STOCHASTIC VARIATION OF INTEREST AND MORTALITY

PENTTI SOININEN
MUTUAL INSURANCE COMPANY KALEVA
KLUUVIKATU 3 - PB 347
SF-00101 HELSINKI - FINLAND
TELEPHONE : 358-0-132 2253
FAX : 358-0-132 2675
E-MAIL: SAMPO.SAM.SOININPE@ELVI.VTKK.FI

Abstract

This article studies the ways in which the stochastic variance of the forces of interest and mortality affect the formulae of discounting, prolongation and term insurance derived with the deterministic model. The stochasticity of the mortality of Finnish life insurance has only a neglible effect on the formulae, while the stochasticity of the force of interest has a considerable effect.

The expected values have been derived by integrating over all the forces of interest on the infinite-dimensional Banach space of the continuous functions C[0,T]. The measure used is the Gauss or Wiener measure. In many articles the Wiener measure is referred to, but not explicitly expressed. In this article the measure and the useful calculation formula of the expectation value are derived by functional analysis and the formula is applied to the life insurance. The idea of the formula is applied to the life insurance. The idea of the formula is similar to that of the Feynman path integral in quantum physics.

In addition the integration with respect to the gamma measure on finite-dimensional spaces is studied. However, no corresponding result as for the Wiener measure can be derived for the gamma measure on any infinite-dimensional space.

Keywords: Stochastic interest and mortality, Integration in Banach spaces, Wiener measure, Discounting, Term insurance.

CONTENTS

NOTATIONS

If function f is distributed according to the distribution function F, the notation $f \div F$ is used.

If it is desired that the previously undefined function f be defined with the known function g, the notation $f \equiv g$ is used.

Let (X,τ) be a topological space. B is the functor mapping the topological space to its Borel sets; $B(X)$ is the σ-algebra generated by the family τ.

INTRODUCTION

In deterministic theory the discounting factor is

$$D(\delta,\mu,x,x+T)=\exp-\int_0^T(\delta(t)+\mu(t))dt \tag{0.1}$$

When using this as the exact discounting factor one must assume exact information on the rates of interest and mortality during the time interval $[0,T]$. Although one may have an exact model of the expectation value of the rate of interest δ, the true rate varies in the neighbourhood of the expected value and can be measured almost continuously. Mortality differs from the rate of interest in such a way that it cannot be measured, even continuously. Depending on the size of the population, it has been measured in one-year or five-year age-groups and then interpolated to be a continuous function. During the separated years the observed mortality in each group varies and thus it is possible to obtain the mortality distribution in each group.

The discounting factor above has been derived from the Thiele differential equation for the reserve V

$$V'(t)=(\delta(t)+\mu(t))\cdot V(t)\ldots \tag{0.2}$$

The theory of stochastic differential equations has the same starting point but with the further addition of the Brownian motions W_δ and W_μ to the interest and mortality rates:

$$V'(t)=\left(\delta(t)+W_\delta(t)+\mu(t)+W_\mu(t)\right)\cdot V(t)\ldots \tag{0.3}$$

In this study the starting point is, however, different. The value of the discounting factor (0.1) solved from the differential equation of Thiele (0.2) depends on the paths of the interest and mortality rates during time interval $[0,T]$. Every pair (δ,μ) of the random functions is mapped to the real value $D(\delta,\mu,x,x+T)$. Thus the discounting factor is a random variable of pair

(δ,μ) and whose moments can be calculated by integrating it with respect to a probability measure of the function space formed by the pair (δ,μ).

The main source of integration theory in infinite-dimensional Banach spaces has been the work of Laurent Schwartz [Schwartz] which deals with integration in arbitrary topological spaces and thus is too general an approach for the present study; nonessential definitions and properties of topological spaces have therefore been ignored. In addition, two other works dealing with integration in infinite-dimensional Banach spaces have been those of Skorohod [Skorohod] and Ledoux and Talagrand [Ledoux and Talagrand] which are too narrow in scope and contain insufficient results for the present study. The prerequisites of these works are the elements of functional analysis, measure and integration theory (especially in topological spaces) and topological vector spaces.

The reader not interested in theory need only read the notations in the second chapter. After construction of the theory the applications of the main theorem will be based only on elementary calculus.

The scope of the present study focuses on a single insurance when the moment of time of the certain premium is known; if it also focuses on the company, the moments of the premiums should be stochastic, as treated by [Norberg]. Embedding this stochasticity in a study is simple in theory, although not necessarily in practice, if one assumes that the premiums, interest and mortality rates are independent stochastic variables.

