An optimization approach to the dynamic allocation of economic capital[*]

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Abstract

We propose an optimization approach to allocating economic capital, distinguishing between an allocation principle and a measure for the risk residual. The approach is applied both at the aggregate (conglomerate) level and at the individual (subsidiary) level and yields an integrated solution to the capital allocation problem. In particular, we formalize a procedure to determine (i) the optimal amount of economic capital to be held physically by a financial conglomerate, (ii) the optimal allocation of this amount among the subsidiaries and (iii) a consistent distribution of the cost of risk-bearing borne by the conglomerate among the subsidiaries. Different degrees of information on the dependence structure between the subsidiaries are considered. Static solutions are generalized to a dynamic setting. The approach is illustrated using an example of a financial conglomerate represented by a multivariate Wiener process.

Keywords: Risk measurement, Capital allocation, Dependence structure, Comonotonicity

JEL-Classification: G22, G31

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1 Introduction

An interesting problem of risk measurement is the allocation of economic capital. The objective of the economic capital allocation problem is twofold:

1. Determination of the amount of economic capital to be held at the aggregate level of a financial conglomerate, and

2. Allocation of this amount of capital among the constituents of the conglomerate. In the following, we will refer to the constituents of a financial conglomerate as subsidiaries, although portfolios or lines of business may just as well be considered.

In this paper, we take the viewpoint of a higher authority within the financial conglomerate by which the economic capital allocation is performed. By *economic capital* –some authors use the terminology *risk capital*–, we mean an amount of capital that serves as solvency or buffer capital. We remark that the second objective mentioned above is important for the purpose of cost allocation and from the perspective of performance evaluation.

Several approaches to the economic capital allocation problem exist in the literature, most of which are axiomatic of nature. Axiomatic approaches impose a set of mathematical properties that should be satisfied by the allocation. For the first objective of the economic capital allocation problem, the axiomatic approach of Artzner (1999b) suggests to use the *tail conditional expectation* (TCE) defined by

$$\text{TCE}_\alpha(X) = \mathbb{E}[X|X > \text{VaR}_\alpha(X)], \qquad \alpha \in (0,1) \tag{1}$$

or the *worst conditional expectation* (WCE) defined by

$$\text{WCE}_\alpha(X) = \sup(\mathbb{E}[X|A]|\mathbb{P}[A] > \alpha), \qquad \alpha \in (0,1) \tag{2}$$

as alternatives to the widely applied *Value-at-Risk* (VaR) measure of risk. In a related paper, Artzner et al. (1999a) argue against the use of the VaR measure of risk. Their main argument is that the VaR does not satisfy the subadditivity property, i.e. it is not in general the case that

$$\text{VaR}_\alpha(X_1 + \ldots + X_n) \leq \text{VaR}_\alpha(X_1) + \ldots + \text{VaR}_\alpha(X_n) \tag{3}$$

in which for a random loss X the VaR at level $1 - \alpha$ is defined by

$$\text{VaR}_\alpha(X) = \inf\{x \in \mathbb{R}|\mathbb{P}[X \leq x] \geq 1 - \alpha\}, \qquad \alpha \in (0,1) \tag{4}$$

Artzner et al. (1999a) consider this subadditivity property to be very desirable. Phrased briefly, they argue that the coalescence of risks does not create any additional risk, while the diversification effect may be beneficial. The subadditivity property is one of four axioms which a risk measure should satisfy to be classified as an *Artner coherent risk measure* (Artzner et al. (1999a)). Both risk measures in

(1) and (2) do satisfy the subadditivity property and even more can be classified as *Artzner coherent* risk measures.[1]

In line with Artzner's (1999b) approach, Panjer (2001) suggests the use of the TCE to set the economic capital at the aggregate (conglomerate) level and to use the related measure

$$\mathbb{E}[X_i | X > \text{VaR}_\alpha(X)] \tag{5}$$

to allocate economic capital at the individual (subsidiary) level. The same allocation procedure is suggested in Overbeck (2000), among other methods of allocation. The game-theoretic axiomatic approach of Denault (2001) yields the same allocation among the subsidiaries as in (5), in case the TCE is used at the aggregate level. More generally, Denault's approach allows for any Artzner coherent risk measure at the aggregate level, and employs the Aumann-Shapley (1974) value, well-known from the theory of cooperative games with fractional players, to establish the allocation among the subsidiaries.

However, we encounter 4 problems when employing an axiomatic approach to allocate economic capital. The first problem concerns the allocation among the subsidiaries of an exogenously given amount of economic capital held by the conglomerate. While indeed an allocation like expression (5) prescribes how to allocate capital among the subsidiaries, we argue that given an amount of economic capital held by the conglomerate, the allocation of capital to the subsidiaries can generally be performed in a better way. In particular, the optimal allocation of capital among the subsidiaries can be derived using an optimization procedure. We will formulate this optimization procedure below.

The second problem occurs in case the dependence structure between the subsidiaries is unknown. We remark that the measure in expression (5) requires the conditional distribution functions to be available. Hence, in case the dependence structure is unknown, the measure cannot be computed in general. Therefore, we will consider the capital allocation problem under different degrees of information on the dependence structure.

To illustrate the third problem, we consider the raising of economic capital by the conglomerate. Indeed, both the TCE and the WCE can be used to set the economic capital at the aggregate level. However, this amount of capital will typically not be held physically. Part of it will be raised through internal finance and part of it through external finance or through (re)insurance. We will provide an optimization procedure to determine the optimal amount of capital to be held physically given current market conditions.

The fourth problem is a little more subtle. We remark that using an upper tail risk measure like the TCE or the WCE to set the economic capital –here interpreted as the sum of the physical and the non-physical economic capital– implies that an upper bound is set to the loss realization. Indeed if the risk realization exceeds the

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[1]Formally, the subadditivity of the TCE holds only for continuous distributions (see Dhaene et al. (2003)). For discontinuous distributions, the TailVar (TVaR) given by

$$\text{TVaR}_\alpha(X) = 1/\alpha \int_{1-\alpha}^1 \text{VaR}_{1-p}(X) dp, \qquad \alpha \in (0, 1)$$

provides a subadditive alternative.

upper tail risk measure, the loss incurred is limited to this upper tail risk measure. We will show below that imposing bounds on the loss realization may provide unjustified incentives to separate risks and to split-up conglomerates.

In the present paper we introduce an optimization approach to the allocation of economic capital, distinguishing between an allocation principle and a measure for the risk residual. Different degrees of information on the dependence structure between the risks are considered. This paper is the first to elaborate on a distinction between an allocation principle and a measure for the risk residual and provides an integrated solution to the problem of capital allocation. Some provisional ideas can be found in Goovaerts, Dhaene and Kaas (2001) and Goovaerts, Dhaene and Kaas (2003). Most economic capital allocation approaches known in the literature are in a static rather than a dynamic setting. In a dynamic setting the effect of information releases, formalized by an information filtration, needs to be explored. Tsanakis (2003) proposes a dynamic approach based on an updating mechanism for non-additive set functions. We propose a generalization to a dynamic setting based on the direct updating of the real world probability measure, closely following the theory of dynamic asset pricing (see e.g. Duffie (1996)).

The outline of this paper is as follows: in section 2, we present a general solution to the problem of capital allocation for an exogenously given amount of economic capital. Next, in section 3 we formalize a procedure to determine the optimal amount of capital to be held physically by a financial conglomerate. Section 4 provides a consistent distribution of the cost of risk-bearing borne by the conglomerate among the subsidiaries. Then, in section 5 the static solutions are generalized to a dynamic setting. Finally, in section 6 the integrated approach is illustrated using an example of a financial conglomerate represented by a multivariate Wiener process. Proofs of the theorems are gathered in the appendix as are all figures and tables.

2 The capital allocation problem for an exogenously given amount of capital

This section presents a general solution to the problem of capital allocation for an exogenously given amount of economic capital. In section 2.1 we will outline the general model and introduce the concept of risk residual. Next, we will solve the capital allocation problem for two extreme cases:

1. The case of complete lack of information on the dependence structure between the risks: section 2.2.

2. The case of known dependence structure between the risks: section 2.3.

To establish the framework, we provide the following three definitions:

Definition 2.1 *Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a risk is a random variable X, defined on a set of states of nature Ω and represents the net loss or deficit at some point in the future of a portfolio or position currently held. If $X > 0$, it represents a loss, while if $X \leq 0$ it represents a gain. In the following, we restrict to the vector*

space $L^p(\Omega, \mathcal{F}, \mathbb{P})$, for $1 \leq p \leq \infty$, consisting of equivalence classes of p-integrable random variables.

Definition 2.2 *A risk measure is a functional $\pi : L^p \to \mathbb{R}$ assigning a real number to a random variable X or its distribution function $F_X(\cdot)$. Henceforth, we restrict to risk measures π which are such that if X_n converges weakly to X then $\pi(X_n)$ converges to $\pi(X)$.*

Definition 2.3 *An allocation principle is a functional $\rho(\cdot)$ assigning an amount of capital u_i to a random variable X_i from the vector $(X_1, \ldots, X_n)$ with joint distribution functions contained in the Fréchet space (Fréchet (1951)) generated by the vector of marginal distributions $(F_{X_1}, \ldots, F_{X_n})$. While $\rho(X_i)$ depends on the specific Fréchet space or on a specific element in the Fréchet space, we do not explicitly denote this dependence, it will be clear from the context.*

2.1 The general framework

In this section we present the general framework to solve the problem of capital allocation for an exogenously given amount of economic capital. We consider a financial conglomerate consisting of $n \in \mathbb{N}$ subsidiaries and let i, $i = 1 \ldots n$, denote the particular subsidiary. The aggregate risk of the conglomerate is denoted by X and equals the sum of the risks of the subsidiaries, i.e. $X = X_1 + \ldots + X_n$. Marginal distributions $(F_{X_1}, \ldots, F_{X_n})$ are assumed to be known, whereas the dependence structure between the risks may or may not be available. For a random variable X, the (pseudo-) inverse distribution function F_X^{-1} is defined as usual by $F_X^{-1}(p) = \inf\{x \in \mathbb{R} | F_X(x) \geq p\}$, $p \in [0, 1]$, with $\inf \emptyset = +\infty$ by convention. Below we will also need a more sophisticated definition of the inverse distribution function, known as the α-*mixed* inverse distribution function (see Dhaene et al. (2002)). Therefore, we first introduce for a random variable X the function F_X^{-1+} defined by $\sup\{x \in \mathbb{R} | F_X(x) \leq p\}$, $p \in [0, 1]$. Herewith we define the α-mixed inverse distribution function $F_X^{-1(\alpha)}$ as follows:

$$F_X^{-1(\alpha)}(p) = \alpha F_X^{-1}(p) + (1 - \alpha) F_X^{-1+}(p), \qquad p \in (0, 1), \ \alpha \in [0, 1] \qquad (6)$$

It is not difficult to see that for any random variable X and for all d with $0 < F_X(d) < 1$, there exists an $\alpha_d \in [0, 1]$ such that $F_X^{-1(\alpha_d)}(F_X(d)) = d$.

