rate i_n we associate an interest factor $r_n = 1 + i_n$ and a discount factor $v_n = 1 / r_n$; we assume that $i_n > -1$ for all n and that i_n is applicable for a_n from time t_1 until time t_n, and consequently the present value of this amount is $a_n \cdot v_n^{t_n - t_1}$; the relationship between this assumption and the conventional framework, in which i_n is applicable from time t_{n-1} upto time t_n is discussed in Babad and Berliner [1].

Certain measures are traditionally associated with cash flows (see, e.g., Bühlmann and Berliner [4]), of which probably the most important is the value of the cash flow at some point in time. In this paper we concentrate on the present and future values of the cash flow; for discussion of intervals of possibilities for other measures of cash flows, see Babad and Berliner [1]. For simplicity, we will fix the present value at the initial time, and the future value at the terminal time. These values are defined as

$$PV = \underline{\text{present value}} = \sum_{n=1}^{N} a_n \cdot v_n^{t_n - t_1} \text{ and}$$

$$FV = \underline{\text{future value}} = \sum_{n=1}^{N} a_n \cdot r_n^{t_N - t_n}.$$

To simplify the presentation let $\tau_n = t_n - t_1$, $\varsigma_n = t_N - t_n$, and perform all summations from 1 to N. Then

$$\left.\begin{array}{ll} PV = \text{present value} = \sum a_n \cdot v_n^{\tau_n} & (7a) \\ FV = \text{future value} = \sum a_n \cdot r_n^{\varsigma_n} & (7b) \end{array}\right\} \quad (7)$$

A <u>portfolio</u> is a collection of several cash flows, each with its own identity. To compute the value of the portfolio, one can apply (7) to the collections of all the constituent amounts, payment times and interest rates (regardless of the specific cash flows that contributed these elements.) For our objectives, as explained below, it often is more appropriate to use an alternative procedure: first compute the values of the individual cash flows, and then combine them. Since summation is a linear operator, it is immediately apparent that the portfolio's present value PV_p is

$$PV_p = \sum_m PV_m \qquad (8)$$

where the PV_m's are present values of the constituent cash flows. Similar result holds, of course, with regard to the future value of the portfolio.

3.2 <u>Intervals of Possibilities and the Values of Cash Flows</u>
Using the statement at the end of section 2.1 we can develop expressions for the values of uncertain and/or imprecise cash flows, by considering the expressions for the values with intervals of possibilities as input variables. The results will provide the desired intervals of possibilities.

From the definition of a cash flow it is apparent that the payment times are non-negative. To be able to exploit the simplifications introduced by positive interval of possibility, as expressed by (3), (5), (6a) and (6b), we need to assure that all the amounts have the same sign and that all the discount and interest factors are either greater than, or less than, 1. The latest requirement is equivalent to specifying that all the interest rates have the same sign. Thus, we consider the original cash flow as a portfolio of five independent cash flows: (I) one with non-negative amounts and interest rates; (II) one with non-negative amounts and non-positive interest rates; (III) one with non-positive amounts and non-negative interest rates; (IV) one with negative amounts and interest rates; and (V) one in which the intervals of possibilities, either for amounts or for interest rates, contain the value of zero. Intervals of type (V) have to be computed using (1) and the expressions for financial measures, and will not be discussed here.

Consider first a cash flow of type I, with a sequence of triplets of positive intervals of possibilities, $\{(A_n, T_n, I_n),$

$n = 1, .., N\}$. Since $I_n = (\underline{i}_n, i_n, \bar{i}_n)$ is a positive interval of possibility it has a corresponding positive intervals of possibilities: an interest factor interval $R_n = 1 + I_n$ and a discount factor interval $V_n = 1 / R_n \le \bar{1}$, which, by (4), is

$$V_n = (\underline{v}_n, v_n, \bar{v}_n) = (1/\bar{r}_n, 1/r_n, 1/\underline{r}_n)$$
$$= (1/(1+\bar{i}_n), 1/(1+i_n), 1/(1+\underline{i}_n))$$

The financial measures' expressions use the time differences $\tau_n = t_n - t_n$ with the corresponding interval expressions $\Upsilon_n = T_n - T_1 = (\underline{\tau}_n, \tau_n, \bar{\tau}_n) = (\underline{t}_n - \bar{t}_1, t_n - t_1, \bar{t}_n - \underline{t}_1)$ which are assumed to be a positive, i.e., $\underline{\tau}_n \ge 0$[3].