We denote by u the amount of economic capital held physically by the financial conglomerate. In this section, we assume u to be given exogenously and to satisfy $\min[X_1] + \ldots + \min[X_n] < u < \infty$. The objective is to determine the allocation principle $\rho(\cdot)$ which assigns in an optimal way an amount of capital u_i to each of the risks X_i, $i = 1, \ldots, n$, while satisfying $\sum_{i=1}^{n} \rho(X_i) = u$.

To define our understanding of an optimal allocation of economic capital, we introduce the concept of *risk residual*, which is the positive risk which remains after the capital allocation has been performed, i.e. for subsidiary i, $i = 1, \ldots, n$ the risk residual is given by $\max(X_i - u_i, 0) = (X_i - u_i)_+$. We consider the following inequality, which holds with probability one

$$(\sum_{i=1}^{n} X_i - u)_+ \leq_1 \sum_{i=1}^{n} (X_i - u_i)_+, \qquad u = \sum_{i=1}^{n} u_i \qquad (7)$$

The above inequality demonstrates that the risk residual representing the conglomerate after raising an amount of economic capital u is first-order stochastically dominated by the sum of the risk residuals representing the subsidiaries after allocation of amounts u_i, $i = 1, \ldots, n$. In fact, inequality (7) indicates a natural and objective preference for the coalescence of risks. Note that we employ the condition of full allocation of the economic capital, i.e. $\sum_{i=1}^{n} u_i = u$. We remark that inequality (7) holds true as long as $\sum_{i=1}^{n} u_i \leq u$, i.e. superadditivity of the allocation. Hence, for any first-order stochastic dominance preserving risk measure $\pi(\cdot)$ we have that

$$\pi(\sum_{i=1}^{n} X_i - u)_+ \leq \pi(\sum_{i=1}^{n} (X_i - u_i)_+), \qquad u = \sum_{i=1}^{n} u_i \tag{8}$$

We then define a capital allocation $(u_1^*, \ldots, u_n^*)$ to be optimal if it minimizes the risk measure applied to the sum of the risk residuals representing the subsidiaries after the capital allocation has been performed. We argue this to be a natural approach since a basic objective of capital allocation is to reduce residual risk exposure. Hence, we establish the following minimization problem

$$\min_{\rho(\cdot)} \pi(\sum_{i=1}^{n} (X_i - \rho(X_i))_+), \text{ subject to } \sum_{i=1}^{n} \rho(X_i) = u \tag{9}$$

for some first-order stochastic dominance preserving risk measure $\pi(\cdot)$.

To guarantee the existence of a solution to the minimization problem for a given risk measure π in a general setting, we need to impose a weak condition. In particular, we restrict to the compact (i.e. closed and bounded) set D defined by

$$D = \{(u_1, \ldots, u_n) \in \mathbb{R}^n \mid u_i \in [a_i, b_i], \forall i = 1, \ldots, n, \sum_{i=1}^{n} u_i = u\} \tag{10}$$

in which we have $a_i, b_i \in \mathbb{R}$ for all i. In case we consider random variables in the vector space $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, a natural choice would be $a_i = \min[X_i]$ and $b_i = \max[X_i]$. Under the continuity assumption on risk measures imposed in definition 2.2, it then holds true that the functional $\pi((X_1 - u_1)_+ + \ldots + (X_n - u_n)_+)$ attains a minimum value on the compact domain D, i.e. there are values $(u_1^*, \ldots, u_n^*) \in D$ which are such that for any $(u_1, \ldots, u_n) \in D$ we have

$$\pi((X_1 - u_1^*)_+ + \ldots + (X_n - u_n^*)_+) \leq \pi((X_1 - u_1)_+ + \ldots + (X_n - u_n)_+)$$

We remark that differentiability of the risk measure and continuity of the distribution functions is not required for existence. Uniqueness of the solution will be considered in further detail in sections 2.2 and 2.3.

2.1.1 Bounds on the loss realizations

We stated above that inequality (7) indicates a natural and objective preference for the coalescence of risks. However, we will now demonstrate that in a slightly different model setting, this preference is no longer guaranteed.

To establish the different model setting, we introduce the real variables $v < \infty$, denoting the sum of the physical and the non-physical economic capital raised by the conglomerate and $v_i < \infty$, $i = 1, \ldots, n$ denoting the sum of the physical and non-physical capital allocation to subsidiary i. Indeed, several approaches in the literature propose to use upper tail risk measures such as VaR_α, CTE_α and WCE_α to set the economic capital v and to use related measures such as the one in expression (5) to set the corresponding allocation v_i. It is important to remark that in that case bounds on the loss realization are present. Indeed, using upper tail risk measures to set the economic capital v implies that if the risk realization $X = x$ exceeds the upper tail risk measure, the loss incurred is limited to the upper tail risk measure and the conglomerate becomes insolvent. If furthermore only part of the economic capital is held physically, which is the usual case, then we have risk residuals which are limited from above. To represent the limited risk residual we introduce random variables Y and Y_i given by

$$Y = \min((X - v)_+, v - u) \tag{11}$$

$$Y_i = \min((X_i - u_i)_+, v_i - u_i) \tag{12}$$

in which u and u_i denote the amount of capital held physically by the financial conglomerate respectively by subsidiary i.

Analogously to inequality (7), we then compare Y with $Y_1 + \ldots + Y_n$. We remark that even for the case in which $v = v_1 + \ldots + v_n$, it does not hold true in general that $Y \leq_1 Y_1 + \ldots + Y_n$. Depending on the values of $(v, v_1, \ldots, v_n)$ the coalescence of risks may or may not be expected to be beneficial. It is obvious that for "relatively low" values of $v_1, \ldots, v_n$ an unjustified incentive to separate risks may occur. We employ the word "unjustified" since the incentive stems from the subjective choice to set a bound on the loss realization rather than from an objective preference. Below we provide an example in which it is expected to be beneficial to separate risk residuals and hence to split-up conglomerates.

Example 2.1 *In this example we will demonstrate that for a specific bivariate case and typical specifications of the bounds on the loss realization, the expectation of the random variable Y defined in (11) is larger than the expectation of the random variables $Y_1 + Y_2$ defined in (12). Let U be a random variable uniformly distributed on $(0, 1)$ and let the random variables $X_1 = F_{X_1}^{-1}(U)$ and $X_2 = F_{X_2}^{-1}(U)$ be defined as in Table (1).*

In this bivariate example we take $X = X_1 + X_2$. We will then use the following typical bounds for the loss realization

$$v = \mathbb{E}[X|X > \text{VaR}_\alpha] = \text{TCE}_\alpha(X),$$

$$v_i = \mathbb{E}[X_i|X > \text{VaR}_\alpha]$$

Indeed these bounds correspond to capital allocation based on conditional tail expectation as suggested in Artzner (1999b), Denault (2001), Overbeck (2000) and Panjer (2001). We will specify $\alpha = 0.10$. We remark that for these specifications of the bounds it holds true that $v = v_1 + v_2$. It can easily be verified that $VaR_{0.10}(X) = 4$

and hence that $CTE_{0.10}(X) = 6.6$. Furthermore, we have $\mathbb{E}[X_1|X > 4] = 1.5$ and $\mathbb{E}[X_2|X > 4] = 5.1$. We will arbitrarily take $u = 4$, which corresponds to $VaR_{0.10}(X)$. Furthermore, we will set $u_1 = 1$ and $u_2 = 3$, which as we will see below in section 2.2 corresponds to the optimal allocation of capital among the subsidiaries. Then it can easily be verified that

$$\mathbb{E}[Y] = 0.156 > 0.025 + 0.111 = \mathbb{E}[Y_1] + \mathbb{E}[Y_2]$$

Hence, we have an unjustified incentive to separate risks and to split-up the financial conglomerate. □

2.2 The case of complete lack of information on the dependence structure

In this section we consider the capital allocation problem for the case of complete lack of information on the dependence structure between the subsidiaries. The model setting is the same as the one described in the previous section and the objective is to solve

$$\min_{\rho(\cdot)} \pi(\sum_{i=1}^{n}(X_i - \rho(X_i))_+), \text{ subject to } \sum_{i=1}^{n}\rho(X_i) = u \tag{13}$$

Given that the dependence structure is unknown, another problem coexists: in order for our risk measurement to be prudent we want to maximize $\pi((X - u)_+)$ over all possible dependence structures between the subsidiaries. Hence, we establish the following maximization problem

$$\max_{(X_1,...,X_n)\in\Gamma} \pi((X_1 + \ldots + X_n - u)_+) \tag{14}$$

in which Γ denotes the set of random vectors with corresponding marginal distributions, i.e. the Fréchet space.

Suppose that we choose $\pi(\cdot)$ equal to the expectation operator. Note that for this specification of $\pi(\cdot)$, the measure of the risk residual will indeed not take into account the (unknown) dependence structure between the risks. In fact, the choice of $\pi(\cdot) = \mathbb{E}[\cdot]$ is not that arbitrary since for problem (13) to yield a solution under complete lack of information on the dependence structure, $\pi(\cdot)$ needs to be additive for any dependence structure. Since imposing additivity of the risk measure for both the comonotonic and independent dependence structure already characterizes the expectation operator, under some additional scaling conditions (see Goovaerts, Dhaene and Kaas (2001)), indeed $\pi(\cdot) = \mathbb{E}[\cdot]$ is not that arbitrary. It is obvious that since the expectation operator is a first-order stochastic dominance preserving risk measure, it holds true that

$$\mathbb{E}[(X_1 + \ldots + X_n - u)_+] \leq \sum_{i=1}^{n}\mathbb{E}[(X_i - u_i)_+], \quad u = \sum_{i=1}^{n}u_i \tag{15}$$

We then state the following theorem:

Theorem 2.1 *If $\pi(\cdot) = \mathbb{E}[\cdot]$, the objective functions in problems (13) and (14) yield the same value at their solution, which is expressed by the equality*

$$\sum_{i=1}^{n} \mathbb{E}[(X_i - F_{X_i}^{-1(\alpha_u)}(F_{X_1^c + \ldots + X_n^c}(u)))_+] = \mathbb{E}[(X_1^c + \ldots + X_n^c - u)_+] \qquad (16)$$

in which $(X_1^c, \ldots, X_n^c)$ denotes the comonotonic random vector in the Fréchet space. Henceforth, we will often denote $X^c = X_1^c + \ldots + X_n^c$, and correspondingly $F_{X^c}(u) = F_{X_1^c + \ldots + X_n^c}(u)$. We tacitly assume that the solution $(u_1^*, \ldots, u_n^*)$ of optimal allocations is internal in the domain D defined in (10). □

Theorem 2.1 demonstrates that for the case of complete lack of information on the dependence structure and the choice of $\pi(\cdot) = \mathbb{E}[\cdot]$, the optimal allocation to risk X_i is given by $u_i^* = F_{X_i}^{-1(\alpha_i)}(F_{X^c}(u))$, for some vector $(\alpha_1, \ldots, \alpha_n)$ in the set A defined in (38). For the case of strictly increasing and continuous marginal distribution functions, the allocation is unique. Note that the solution is a percentile. Note furthermore, that the solution takes into account all available risk characteristics, i.e. all marginal distributions, since the level of the percentile is based on the comonotonic random vector of the Fréchet space. The safe-best upper bound for problem (14) is obtained by using the comonotonic dependence structure.

2.3 The case of known dependence structure

In this section we will consider the capital allocation problem for the case of known dependence structure between the risks. In particular, we will solve the capital allocation problem for 3 specifications of the risk measure $\pi(\cdot)$ and demonstrate how the optimal allocation of capital among the subsidiaries may depend on the dependence structure between the subsidiaries.

2.3.1 π is the expectation operator

We state the following trivial result:

Theorem 2.2 *If $\pi(\cdot) = \mathbb{E}[\cdot]$, the optimal allocation among the subsidiaries for the case of complete information on the dependence structure is the same as for the case of complete lack of information on the dependence structure, see Theorem 2.1.* □

The following corollary, concerns a reorganization of the financial conglomerate, in which the subsidiaries are split-up in several sub-subsidiaries but the aggregate risk remains the same.

Corollary 2.1 *For comonotonic risks, the solution in Theorem 2.2 satisfies the property of hierarchical additivity, i.e. the allocation to each subsidiary is invariant for different subdivisions of the financial conglomerate. For non-comonotonic risks the solution in Theorem 2.2 is generally discriminating between the hierarchical splitting, i.e. the allocation to each subsidiary differs for different subdivisions of the financial conglomerate.* □

Some authors (e.g. Panjer (2001)) consider the property of hierarchical additivity to be very desirable. They argue in favor of an allocation principle which is additive and hence, invariant for all possible hierarchial subdivisions. We are reluctant to require this hierarchical additivity and argue that the cost of imposing it, leading to an allocation of capital which is inferior in terms of risk residual, is greater than the benefit of invariance of the allocation for all possible subdivisions of the financial conglomerate. The problem is related to a heritage problem. Should a grandparent be fair towards his children or towards his grand children? Hence, in our approach it makes a difference whether at the moment of a subdivision of a subsidiary into several sub-subsidiaries, the capital allocation takes place from the point of view of the financial conglomerate or from the point of view of the subsidiary. In case the capital allocation takes place from the point of view of the subsidiary only the economic capital of the particular subsidiary needs to be distributed among the new juridical entities, whereas if the capital allocation takes place from the point of view of the financial conglomerate a complete reallocation is needed, which will in general yield different allocations to the sub-subsidiaries.

2.3.2 π is a mean value principle

We will now more generically specify π as a *mean value principle*, i.e. $\pi(\cdot) = v^{-1}(\mathbb{E}[v(\cdot)])$, for some strictly increasing, strictly convex and differentiable function $v(\cdot)$. The mean value principle was first introduced in Hardy, Littlewood and Pólya (1952), and was justified by means of an axiomatic representation; see also Goovaerts, De Vylder and Haezendonck (1984) for mathematical properties and generalizations. Obviously, the mean value principle satisfies the first-order stochastic dominance preserving property. We will show for the bivariate case that the optimal solution for the allocation principle $\rho(\cdot)$ will depend on the dependence structure.

Theorem 2.3 *If π is a mean value principle, the solution to the capital allocation problem*

$$\min_{\rho(\cdot)} v^{-1}\left(\mathbb{E}[v((X_1 - \rho(X_1))_+ + (X_2 - \rho(X_2))_+)]\right), \text{ subject to } \rho(X_1) + \rho(X_2) = u$$

for independent risks generally differs from the solution for comonotonic risks. Suppose that the solution is internal in the domain D and assume that all distribution functions are absolutely continuous. Then, the solution for independent risks is characterized by

$$\frac{\int_{u_1}^{\infty} v'(x_1 - u_1) dF_{X_1}(x_1)}{F_{X_1}(u_1)} = \frac{\int_{u-u_1}^{\infty} v'(x_2 - u + u_1) dF_{X_2}(x_2)}{F_{X_2}(u - u_1)} \tag{17}$$

The strict convexity of $v(\cdot)$ guarantees the uniqueness of the optimal solution. For comonotonic risks the unique solution is given by

$$u_i^* = F_{X_i}^{-1}(F_{X_j}(u - u_i)) = F_{X_i}^{-1}(F_{X^c}(u)) \tag{18}$$

$\square$

2.3.3 π is a variance principle

As a third specification, we take π to be a *variance principle*, i.e. $\pi(X) = \mathbb{E}[X] + \beta\mathrm{Var}[X]$, for some $\beta > 0$. The interested reader is referred to Goovaerts, De Vylder and Haezendonck (1984) for an elaborate treatment of the variance principle. We consider the bivariate capital allocation problem given by

$$\min_{\rho(\cdot)} \mathbb{E}[(X_1 - \rho(X_1))_+ + (X_2 - \rho(X_2))_+] + \beta\mathrm{Var}[(X_1 - \rho(X_1))_+ + (X_2 - \rho(X_2))_+],$$

$$\text{subject to } \rho(X_1) + \rho(X_2) = u \qquad (19)$$

We state the following theorem:

Theorem 2.4 *If π is a variance principle, the solution to the bivariate capital allocation problem is characterized by the one-variable equation*

$$F_{X_1}(u_1) - F_{X_2}(u - u_1) +$$

$$2\beta((F_{X_2}(u - u_1) - F_{X_1}(u_1))(\int_{u_1}^{\infty}(1 - F_{X_1}(x))dx + \int_{u-u_1}^{\infty}(1 - F_{X_2}(x))dx +$$

$$\int_{u-u_1}^{\infty}(x_2 - (u - u_1))dF_{X_1,X_2}(u_1, x_2) -$$

$$\int_{u_1}^{\infty}(x_1 - u_1)dF_{X_1,X_2}(x_1, (u - u_1))) = 0 \quad (20)$$

tacitly assuming that the solution is internal in the domain D and that F_{X_1,X_2} is absolutely continuous. $\square$

Below we provide an example in which we solve the bivariate capital allocation problem for two exponential distributed constituents, using a variance principle as the risk measure.