Assume first that the number of payments N is known. Then the resulting present value is contained in

$$PV_I = \sum A_n \cdot V_n^{\Upsilon_n} = \sum (\underline{a}_n, a_n, \bar{a}_n) \cdot (\underline{v}_n, v_n, \bar{v}_n)^{(\underline{\tau}_n, \tau_n, \bar{\tau}_n)}$$
$$= \sum (\underline{a}_n \cdot \underline{v}_n^{\bar{\tau}_n}, a_n \cdot v_n^{\tau_n}, \bar{a}_n \cdot \bar{v}_n^{\underline{\tau}_n})$$
$$= (\sum \underline{a}_n \cdot \underline{v}_n^{\bar{\tau}_n}, \sum a_n \cdot v_n^{\tau_n}, \sum \bar{a}_n \cdot \bar{v}_n^{\underline{\tau}_n})$$

The result is a positive interval of possibility whose components, the infimum, plausible and supremum values, are again present values. The only distinction between the component present values is in their constituencies. The infimum present value, for example, uses the infimum amounts, the infimum discount factors and consequently the supremum interest factors and rates, and the supremum times. This will be denoted symbolically as $PV(\underline{a}, \underline{v}, \bar{r}, \bar{i}, \bar{t}, N)$. Thus

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[3] For n=1, $\underline{t}_1 = t_1 = \bar{t}_1$ is known precisely, and therefore, $T_1 = \bar{0}$. For $n > 1$, it is reasonable to assume that the infimum time will occur in the future, assuring that $\underline{\tau}_n$, and T_n, are non-negative.

$$PV_I = (PV(\underline{a},\underline{v},\bar{r},\bar{i},\bar{t},N),\ PV(a,v,r,i,t,N),\ PV(\bar{a},\bar{v},\underline{r},\underline{i},\underline{t},N)) \quad (9)$$

The number of payments, N, may itself be uncertain; e.g., a bond may be called prior to its scheduled maturity. Thus we may consider N to be itself a positive interval of possibility. However, for any sequence of non-negative amounts, interest rates and payment times, if we truncate the sequence at, say, $\underline{N}$ and later at N and $\bar{N}$ payments, we have (when keeping "a,v,r,i,t" unchanged whenever they are used for the same payment)

$$PV(a,v,r,i,t,\underline{N}) \le PV(a,v,r,i,t,N) \le PV(a,v,r,i,t,\bar{N})$$

Further, we know from (9) that for any number of payments

$$PV(\underline{a},\underline{v},\bar{r},\bar{i},\bar{t},N) \le PV(a,v,r,i,t,N) \le PV(\bar{a},\bar{v},\underline{r},\underline{i},\underline{t},N)$$

When we combine these two inequalities, we finally get

$$PV_I = (PV(\underline{a},\underline{v},\bar{r},\bar{i},\bar{t},\underline{N}),PV(a,v,r,i,t,N),PV(\bar{a},\bar{v},\underline{r},\underline{i},\underline{t},\bar{N})) \quad (10a)$$