Example 2.2 *In this example we consider the joint tail distribution function $\mathbb{P}[X_1 > x_1, X_2 > x_2] = \exp(-x_1 - 2x_2 - \epsilon x_1 x_2)$ for some $\epsilon > 0$. It is not difficult to see that in that case $X_1 \sim \mathrm{EXP}(1)$ and $X_2 \sim \mathrm{EXP}(2)$ and that ϵ determines the dependence structure between the random variables. We remark that since exponential random variables are supported on $(0, \infty)$, the constituents represent claim sizes or losses instead of net payouts. All terms in equation (20) can now be determined easily. Note that $X_2 \leq_1 X_1$ and hence that X_2 precedes X_1 in stop-loss order in the sense that $\mathbb{E}[(X_2 - d)_+] \leq \mathbb{E}[(X_1 - d)_+]$, for all $d \in \mathbb{R}_+$. Furthermore, it is not difficult to see that $\mathrm{Var}[(X_1 - d)_+] \leq \mathrm{Var}[(X_2 - d)_+]$, for all $d \in \mathbb{R}_+$. In Table 2 we present the solution to the bivariate capital allocation problem for various specifications of the parameters β and ϵ. To guarantee uniqueness of the solution (u_1^*, u_2^*), internal in the domain D defined in (10), Figure 1 verifies for various parameter specifications that condition (40) is satisfied. We see that an increase in the parameter β for the variance principle, naturally leads to a relative increase in the allocation to X_1. Furthermore, an increase in the dependence parameter ϵ also leads to a relative increase in the allocation to X_1.* $\square$

3 The optimal amount of economic capital

In this section we present a minimization procedure to determine the optimal amount of economic capital to be held physically by a financial conglomerate. The thus obtained amount of capital can then be distributed among the subsidiaries using the approach described in the previous section. We emphasize that we take the viewpoint of a higher authority within the conglomerate, and not of e.g. a regulatory authority. Therefore, the approach may be regarded as an internal decision model. To establish the framework, we provide the following definition:

Definition 3.1 *A raising principle is a functional* $\varrho : L^p \to \mathbb{R}$ *assigning an amount of economic capital* u *to a random variable* X, *given a set of information reflecting current market conditions.*

We will again formulate an optimization approach. The optimization procedure establishes an exchange between the cost of economic capital on the one hand and the market price of the risk residual on the other hand and is driven by current market conditions. In particular, we suggest the following minimization problem

$$\min_{\varrho(\cdot)} \pi_v((X - \varrho(X))_+) + (r_c - r_f)\varrho(X) \tag{21}$$

in which r_c denotes the cost of raising economic capital, r_f denotes the risk-free rate of interest and $\pi_v(\cdot)$ is a valuation measure for the risk residual. We use the subscript v to distinguish between the risk measure $\pi(\cdot)$ used to establish the capital allocation and the above valuation measure $\pi_v(\cdot)$. Typically, r_c equals the (opportunity) cost of capital charged by the shareholders. We naturally assume $r_c \geq r_f$. We remark that while it may be more realistic to let r_c be an increasing function of $\varrho(X)$, as a first approach we assume r_c to be constant. We note that $\{\pi_v(\cdot), r_c, r_f\}$ is the set of information to which definition 3.1 refers.

To guarantee the existence of a solution to the minimization problem we restrict to the domain D' defined by

$$D' = \{u \in \mathbb{R} | u \in [a, b]\} \tag{22}$$

in which $a, b \in \mathbb{R}$. In case we restrict to the vector space $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ a natural choice would be $a = \min[X_1 + \ldots + X_n]$ and $b = \max[X_1 + \ldots + X_n]$. Furthermore, in case lending is not allowed, we should restrict to $a \geq 0$.

The valuation measure $\pi_v(\cdot)$ reflects the market or transfer price of risk. A relevant question is which valuation measure to choose. Valuation measures are typically expressed as expected values calculated with respect to a transformed probability measure. Although various measure transformations exist in the literature, we will henceforth use *distorted probability measures* to establish the valuation measure. A distorted probability measure is obtained by transforming the decumulative distribution function $\overline{F}_X(x) = \mathbb{P}[X > x] = 1 - F_X(x)$ using a distortion or probability weighting function. We refer to Denneberg (1994) for an elaborate treatment of distorted probability measures. We state the following definition:

Definition 3.2 *A distortion function* $g : [0,1] \rightarrow [0,1]$ *is a continuous, strictly increasing function, satisfying* $g(0) = 0$ *and* $g(1) = 1$.

The transformed distribution function obtained by applying a distortion function to a decumulative distribution function, is given by

$$F_X^*(x) = 1 - g(\overline{F}_X(x))$$

In case both the distortion function $g(\cdot)$ and the distribution function F_X are differentiable, the transformed density function $f^*(\cdot)$ exists and is given by

$$f^*(x) = f(x)g'(\overline{F}_X(x))$$

The following definition introduces the distortion risk measure:

Definition 3.3 *For a non-negative random variable* X, *the distortion risk measure* $H_g(X)$ *with respect to the distortion function* $g(\cdot)$ *is given by*

$$H_g(X) = \int_0^\infty g(\overline{F}_X(x))dx = \int_0^\infty (1 - F^*(x))dx \qquad (23)$$

For an axiomatic representation of the distortion risk measure, we refer to Yaari's (1987) dual theory of choice. For its use as a valuation measure we refer to Chateauneuf, Kast and Lapied (1996) and Wang, Young and Panjer (1997). For further mention, we state without proof that the distortion risk measure is additive for comonotonic risks, i.e.

$$H_g(\sum_{i=1}^n X_i^c) = \sum_{i=1}^n H_g(X_i) \qquad (24)$$

We then state the following theorem:

Theorem 3.1 *The solution to the minimization problem in (21) using* $\pi_v(\cdot) = H_g(\cdot)$ *as the valuation measure, is given by*

$$u^* = F_X^{-1(\alpha)}(1 - g^{-1}(r_c - r_f)), \qquad \alpha \in [0,1] \qquad (25)$$

For the special case when $g(x) = x$ *and hence* $\pi_v(\cdot) = \mathbb{E}[\cdot]$, *we obtain*

$$u^* = F_X^{-1(\alpha)}(1 - (r_c - r_f)), \qquad \alpha \in [0,1] \qquad (26)$$

$$\square$$

Note that the thus obtained solution in Theorem 3.1 is a percentile under a transformed probability measure, i.e.

$$F_X^{-1(\alpha)}(1 - g^{-1}(1-s)) = \overline{F}_X^{-1(\alpha)}(g^{-1}(1-s)) = F_X^{*\,-1(\alpha)}(s) \qquad (27)$$

For an elaboration on percentile estimation with respect to transformed probability measures we refer to Aït-Sahalia and Lo (2000).

4   Allocating the cost of risk-bearing

A main objective of the capital allocation problem is to gain insight in the question how to allocate among the subsidiaries the cost of risk-bearing *borne by* the financial conglomerate. In this section we provide a consistent approach. The cost of risk-bearing borne by the financial conglomerate is given by

$$\pi_v((X - u^*)_+) + (r_c - r_f)u^* \tag{28}$$

Analogously, the cost of risk-bearing *due to* subsidiary i is given by

$$\pi_v((X_i - u_i^*)_+) + (r_c - r_f)u_i^* \tag{29}$$

As a first approach one may consider to allocate to each of the subsidiaries the fraction u_i^*/u^* of the cost of risk-bearing borne by the conglomerate, i.e.

$$\frac{u_i^*}{u^*}(\pi_v((X - u^*)_+) + (r_c - r_f)u^*) = \frac{u_i^*}{u^*}\pi_v((X - u^*)_+) + (r_c - r_f)u_i^*$$

However, in that case an increase in u_i^* would not only lead to an increase in $(r_c - r_f)u_i^*$ but also to an increase in $u_i^*/u^*\pi_v((X - u^*)_+)$.

A more appropriate approach would be to allocate to each of the subsidiaries the fraction $\pi((X - u^*)_+)/\sum_{i=1}^n \pi((X_i - u_i^*)_+)$ of the risk residual cost $\pi((X_i - u_i^*)_+)$ and in addition the capital cost $(r_c - r_f)u_i^*$. Note that the fraction $\pi((X - u^*)_+)/\sum_{i=1}^n \pi((X_i - u_i^*)_+)$ reflects the diversification benefit to the subsidiaries of being embedded in the financial conglomerate. In case of comonotone subsidiaries there does not exist a diversification benefit and therefore it is natural to require $\pi((X^c - u^*)_+)/\sum_{i=1}^n \pi((X_i - u_i^*)_+) = 1$.

As before, we will use a distortion risk measure as the valuation measure, i.e. $\pi_v(\cdot) = H_g(\cdot)$ for some distortion function $g(\cdot)$. A convenient property of the distortion risk measure is its additivity for comonotonic risks and hence that indeed $\pi((X^c - u^*)_+)/\sum_{i=1}^n \pi((X_i - u_i^*)_+) = 1$.

In some situations it is arguable that the cost of risk-bearing allocated to a subsidiary should not depend on its relative dependence within the conglomerate. Henceforth, we assume that the financial conglomerate has solved the capital allocation problem using $\pi(\cdot) = \mathbb{E}[\cdot]$ and hence that $u_i^* = F_{X_i}^{-1(\alpha_u)}(F_{X^c}(u))$. We remind that this specification of $\pi(\cdot)$ yields an optimal solution for u_i^* which does not take into account the dependence structure and is not arbitrary in the case of unknown dependence structure. In case the dependency structure between the subsidiaries is not taken into account, it is plausible to require monotonicity of the allocation of the cost of risk bearing, i.e. if $X_1 \leq X_2$ then the cost of risk-bearing charged to subsidiary 1 should be smaller than the cost of risk-bearing charged to subsidiary 2.

Therefore, we introduce an internal parameter γ denoting the fraction of the risk residual cost that the financial conglomerate charges to its subsidiaries. Furthermore, we suggest to introduce an internal parameter κ denoting the fraction of the capital allocation that the financial conglomerate charges to its subsidiaries in addition to the capital cost $(r_c - r_f)u_i^*$. Then,

$$\gamma\pi_v((X_i - u_i^*)_+) + (r_c - r_f + \kappa)u_i^* \tag{30}$$

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is the total cost of risk-bearing charged to subsidiary i. In particular, we suggest the following specifications of γ and κ:

$$\gamma = \begin{cases} \frac{\pi_v((X-u^*)_+)}{\pi_v((X^c-u^*)_+)}, & \text{if } \frac{g(1-F_X(u^*))}{g(1-F_{X^c}(u^*))} \geq \frac{\pi_v((X-u^*)_+)}{\pi_v((X^c-u^*)_+)} \\ \frac{g(1-F_X(u^*))}{g(1-F_{X^c}(u^*))}, & \text{if } \frac{g(1-F_X(u^*))}{g(1-F_{X^c}(u^*))} < \frac{\pi_v((X-u^*)_+)}{\pi_v((X^c-u^*)_+)} \end{cases} \tag{31}$$

and

$$\kappa = \begin{cases} 0, & \text{if } \frac{g(1-F_X(u^*))}{g(1-F_{X^c}(u^*))} \geq \frac{\pi_v((X-u^*)_+)}{\pi_v((X^c-u^*)_+)} \\ (\pi_v((X-u^*)_+) - \gamma\pi_v((X^c-u^*)_+))\frac{1}{u^*}, & \text{if } \frac{g(1-F_X(u^*))}{g(1-F_{X^c}(u^*))} < \frac{\pi_v((X-u^*)_+)}{\pi_v((X^c-u^*)_+)} \end{cases} \tag{32}$$

As stated above, the parameter γ reflects the diversification benefit to the subsidiaries of being embedded in the conglomerate. It follows immediately that for the case of comonotonic subsidiaries or for the case of complete lack of information on the dependence structure, the diversification benefit does not exist and we obtain $\gamma = 1$ (and $\kappa = 0$). The parameter κ has been added to prevent that, in times when the market price for risk residuals is high relatively to the cost of economic capital, risky subsidiaries with large percentiles benefit more than less risky subsidiaries from a high use of the economic capital. We then state the following theorem:

Theorem 4.1 *Let γ and κ be as defined in (31) and (32). Then we have monotonicity of the allocation of the cost of risk-bearing, i.e. if $X_1 \leq_1 X_2$ then*

$$\gamma\pi_v((X_1 - u_1^*)_+) + (r_c - r_f + \kappa)u_1^* \leq \gamma\pi_v((X_2 - u_2^*)_+) + (r_c - r_f + \kappa)u_2^* \tag{33}$$

$\square$

5 Dynamic capital allocation

In this section we present a generalization of the integrated capital allocation solution to a dynamic setting. We consider a finite and fixed time horizon $T < \infty$. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we let $X = (X_t : 0 \leq t \leq T)$ denote a stochastic process representing the net loss process of a position currently held. The flow of information is formalized by the augmented (i.e. completed) filtration $\mathbb{F} = (\mathcal{F}_t : 0 \leq t \leq T)$ of σ-algebra's with $\mathcal{F}_t \subset \mathcal{F}_T = \mathcal{F}$, where for all $0 \leq t \leq T$, $\mathcal{F}_t$ contains all $\mathcal{F}$-measurable events that are of $\mathbb{P}$-measure 0. We assume that the augmented filtration is right-continuous. In addition, we continue to restrict to the vector space $L(\Omega, \mathcal{F}, \mathbb{P})$, with $1 \leq p \leq \infty$.

Let $\mathbf{X} = (X_1, \ldots, X_n)$ denote a vector of $\mathcal{F}$-adapted stochastic processes, representing individual subsidiaries and let $X = X_1 + \ldots + X_n$ represent the financial conglomerate. In a dynamic setting, information releases formalized by the filtration $\mathbb{F}$, require an updating mechanism for the probability distribution of the terminal values $\mathbf{X}_T = (X_{T,1}, \ldots, X_{T,n})$. We propose Bayesian updating of the probability measure, which is a common approach in the theory of dynamic asset pricing. We introduce the notion of a conditional risk measure $\pi(\cdot|A)$, $A \in \mathcal{F}$, which is to be interpreted as the risk measure calculated with respect to the conditional distribution

function $F_{\cdot|A}$. Here, for a random variable X_t the conditional distribution function is defined as usual by

$$F_{X_t|A}(B) = \frac{\mathbb{P}[\{X_t \in B\} \cap A]}{\mathbb{P}[A]}, \qquad A, B \in \mathcal{F}$$

Now we can generalize the static approach to the dynamic setting. In particular, we replace the random variables in expressions (9), (21), (28) and (29) by their equivalents at time T in the dynamic setting, conditioned on the filtration $\mathcal{F}_t$, representing the information which is available at the time t at which the allocation is performed. Furthermore, the static solutions $(u_1^*, \ldots, u_n^*)$ in (28) and (29) are replaced by the dynamic ones.

6   A dynamic example

In this section we apply the integrated capital allocation approach developed in the previous sections, to a financial conglomerate consisting of 4 subsidiaries (or lines of business or portfolios). We assume that the net losses of the subsidiaries are generated by a multivariate Wiener process. The particular choice of the multivariate Wiener process is arbitrary. A convenient property of a Wiener process is that it is a Markov process and hence that

$$\pi(X_T|\mathcal{F}_t) = \pi(X_T|X_t)$$

Furthermore, a Wiener process is closed under summation, i.e. if all X_i, $i = 1, \ldots, n$ are Wiener processes then so is $X = X_1 + \ldots + X_n$. In addition, a multivariate Wiener process is completely characterized by the vector of drifts and the matrix of covariance parameters. For an elaborate treatment of (multivariate) Wiener processes, we refer to Karatzas and Shreve (1988) or Revuz and Yor (1991).

The multivariate Wiener process is as usual defined by the stochastic differential equation

$$d\mathbf{X}_t = \mu dt + \boldsymbol{\Sigma} d\mathbf{W}_t, \qquad \mathbf{X}_0 = \mathbf{0} \tag{34}$$

in which $\mathbf{X}_t \in \mathbb{R}^n$ is the random vector of net losses of the subsidiaries, $\mu \in \mathbb{R}^n$ is the vector of drifts, $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times m}$ is the matrix of covariance parameters and $\mathbf{W}_t \in \mathbb{R}^m$ is an m-dimensional standard Brownian motion. Then, the individual X_i's are multivariate Wiener processes with drift $\mu_i \in \mathbb{R}$ and vector of covariance parameters $(\Sigma_{i,1}, \ldots, \Sigma_{i,m})$, $i = 1, \ldots, n$. Furthermore, we have that

$$dX_t = \sum_{i=1}^{n} dX_{t,i} = \tilde{\mu} dt + \tilde{\Sigma} d\tilde{W}_t, \qquad X_0 = 0 \tag{35}$$

in which $\tilde{\mu} = \sum_{i=1}^{n} \mu_i$, $\tilde{\Sigma} = \sqrt{\sum_{j=1}^{m} (\sum_{i=1}^{n} \Sigma_{i,j})^2}$ and $\tilde{W}_t$ is 1-dimensional standard Brownian motion satisfying $\tilde{\Sigma} \tilde{W}_t = \sum_{i=1}^{n} \sum_{j=1}^{m} \Sigma_{i,j} W_{t,j}$. It is well-known that

$$X_T|\mathcal{F}_t \sim N(x_t + (T-t)\tilde{\mu}, (T-t)\tilde{\Sigma}^2)$$

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and

$$X_{T,i}|\mathcal{F}_t \sim N(x_{t,i} + (T-t)\mu_i, (T-t)\sum_{j=1}^{m}\Sigma_{i,j}^2), \qquad i = 1, \ldots, n$$

We set $n = 4$ (i.e. 4 subsidiaries) and $m = 4$ (i.e. a 4-dimensional Brownian motion) and consider a time horizon $T = 3$. The vector of drifts and the matrix of covariance parameters are given by

$$\mu = \begin{pmatrix} -0.4 \\ 0.5 \\ 0.9 \\ -1.4 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} 0.5 & 0.5 & 0.25 & 0 \\ 0 & 0 & 0.25 & 0.5 \\ 0 & 0.75 & 0.25 & -0.5 \\ 0.5 & 0 & -0.5 & 0.5 \end{pmatrix}$$

Then,

$$\begin{pmatrix} \text{Var}[X_1] \\ \text{Var}[X_2] \\ \text{Var}[X_3] \\ \text{Var}[X_4] \\ \text{Var}[X] \end{pmatrix} = \begin{pmatrix} 0.56 \\ 0.31 \\ 0.88 \\ 0.75 \\ 2.88 \end{pmatrix}$$

Figure 2 presents simulated sample paths of the multivariate Wiener process.

First, we determine the optimal amount of economic capital at time $t, 0 \le t \le T$, denoted by u_t^*, given the information available at time t. We will employ the distortion risk measure $H_g(\cdot)$ defined in (23) as the valuation measure. For the static case the optimal solution is obtained in Theorem 3.1. For the dynamic case, given the Markov property of the Wiener process, the solution is given by

$$u_t^* = F_{X_T|X_t}^{-1}(1 - g^{-1}(r_c - r_f)), \qquad t \in [0, T]$$

We will employ two different distortion functions to specify the valuation measure. The first distortion function is the proportional hazards (PH) distortion given by

$$g_{\text{ph}}(s) = s^{1/a}, \ a \ge 1 \tag{36}$$

The second distortion function is the dual power (DP) distortion given by

$$g_{\text{dp}}(s) = 1 - (1 - s)^b, \ b \ge 1 \tag{37}$$

For a general discussion on distortion functions we refer to Wang (1996). We remark that for the stated domains of a and b both distortion functions satisfy $g(s) \ge s$ for all $s \in [0, 1]$. Hence, we have that $H_g(X) \ge \mathbb{E}[X]$ for both specifications of the distortion function. We arbitrarily set $r_c = 0.07$ and $r_f = 0.04$. Dynamic solutions for the optimal amount of economic capital for different parameter values a and b are presented in Figure 3. We remark that for equal values of a and b, the optimal values u^* under the PH-transform are larger (or equal for $a = b = 1$) than the optimal values under the dual power transform. Although this does not hold true in general, it will be the usual case since $g'_{\text{ph}}(s|s \downarrow 0) = \infty$ for $a > 1$ whereas $g'_{\text{dp}}(s|s \downarrow 0) = b$ for $b \ge 1$ and therefore the PH-distortion attaches more weighting

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to very large values of the loss distribution than the DP-distortion does. Henceforth we will specify the valuation measure using the PH-distortion with parameter value $a = 1.25$.