In a similar fashion we can now express the results for the other present and future values. Note that in cases III and IV the amounts are negative, while the discount factors are positive, and thus we must use equation (3c). For the present values we get

$$PV_{II} = (PV(\underline{a},\underline{v},\bar{r},\bar{i},\underline{t},\underline{N}),PV(a,v,r,i,t,N),PV(\bar{a},\bar{v},\underline{r},\underline{i},\bar{t},\bar{N})) \quad (10b)$$

$$PV_{III} = (PV(\underline{a},\bar{v},\underline{r},\underline{i},\underline{t},\underline{N}),PV(a,v,r,i,t,N),PV(\bar{a},\underline{v},\bar{r},\bar{i},\bar{t},\bar{N})) \quad (10c)$$

$$PV_{IV} = (PV(\underline{a},\bar{v},\underline{r},\underline{i},\bar{t},\underline{N}),PV(a,v,r,i,t,N),PV(\bar{a},\underline{v},\bar{r},\bar{i},\underline{t},\bar{N})) \quad (10d)$$

while for the future values we have

$$FV_I = (FV(\underline{a},\bar{v},\underline{r},\underline{i},\underline{t},\underline{N}),FV(a,v,r,i,t,N),FV(\bar{a},\underline{v},\bar{r},\bar{i},\bar{t},\bar{N})) \quad (10e)$$

$$FV_{II} = (FV(\underline{a},\bar{v},\underline{r},\underline{i},\bar{t},\underline{N}),FV(a,v,r,i,t,N),FV(\bar{a},\underline{v},\bar{r},\bar{i},\underline{t},\bar{N})) \quad (10f)$$

$$FV_{III} = (FV(\underline{a},\underline{v},\bar{r},\bar{i},\bar{t},\underline{N}),FV(a,v,r,i,t,N),FV(\bar{a},\bar{v},\underline{r},\underline{i},\underline{t},\bar{N})) \quad (10g)$$

$$FV_{IV} = (FV(\underline{a},\underline{v},\bar{r},\bar{i},\underline{t},\underline{N}),FV(a,v,r,i,t,N),FV(\bar{a},\bar{v},\underline{r},\underline{i},\bar{t},\bar{N})) \quad (10h)$$

Consider now the meaning and usefulness of these results. First, it is apparent that in each case in (10), the resulting interval of possibilities can directly and exactly be computed; thus, one need not consider and try various combinations of possible values of the components, as is required by (1) in the general case. This is not always the case with interval expressions; often the application of the arithmetical operations of section 2.1 only leads to bounds for the resulting interval, rather than its exact determination.

Second, the infima and suprema values of the resulting intervals of possibilities for the present (future) values have a special form. They are present (future) values that are determined by the infima and/or suprema values of the intervals of possibilities that participate in the cash flows. The resulting plausible values, on the other hand, include only the plausible values of the participating intervals. Thus, one need not consider other intermediate values within the component intervals of possibilities.

Third, it is apparent that one must be very careful in the selection of component infima and suprema to be used. One cannot apply the intuitive approach of using, say, only the infima of the components to compute the infimum of the result; in general, a combination of the most "troublesome" components usually does not deliver the most "troublesome" result. Rather, as seen in (10), one has in each case to carefully select and match infima and suprema component values.

Fourth, the combination of infima and suprema in (10) clearly demonstrate that the arithmetical operations on intervals of possibilities tend to increase the width of the resulting interval, and thus - as can be intuitively be expected - the uncertainty of the result. This is due to the need to include in the resulting interval all possible combination of values from the component intervals.

Fifth, note that in (10) we have already taken into account negative amounts. Thus, the results for the various cases can directly be added together, according to (2a) and (8), and one need not consider substraction. It should also be noted that while (10) includes expressions for four separate cases, one need not memorize all of these results. They can rather be embedded within a software package, or a spreadsheet template, that will automatically produce the results for the appropriate cases.

Furthermore, in many practical situations, only few of these cases will be realized. In most common situations positive interest rates can be assumed, thus leading to cases I and III. While situations with negative interest rates are not common, they still exist. An example is the negative interest rate that was applied in Switzerland to foreign deposits at the beginning of the eighties, to protect the Swiss Franc.

Finally, (10) demonstrates how imprecision and uncertainty can be propagated <u>before</u> any decisions are made. This assures that the "largest" amount of <u>information will be available at the moment of actual decision</u>, and in turn improves the quality of the decisions being made.

Intervals of possibilities support a new measures of a cash flow, in addition to dispersion and similar measures: the width of the resulting interval of possibilities. The width is related to the uncertainty, or imprecision, of the result; the smaller it is, the higher the precision. To illustrate, for the present value of case I, we have from (10a):

$$\text{width} = PV(\bar{a},\bar{v},\underline{r},\underline{i},\underline{t},\bar{N}) - PV(\underline{a},\underline{v},\bar{r},\bar{i},\bar{t},\underline{N}).$$

4. Intervals of Possibilities of Common Cash Flows

To demonstrate the usefulness of our approach, we apply it to several common cash flows and insurance-related activities. These include (1) the price of a bond when the cost-of-money varies, and thus the interest rate to be used is uncertain; (2) the price of a callable bond, where both the amounts and number of payments are uncertain; and (3) the values of an annuity and a life insurance payment, when the underlying mortality table is subjected to projected changes.

We start by noting that a common reinsurance treaty, the sliding scale excess of loss treaty, is actually based on an interval of possibilities. Here the reinsurer and the ceding insurance company define minimum and maximum (i.e., an interval) prices, and allow experience to show what is the right value, thus eliminating to some degree the chance element which is involved in the transaction.

A sliding scale excess of loss treaty (see, e.g., Reinarz [10]) is from the coverage point of view a usual excess of loss reinsurance coverage. On the reinsurance premium side it is, however, unusal. The premium is not a-priori fixed, because both sides to the treaty evaluate the value of the coverage very differently and "agree to disagree" in order to make a reinsurtance coverage possible that satisfies both sides. The premium can thus slide between a minimum that is fixed by the ceding company and agreed to by the reinsurer, and a maximum fixed by the reinsurance company and agreed to by the ceding company. The actual reinsurance premium usually equals a loading facter times the burning cost, i.e., the ratio of the reinsurer's claims amount including reserves divided by the ceding company's net premium income before the sliding scale reinsurance costs.

Note that the premium of a sliding scale treaty is often known only after several years, whereas the period of coverage is usually one year. Therefore several intervals of possibilities must be treated simulatenously. Further, often a sequence of sliding scales treaties is used, each with its own interval; the arithmetic of intervals of possibilities can then be used to determine the combined scale.

4.1 <u>The Price of a Bond with an Uncertain Cost-of-Money</u>

Consider a bond with a face value of $1, which is due to pay n semi-annual coupons of $c each. The face value of the bond will be paid with the last coupon. An investor who can borrow money at a prime-linked interest rate (e.g., prime plus some percentage point) wishes to buy the bond and hold it to maturity; thus, the prime rate determines the investor's cost-of-money which, in turn, will determine the price s/he is ready to pay for the bond. From economical projections the investor can deduce intervals of possibilities for the future prime rate values, and thus also for his cost-of-money. For simplicity sake, assume that the pricing is done immediately after the payment of a coupon, and that semi-annual interest is used.

The sequence of coupons to be received is a cash flow of type I of an annuity with n payments of $c and with uncertain cost-of-money positive interest rates of I_1, .., I_n. The face value of $1 will be received with the last coupon, and is thus subjected to the uncertain cost-of-money rate I_n. The price of the bond, excluding taxes and commissions, is therefore by (10a) the interval of possibilities

$[1/(1+\bar{i}_n)^n + c\cdot\sum_{k=1}^n 1/(1+\bar{i}_k)^k,\ 1/(1+\underline{i}_n)^n + c\cdot\sum_{k=1}^n 1/(1+\underline{i}_k)^k]$

$= [(1-c/\bar{i})/(1+\bar{i})^n + c/\bar{i},\ \ (1-c/\underline{i})/(1+\underline{i})^n + c/\underline{i}\ ]$, with a plausible value of $1/(1+i_n)^n + c\cdot\sum_{k=1}^n 1/(1+i_k)^k$, or

$(1-c/i)/(1+i)^n + c/i$.