Secondly, we dynamically allocate the optimal amount of economic capital among the subsidiaries. We choose $\pi(\cdot) = \mathbb{E}[\cdot]$ to perform the allocation. For the static case, the optimal allocation is obtained in Theorem 2.2. For the dynamic case, the optimal allocation is given by

$$u_{t,i}^* = F_{X_{T,i}|X_{t,i}}(F_{X_T^c|X_t}(u_t^*)), \qquad t \in [0, T], \ i = 1, \ldots, n$$

using the Markov property of the Wiener process. In order to compute $F_{X_T^c|X_t}(u_t^*)$ we note that

$$X_T^c|\mathcal{F}_t \sim N(x_t + (T - t)\tilde{\mu}, (T - t)(\sum_{i=1}^n \sqrt{\sum_{j=1}^m \Sigma_{i,j}^2})^2)$$

Optimal allocations are presented in Figure 4. We remark that because of its "prosperous" loss distribution, the economic capital allocated to subsidiary 4 is negative. The *cost* of this allocation, i.e. $(r_c - r_f)u_4^*$, which is actually a benefit, will be deducted from the cost of its risk residual.

Finally, we allocate the cost of risk-bearing. For the static case, consistent allocations which do not take into account the dependence structure are given in (30), with $\pi_v(\cdot) = H_{g_{\text{ph}}}(\cdot)$ and γ and κ as defined in (31) and (32). Although dynamic allocations can be given analogously, we will in this example restrict to a static allocation of the cost of risk-bearing, due to the memory-consuming calculation of the risk measure $\pi_v(\cdot)$. The allocation at time 0 is presented in Table 3, as are the values of γ and κ. The low value of γ represents the strong diversification effect within the multivariate normal distribution.

7   Concluding Remarks

We proposed a dynamic optimization approach to economic capital allocation, distinguishing between an allocation principle and a measure for the risk residual. The approach provides an integrated solution to the allocation problem and solves the 4 problems encountered in previous axiomatic approaches, mentioned in the introduction. In particular, we propose an optimization procedure to solve for the optimal amount of economic capital to be held physically by a financial conglomerate. The optimization procedure establishes an exchange between the cost of economic capital and the market price of the *entire* risk residual. We demonstrated that imposing bounds on the risk residual may provide unjustified incentives to split-up conglomerates. Furthermore, we formulated an optimization procedure for the optimal allocation of the economic capital among the subsidiaries under different degrees of information on the dependence structure between the subsidiaries. Finally, we suggested a consistent allocation among the subsidiaries of the cost of risk-bearing borne by the conglomerate.

We conclude with a final remark on diversification and the property of subadditivity. Whereas axiomatic approaches typically impose subadditivity of the economic

capital to represent the diversification effect, i.e. the economic capital u_{X+Y} of the pooled conglomerate $X+Y$ should be less or equal than the economic capital $u_X + u_Y$ of the conglomerates X and Y when considered separately, our approach exploits the natural inequality

$$(X + Y - (u_X + u_Y))_+ \leq_1 (X - u_X)_+ + (Y - u_Y)_+$$

We are reluctant to impose subadditivity of the economic capital. To see why, note that while the above inequality provides a natural preference for the coalescence of risks, it does not in general imply that u^*_{X+Y} obtained as the solution to the problem

$$\min_{u_{X+Y}} \pi_v((X + Y - u_{X+Y})_+) + iu_{X+Y}$$

is less or equal than $u^*_X + u^*_Y$, obtained as the solution to

$$\min_{u_X, u_Y} \pi_v((X - u_X)_+) + \pi_v((Y - u_Y)_+) + i(u_X + u_Y)$$

even not in the case in which the valuation measure $\pi_v(\cdot)$ is subadditive and

$$\pi_v((X - u_X)_+) + \pi_v((Y - u_Y)_+) \geq \pi_v((X + Y - (u_X + u_Y))_+)$$

Proofs

Proof of Theorem 2.1 We make use of a Lagrange multiplier to restate problem (13) as follows

$$\mathcal{L}((u_1, \ldots, u_n), \lambda) = \sum_{i=1}^n \mathbb{E}[(X_i - u_i)_+] + \lambda(u_1 + \ldots + u_n - u)$$

in which $\lambda \in \mathbb{R}$. The first-order conditions for an internal solution, yield for some constant $c \in (0, 1)$ that

$$F_{X_i}(u_i) = c, \text{ for all } i = 1, \ldots, n$$

or equivalently

$$u_i = F_{X_i}^{-1(\alpha_i)}(c), \text{ for all } i = 1, \ldots, n$$

for some (not necessarily unique) vector $(\alpha_1, \ldots, \alpha_n)$. Since $\sum_{i=1}^n u_i = u$ by assumption, we restrict to vectors $(\alpha_1, \ldots, \alpha_n)$ in the set A defined by

$$A = \{(\alpha_1, \ldots, \alpha_n) : \sum_{i=1}^n F_{X_i}^{-1(\alpha_i)}(c) = u\} \tag{38}$$

A proper choice is $\alpha_i = \alpha_u$, for all $i = 1, \ldots, n$ in which α_u is determined such that $F_{X^c}^{-1(\alpha_u)} = u$. In case of strictly increasing and continuous marginal distribution functions, and hence $F_X^{-1(\alpha)}(p) = F_X^{-1}(p)$ for all $p \in (0, 1)$ and all $\alpha \in [0, 1]$, we obtain the unique allocation

$$u_i^* = F_{X_i}^{-1}(F_{X^c}(u))$$

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It is not difficult to see that the Hessian matrix of $\mathcal{L}((u_1^*, \ldots, u_n^*), \lambda)$ given by

$$
\begin{pmatrix}
0 & -1 & \cdots & -1 \\
-1 & \frac{F_{X_1}(u_1^* + \Delta x_1) - F_{X_1}(u_1^*)}{\Delta x_1} & 0 & 0 \\
\vdots & 0 & \ddots & 0 \\
-1 & 0 & 0 & \frac{F_{X_n}(u_n^* + \Delta x_n) - F_{X_n}(u_n^*)}{\Delta x_n}
\end{pmatrix}
$$

in which $\Delta x_i = \lim_{\delta \downarrow 0} \sup\{x : F_{X_i}(x) \leq F_{X^c}(u)\} + \delta - u_i^*$, $i = 1, \ldots, n$, is positive definite and that therefore the obtained solution is a minimum. We refer to Du, Qi and Womersley (1995) for an elaboration on multivariate optimization problems with non-differentiable functions. The solution to problem (14) and the equality of the left-hand side and the right-hand side of expression (16) immediately follows from the theory of comonotonic risks. In particular, we refer to Kaas et al. (2001, section 10.6) and Dhaene et al. (2002).

Proof of Theorem 2.2 Follows immediately, since the expectation operator does only take into account marginal distributions and not the dependence structure.

Proof of corollary 2.1 Consider the following two subdivisions of a financial conglomerate:
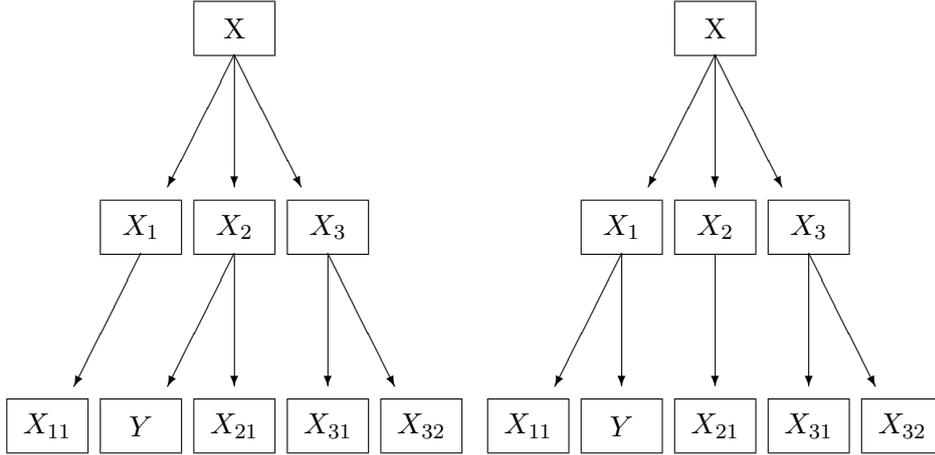
By Theorem 2.1, we have the following allocations at the subsidiary level:

$$
u_i^* = F_{X_i}^{-1(\alpha_u)}(F_{X^c}(u)), \qquad i = 1, 2, 3
$$

Suppose the random vector $(X_{11}, Y, X_{21}, X_{31}, X_{32})$ consists of comonotone elements. Suppose furthermore, that α_u, satisfying $F_{X^c}^{-1(\alpha_u)}(F_{X^c}(u)) = u$, is used to perform the allocation at *each* (sub)sublevel. Then, it holds true that the allocation to Y for subdivision 1 equals the allocation to Y for subdivision 2, i.e.

$$
F_Y^{-1(\alpha_u)} F_{X_2^c}(F_{X_2}^{-1(\alpha_u)}(F_{X^c}(u))) = F_Y^{-1(\alpha_u)} F_{X_1^c}(F_{X_1}^{-1(\alpha_u)}(F_{X^c}(u))) = F_Y^{-1(\alpha_u)}(F_{X^c}(u))
$$

The above equality will not in general hold true for non-comonotonic random vectors $(X_{11}, Y, X_{21}, X_{31}, X_{32})$.