Table 1 illustrates the intervals of possibilities and widths for several combinations of coupons, number of payments, and cost-of-money. Two intervals of possibilities are used for the cost-of-money, both with the same plausible value. As can be expected, the wider the interval of possibilities, or the longer the time span of the bond (as measured by the number of payments), the wider the resulting interval of possibilities. Furthermore, the higher the coupon, the wider the interval for the result; while this phenomenon can be explained mathematically, it is not intuitively apparent.

To highlight these observations, we included small graphical representations of the intervals of possibilities for the prices of the bond. In each of these graphs the x-axis is the number of payments, while the y-axis is the price of the bond. The three lines that are included in each graph display the infimum, plausible and supremum value. For each point on the x-axis that corresponds to a payment period, the spread between the two extreme lines is the width of the interval of possibilities for this number of payments.

Table 1 quantifies the imprecision in the resulting price of the bond that is due to the uncertainty in the cost-of-money. It demonstrates the futility of a single value as the "price" of the bond; rather, our approach provides the decision maker with a spectrum of alternatives. The selection of a particular value by the decision maker may depend on many other concerns, such as his/her level of risk-aversion or alternative uses for his/her resources.

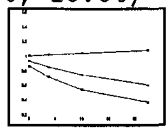
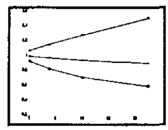
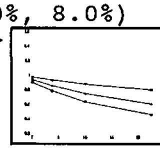
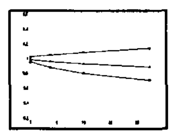
Coupon Rate	No. of Pmnts	Infimum Value	Plausible Value	Supremum Value	Width	Graph: Supremum Plausible Infimum
Cost-of-Money interval of possibilities (2.0%, 6.0%, 10.0%)						
2.5%	2	0.8698	0.9358	1.0097	0.1399	
2.5%	5	0.7157	0.8526	1.0236	0.3079	
2.5%	10	0.5392	0.7424	1.0449	0.5058	
2.5%	20	0.3615	0.5986	1.0818	0.7203	
5.0%	2	0.9132	0.9817	1.0582	0.1450	
5.0%	5	0.8105	0.9579	1.1414	0.3309	
5.0%	10	0.6928	0.9264	1.2695	0.5767	
5.0%	20	0.5743	0.8853	1.4905	0.9162	
Cost-of-Money interval of possibilities (4.0% ,6.0%, 8.0%)						
2.5%	2	0.9019	0.9358	0.9717	0.0698	
2.5%	5	0.7804	0.8526	0.9332	0.1528	
2.5%	10	0.6309	0.7424	0.8783	0.2474	
2.5%	20	0.4600	0.5986	0.7961	0.3361	
5.0%	2	0.9465	0.9817	1.0189	0.0724	
5.0%	5	0.8802	0.9579	1.0445	0.1643	
5.0%	10	0.7987	0.9264	1.0811	0.2824	
5.0%	20	0.7055	0.8853	1.1359	0.4304	

Table 1: Prices of Bonds with Uncertain Cost-of-Money

4.2 Callable Bond

Now consider a more complex example, in which both the amounts and number of payments are uncertain. At issue is the price of the same bond, which may be called prior to the n^{th} coupon payment; in this case the face value is paid together with some call premium. Several call times may be specified, $n_1 < .. < n_m < n$, usually with non-increasing call premiums $p_1 \geq .. \geq p_m \geq 0$. The callable bond is again a cash flow of type I. To simplify the presentation, we assume a certain cost-of money, i say, while the number of coupons and the call premium are uncertain. The price of the bond is then, excluding taxes and commissions, given by the interval of possibilities

$$\Big[\min_{1\leq s\leq m} \{(1+p_s)^{n_s}/(1+i)^{n_s} + c\cdot\textstyle\sum_{k=1}^{n_s} 1/(1+i)^k,$$
$$1/(1+i)^n + c\cdot\textstyle\sum_{k=1}^{n} 1/(1+i)^k\},$$
$$\max_{1\leq s\leq m} \{(1+p_s)^{n_s}/(1+i)^{n_s} + c\cdot\textstyle\sum_{k=1}^{n_s} 1/(1+i)^k,$$
$$1/(1+i)^n + c\cdot\textstyle\sum_{k=1}^{n} 1/(1+i)^k\} \Big]$$

$$= \Big[\min_{1\leq s\leq m} \{(1+p_s-c/i)/(1+i)^{n_s} + c/i,\ (1-c/i)/(1+i)^n + c/i\},$$
$$\max_{1\leq s\leq m} \{(1+p_s-c/i)/(1+i)^{n_s} + c/i,\ (1-c/i)/(1+i)^n + c/i\}\Big]$$

with the plausible value corresponding to one of the intermediate call times. Note that in this evaluation we implicitely used (1), because the extreme values depend both on the sequence of call premiums and on the cost-of-money (and, therefore, it cannot be assumed that the infimum, say, is on the first call).

To illustrate this result, the price of this bond is given in Table 2, under various assumptions about the cost-of-money, number of payments and coupon value. We assume two call times: the first is 2 payments before the full duration of the

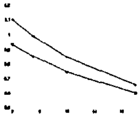
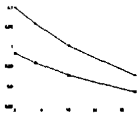
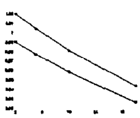
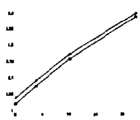
Coupon Rate	No. of Pmnts	Call n-2 Price	Call n-1 Price	No Call Price	Infimum Price	Supremum Price	Graph: Infimum Supremum
Call Premium:		10.0%	5.0%	Cost-of-Money:		6.0%	
2.5%	2	1.1000	1.0142	0.9358	0.9358	1.1000	
2.5%	5	0.9904	0.9183	0.8526	0.8526	0.9904	
2.5%	10	0.8454	0.7915	0.7424	0.7424	0.8454	
2.5%	20	0.6561	0.6260	0.5986	0.5986	0.6561	
5.0%	2	1.1000	1.0377	0.9817	0.9817	1.1000	
5.0%	5	1.0572	1.0050	0.9579	0.9579	1.0572	
5.0%	10	1.0006	0.9616	0.9264	0.9264	1.0006	
5.0%	20	0.9268	0.9049	0.8853	0.8853	0.9268	
Call Premium:		2.0%	1.0%	Cost-of-Money:		3.0%	
2.5%	2	1.0200	1.0049	0.9904	0.9904	1.0200	
2.5%	5	1.0042	0.9903	0.9771	0.9771	1.0042	
2.5%	10	0.9807	0.9687	0.9573	0.9573	0.9807	
2.5%	20	0.9430	0.9341	0.9256	0.9256	0.9430	
5.0%	2	1.0200	1.0291	1.0383	1.0200	1.0383	
5.0%	5	1.0749	1.0832	1.0916	1.0749	1.0916	
5.0%	10	1.1562	1.1634	1.1706	1.1562	1.1706	
5.0%	20	1.2868	1.2922	1.2975	1.2868	1.2975	

Table 2: Prices of Callable Bonds

bond, and the second is 1 payment before the duration of the bond. Table 2 shows the prices for the bond when it is called under the two options, and when it is not called and redeemed at the full duration. As before, we included graphical representations of the infimum and supremum prices, where the x-axis is the number of payments up to the redemption of the bond, and the y-axis is the price. In all of the illustrations in Table 2 the supremum price was for the earliest called bond, with the only exception being the case where the coupon exceeded the cost-of-money and the call premiums. This last case is also exceptional for other reasons: the resulting interval of possibilities is the narrowest, and the price increased as the duration of the bond increased. It is also worth noting that in all the cases demonstrated in Table 2, the resulting interval of possibilities for the price becomes narrower as the duration increases, while in Table 1 the opposite is true.