Proof of Theorem 2.3 First note that since $v(\cdot)$ is strictly increasing, the mean value principle yields the same optimal capital allocation as the risk measure $\pi(X) = \mathbb{E}[v(X)]$. For the case of independent risks we make use of a Lagrange multiplier. Since F_{X_1} and F_{X_2} are absolute continuous by assumption, the first-order conditions for an internal solution yield for $i, j = 1, 2, i \neq j$

$$\int_{u_i}^{\infty} \int_{-\infty}^{\infty} v'((x_i - u_i) + (x_j - u_j)_+)d(F_{X_i}(x_i)F_{X_j}(x_j)) = \lambda$$

for some $\lambda \in \mathbb{R}_+/\{0\}$, or equivalently

$$\int_{u_1}^{\infty} \int_{-\infty}^{u_2} v'(x_1 - u_1)dF_{X_1}(x_1)F_{X_2}(x_2) = \int_{-\infty}^{u_1} \int_{u_2}^{\infty} v'(x_2 - u_2)dF_{X_1}(x_1)F_{X_2}(x_2)$$

This can be rewritten as

$$F_{X_2}(u_2) \int_{u_1}^{\infty} v'(x_1 - u_1)dF_{X_1}(x_1) = F_{X_1}(u_1) \int_{u_2}^{\infty} v'(x_2 - u_2)dF_{X_2}(x_2)$$

Hence, the result follows. The Hessian matrix of $\mathcal{L}((u_1^*, \ldots, u_n^*), \lambda)$ is positive definite and therefore the obtained solution is a minimum. Since by definition $v''(x) > 0$ for all x, the optimal solution is unique.

For the case of comonotonic risks we rewrite the optimization problem as follows:

$$\min_{u_1} \int_0^1 v((g(s) - u_1)_+ + (h(s) - u + u_1)_+)ds$$

in which $g(\cdot) = F_{X_1}^{-1}(\cdot)$ and $h(\cdot) = F_{X_2}^{-1}(\cdot)$. The first-order condition yields

$$-\int_0^1 v'((g(s) - u_1) + (h(s) - u + u_1)_+)1_{\{u_1 < g(s)\}}ds +$$

$$\int_0^1 v'((g(s) - u_1)_+ + (h(s) - u + u_1))1_{\{u_1 > u - h(s)\}}ds = 0$$

For $g^{-1}(u_1) > h^{-1}(u - u_1)$ this reduces to

$$\int_{h^{-1}(u-u_1)}^{g^{-1}(u_1)} v'(h(s) - u + u_1)ds = 0$$

or

$$\int_{F_{X_2}(u-u_1)}^{F_{X_1}(u_1)} v'(F_{X_2}^{-1}(s) - u + u_1)ds = 0$$

while for $g^{-1}(u_1) \leq h^{-1}(u - u_1)$ it reduces to

$$\int_{g^{-1}(u_1)}^{h^{-1}(u-u_1)} v'(g(s) - u_1)ds = 0$$

or

$$\int_{F_{X_1}(u_1)}^{F_{X_2}(u-u_1)} v'(F_{X_1}^{-1}(s) - u_1)ds = 0$$

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Since by definition $v'(x) > 0$ for all x, we obtain $F_{X_1}(u_1) = F_{X_2}(u - u_1)$. Hence, for $i \neq j$ we have

$$u_i^* = F_{X_i}^{-1}(F_{X_j}(u - u_i)) = F_{X_i}^{-1}(F_{X^c}(u))$$

Since $v''(x) > 0$ for all x, the obtained solution is a minimum.

Proof of Theorem 2.4 We rewrite the objective function as follows:

$$\int_{u_1}^{\infty} (1 - F_{X_1}(x))dx + \int_{u_2}^{\infty} (1 - F_{X_2}(x))dx +$$

$$\beta(\int_{u_1}^{\infty} (1 - F_{X_1}(x))2(x - u_1)dx - (\int_{u_1}^{\infty} (1 - F_{X_1}(x))dx)^2 +$$

$$\int_{u_2}^{\infty} (1 - F_{X_2}(x))2(x - u_2)dx - (\int_{u_2}^{\infty} (1 - F_{X_2}(x))dx)^2 +$$

$$2\int_{u_1}^{\infty} \int_{u_2}^{\infty} (x_1 - u_1)(x_2 - u_2)dF_{X_1,X_2}(x_1, x_2) -$$

$$2\int_{u_1}^{\infty} (1 - F_{X_1}(x))dx \int_{u_2}^{\infty} (1 - F_{X_2}(x))dx) \qquad (39)$$

and substitute $u_2 = u - u_1$. Then the first-order condition for an internal solution, yields the stated result. A sufficient condition for convexity of the objective function and hence for a unique minimum is given by

$$f_{X_1}(u_1) + f_{X_2}(u - u_1) + 2\beta(F_{X_1}(u_1) + 2F_{X_1}(u_1)F_{X_2}(u - u_1) +$$

$$F_{X_2}(u - u_1) - (F_{X_1}(u_1))^2 - (F_{X_2}(u - u_1))^2 - 2F_{X_1,X_2}(u_1, u_2) -$$

$$(f_{X_1}(u_1) + f_{X_2}(u - u_1))(\int_{u_1}^{\infty} (1 - F_{X_1}(x))dx + \int_{u - u_1}^{\infty} (1 - F_{X_2}(x))dx)) > 0 \quad (40)$$

Hence, for sufficiently low values of β a unique minimum is guaranteed, see example 2.2.

Proof of Theorem 3.1 Rearranging the first-order condition yields the stated result.

Proof of Theorem 4.1 If $X_1 \leq_1 X_2$ then $u_1^* = F_{X_1}^{-1(\alpha_u)}(F_{X^c}(u^*)) \leq F_{X_2}^{-1(\alpha_u)}(F_{X^c}(u^*)) = u_2^*$. Hence, we have the following inequalities

$$\gamma\pi_v((X_1 - u_1^*)_+) + (r_c - r_f + \kappa)u_1^* \leq$$

$$\gamma g(1 - F_{X_1}(u_1^*))(u_2^* - u_1^*) + \gamma\pi_v((X_2 - u_2^*)_+) + (r_c - r_f + \kappa)u_1^* \leq$$

$$(r_c - r_f)(u_2^* - u_1^*) + \gamma\pi_v((X_2 - u_2^*)_+) + (r_c - r_f + \kappa)u_1^* \leq$$

$$\gamma\pi_v((X_2 - u_2^*)_+) + (r_c - r_f + \kappa)u_2^* \qquad (41)$$

The second inequality holds true since

$$\gamma g(1 - F_{X_1}(u_1^*)) = \gamma g(1 - F_{X^c}(u^*)) \leq$$

$$g(1 - F_X(u^*)) = r_c - r_f \qquad (42)$$

in which the first inequality follows from substitution of γ and the last equality follows from Theorem 3.1.

Tables and Figures

**Table 1:** Specification of the random vector (Example 2.1)

|	X_1	X_2
$U \in (0, 0.50]$	-10	-15
$U \in (0.50, 0.90]$	1	3
$U \in (0.90, 0.95]$	1	4
$U \in (0.95, 0.99]$	2	4
$U \in (0.99, 1)$	2	15

Table 2: Optimal allocation of the economic capital for various parameter specifications (Example 2.2)

| β	ϵ	u	u_1^*	u_2^*
0.1	1	1	0.706	0.294
0.1	2	1	0.710	0.290
0.1	10	1	0.717	0.283
0.2	1	1	0.746	0.254
0.2	2	1	0.754	0.246
0.2	10	1	0.771	0.229
0.3	1	1	0.787	0.213
0.3	2	1	0.797	0.203
0.3	10	1	0.829	0.171
0.4	1	1	0.828	0.172
0.4	2	1	0.840	0.160
0.4	10	1	0.885	0.115