4.3 Life Insurance with an Uncertain Mortality Table

This example illustrates the value of using intervals of possibilities to delay the elimination of uncertainty to the latest possible moment in the decision making process. The example also demonstrates the usefulness of intervals of possibilities in a situation where the extreme values of the intervals are determined directly, rather than through interval arithmetics. These intervals, once determined, are used in conjunction with marketing, competition and other business consideration to select a premium scale for all insured ages.

It is well known that due to medical and other quality of life improvements, mortality rates over the last few decades are decreasing, and this trend is expected to continue. Certain modern life table enable the actuary to project mortality rates into the future. This decrease in mortality rates leads, in turn, to decreasing pure premium for insurance coverage, and increasing costs of annuities and retirement pensions.

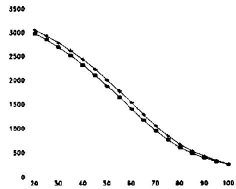
Actuarial conservativism would suggest non-projected pure premiums for life insurance, and projected premiums for annuities. Marketing pressures, on the other hand, would push for smaller, and from the insured's point of view more attractive, premiums, and thus to projected pure premium for life insurance and non-projected premium for annuities. Further, it is desirable to have a "smooth" premium curve for the various ages, without sudden jumps that may scare potential insureds or lead to anti-selection. Finally, administrative and managerial consideration would often favor the same mortality table for both coverages, especially if they are sold together. It is thus apparent that there is no "best" or "correct" projection of mortality rates; rather, a "compromise" projection should be used, which depends on the overall point of view of the decision makers. While reaching this compromise, the decision maker should be aware, at least, of the intervals of possibilities that reflect the projected and non-projected premiums, and their implications to the decision.

To illustrate these concepts, we use the Pensioners, Males, Lives (PML80) life table of the Standard Tables of Mortality - The "80" Series [6]. We present in Table 3 the pure single premium for a $100 life annuity, with payments commencing immediately, and for a $100 life insurance payable at the moment of death. These were computed for various ages, for the current mortality rates and for rates projected to the year 2000, with an interest rate of 2.5 percent.

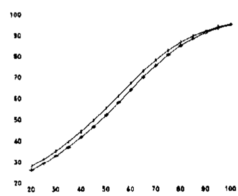
Table 3 includes graphical representation of the resulting intervals of possibilities. In these graphs, the x-axis is the age, while the y-axis is the premium for each age. It is apparent that the intervals of possibilities become narrower as the age increases, for both the annuities and the life insurance. It is also apparent that for the annuities the infimum premiums are achieved for the original mortality table, while the supremum is attained for the projected values. The opposite is true for the life insurance.

Age	Annuity Original	Annuity Projected	Life Insurance Original	Life Insurance Projected
20	2,972.50	3,050.70	27.84	25.91
25	2,838.24	2,922.82	31.16	29.07
30	2,683.10	2,775.83	34.99	32.70
35	2,507.11	2,609.24	39.33	36.81
40	2,311.21	2,423.17	44.17	41.41
45	2,097.52	2,218.59	49.45	46.46
50	1,869.59	1,997.59	55.08	51.92
55	1,632.57	1,763.54	60.93	57.69
60	1,391.93	1,520.16	66.87	63.70
65	1,158.17	1,276.74	72.64	69.72
70	942.31	1,046.09	77.97	75.41
75	752.71	838.47	82.66	80.54
80	596.60	663.25	86.51	84.86
85	474.73	523.34	89.52	88.32
90	380.58	413.53	91.84	91.03
95	303.08	323.63	93.76	93.25
100	243.41	254.59	95.23	94.96

Graphs:
Infimum
Supremum



Annuity



LifeInsurance

Table 3: Premiums for $100 Annuity and Life Insurance

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