Figure 1: Verification of the convexity condition given in expression (40) for various specifications of β while $\epsilon = 2$ (Example 2.2)
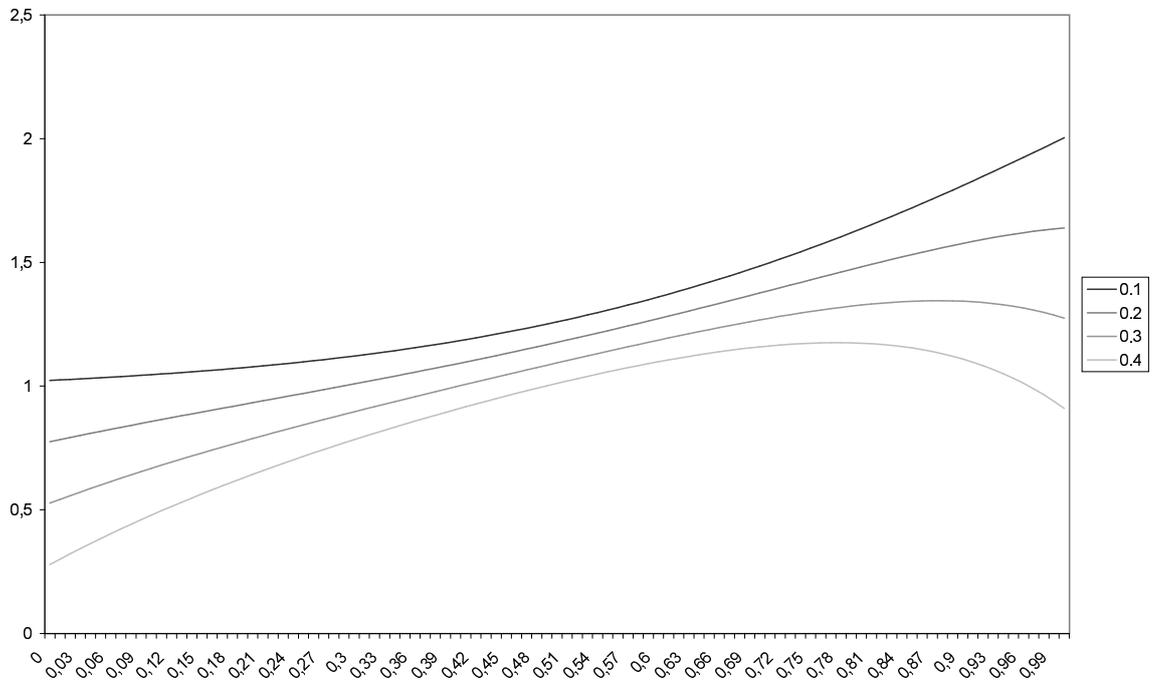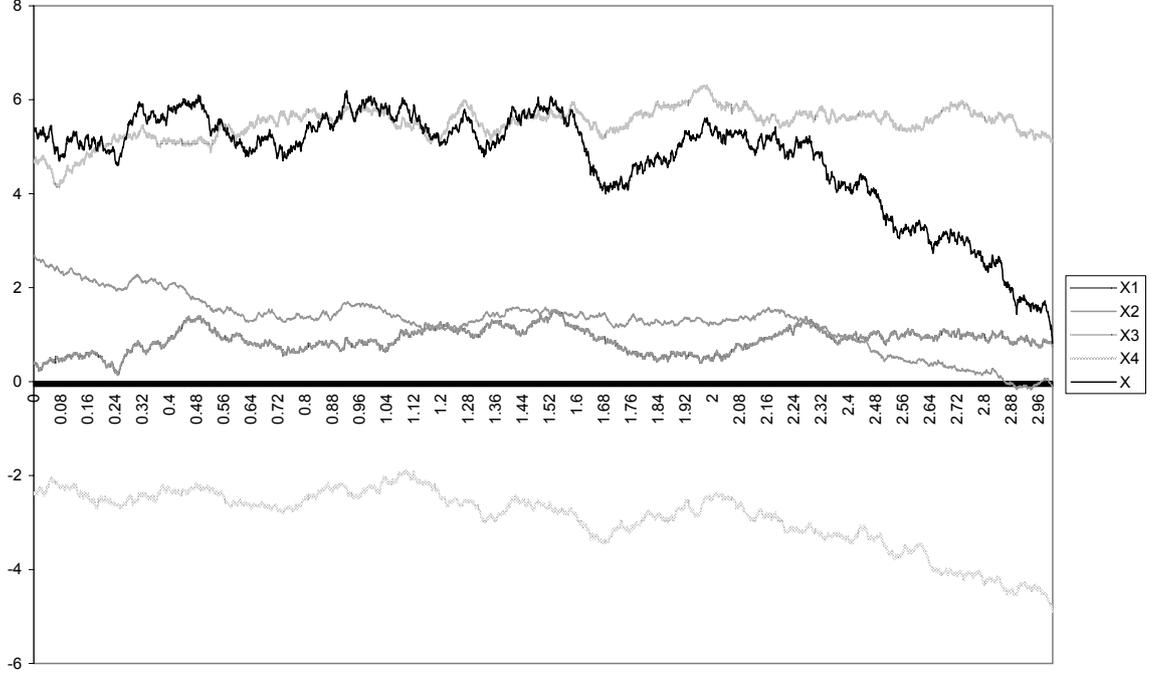
Figure 2: Sample paths of the multivariate Wiener process (Section 6)
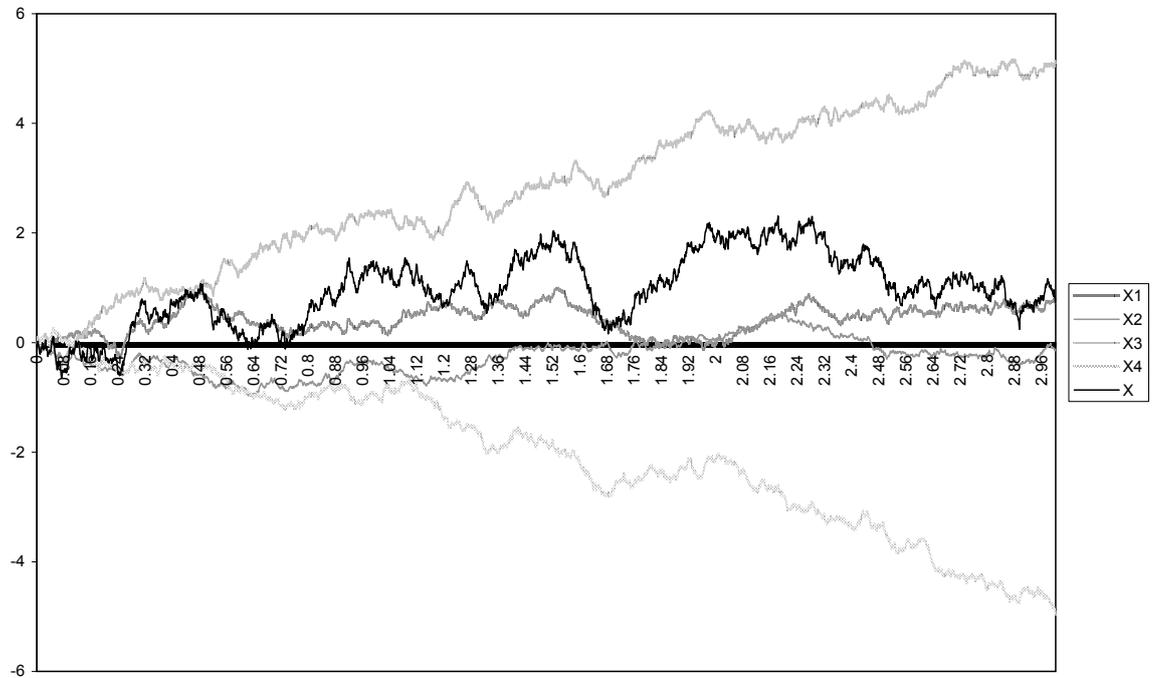
Figure 3: Dynamic solution of the optimal amount of economic capital for various specifications of the valuation measure $\pi_v(\cdot) = H_g(\cdot)$ with $r_c = 0.07$ and $r_f = 0.04$ (Section 6)
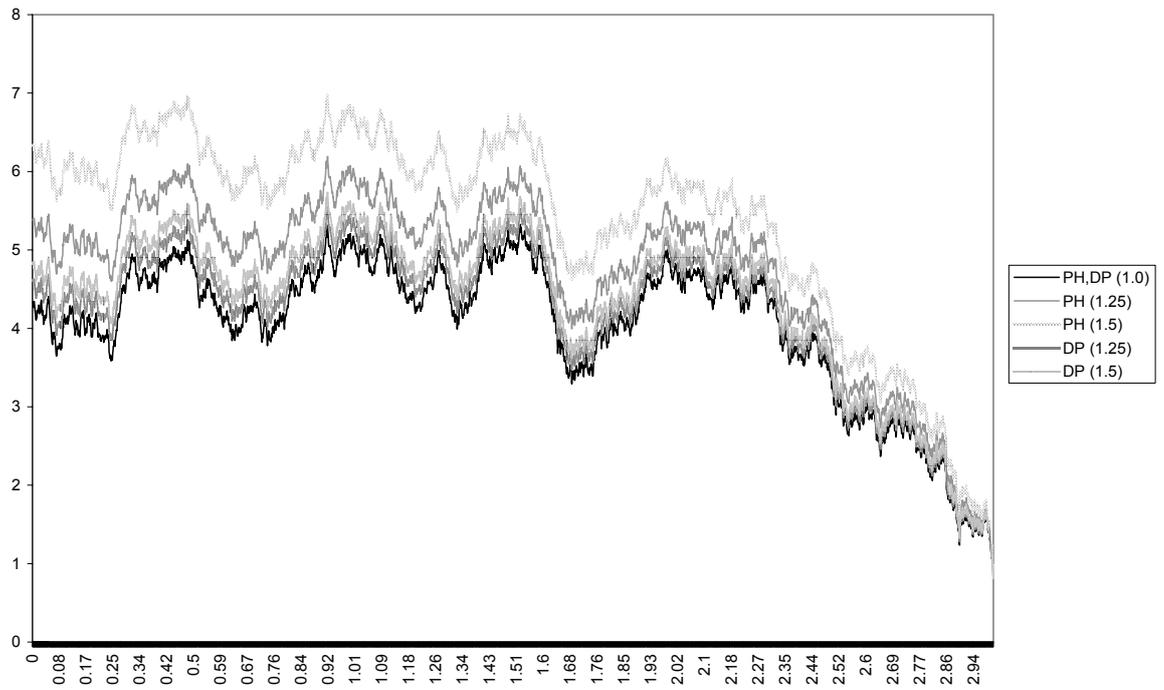
Figure 4: Dynamic allocation among the subsidiaries with $\pi_v(\cdot) = H_{g_{\mathrm{ph}}}(\cdot)$, a=1.25, $r_c = 0.07$, $r_f = 0.04$ and $\pi(\cdot) = \mathbb{E}[\cdot]$ (Section 6)

Table 3: Allocations at time $t = 0$ of the cost of risk-bearing with $\pi_v(\cdot) = H_{g_{\mathrm{ph}}}(\cdot)$, a=1.25, $r_c = 0.07$, $r_f = 0.04$ and $\pi(\cdot) = \mathbb{E}[\cdot]$ (Section 6)

	$X_{1,T}$	$X_{2,T}$	$X_{3,T}$	$X_{4,T}$	$\sum_{i=1}^{4}$
$\pi_v((X_{T,i} - u_{0,i}^*)_+)$	0.13	0.10	0.16	0.15	0.54
$\pi_v((X_{T,i} - u_{0,i}^*)_+) + (r_c - r_f)u_{0,i}^*$	0.14	0.18	0.30	0.08	0.70
$\gamma\pi_v((X_{T,i} - u_{0,i}^*)_+) + (r_c - r_f)u_{0,i}^*$	0.02	0.09	0.15	-0.06	**0.20**
$\pi_v((X_T - u_0^*)_+)$	0.04				
$\pi_v((X_T - u_0^*)_+) + (r_c - r_f)u_0^*$	**0.20**				
$\frac{\pi_v((X - u^*)_+)}{\pi_v((X^c - u^*)_+)}$	0.07				
$\frac{g(1 - F_X(u^*))}{g(1 - F_{X^c}(u^*))}$	0.17				
γ	0.07				
κ	0				

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