1. PROBABILITY IN BANACH SPACE

DEFINITION 1.1. Let X be a Banach space. The probability measure P of space X on the Borel-σ-algebra $B(X)$ is a **Radon (probability) measure** if for every $\varepsilon > 0$ and every measurable set $Y \in B(X)$ a compact set K of space X exists such that $K \subseteq Y$ and

$$P(X - K) < \varepsilon . \tag{1.1}$$

The theory of integration in locally compact Hausdorff spaces and their Radon measures can be applied in infinite-dimensional Banach spaces even though the latter are not locally compact, but since they are regular every regular space can be embedded in an open set of a compact Hausdorff space: the Stone-Cech compactification [Berberian]. The open sets of the compact space are locally compact Hausdorff subspaces, and there is a bijective map between the measures of the Banach space and those of the respective open set of the Stone-Cech compactification.

Let X and Y be measurable spaces, f a measurable function $X \to Y$ and P a measure of the space X. Thus $f(P)$ is the measure of space Y whose value in set $A \in B(Y)$ is $(f(P))(A) = P(f^{-1}(A))$.

Let I be a directed set and $i \to X_i, i \in I$, the decreasing net of the closed subspaces of Banach space X.

Term π_i is the canonical surjection $X \to X_i, x \mapsto x + X_i$ for every $i \in I$. Let $X_j, j \in I$, be another closed subspace of the Banach space contained in space X_i or $X_j \subseteq X_i$. The function $\pi_{ij}: X / X_j \to X / X_i$, is defined as $x + X_j \to x + X_i$; thus evidently $\pi_i = \pi_{ij} \circ \pi_j$ when $i \leq j$.

DEFINITION 1.2. Let X be a Banach space, $(X_n)_{n \in I}$ a decreasing net of finite-codimensional closed subspaces and P_n a probability in the Borel-σ-algebra $B_n = B(X / X_n)$ of the cospace X / X_n. If $m \leq n$ or $X_m \supseteq X_n$ and the consistency formula $P_m = \pi_{mn}(P_n)$ holds the family $(P_n, X / X_n)_{n \in I}$ or, in its abbreviated form, $(P_n)_{n \in I}$ is called the **cylindrical probability (measure)**.

The origin of the concept cylindrical probability can be explained as follows. Set $A \subseteq X$ is called a **cylindrical set** if there exists both a finite-codimensional closed subspace X_i and $B \subseteq X/X_i$ such that $A = \pi_i^{-1}(B)$. Then cylinder A is the set of those vectors by which $a + X_i = B$, while set B is called the cross section or the base of cylinder A. The 'measure' for cylinder A can be defined by the formula $P(A) = P_i(B)$ which in other words is the 'measure' of the cross-sectional area. With the consistency formula it can

easily be proved that $P(A)$ is independent of the pair (X_i,B). The quotation marks around the word 'measure' indicate that P does not necessarily mean measure as used in the measure and integration theory; i.e. it is not necessarily σ-additive.

The cylindrical probability of the previous definition defines a projective system of probability spaces $\left\{X / X_i, B_i, P_i, \pi_{ij} \,|\, i, j \in I\right\}$ ([Rao], p.14) which can be satisfied if the following three conditions are met:

(i) $\pi_{ij}^{-1}(B_j) \subseteq B_j$ for every i ≤ j because π_{ij} is continuous.

(ii) $\pi_{ik} = \pi_{ij} \circ \pi_{jk}$ when $i \leq j \leq k$ or $X_i \supseteq X_j \supseteq X_k$. This is the consequence of the definition of functions π_{ij}.

(iii) $P_i = \pi_{ij}(P_j)$ is contained in the definition of the cylindrical measure. Schwartz uses the term exactly as for the previous system ([Schwartz], p.74).

A natural question is when in Banach space X does a Radon probability P exist from which can be derived the probabilities of the finite-dimensional cospaces with canonical surjection or with the formula $P_n = \pi_n(P)$. The following theorem gives a necessary and sufficient condition.

THEOREM 1.3. (Prokhorov) Let X be a Banach space and $\left\{X / X_i, B_i, P_i, \pi_{ij} \,|\, i, j \in I\right\}$ the projective system of probability spaces. The necessary and sufficient condition for a Radon measure P existing in $B(X)$ for which $P_n = \pi_n(P)$ for every $n \in I$ is that for every real number $0 < \varepsilon \leq 1$ a compact set K_ε of space X exists such that $P_n(\pi_n K_\varepsilon) \geq 1 - \varepsilon$ for every $n \in I$. P is unique if the net $(X_i)_{i \in I}$ is such that the functions $(\pi_i)_{i \in I}$ separate the points of space X.

Proof. See ([Schwartz], p.81), ([Skorohod], p.5).

The theorem is not very easily applicable but is central to development of the theory and it gives an understanding of the existence of Radon measures and their concentrations. The following definition refers to different types of concentrations.

DEFINITION 1.4. Let X be a Banach space, $(P_n, X / X_n)_{n\in I}$ its cylindrical probability, $A \in B(X)$ and $0 < \varepsilon \leq 1$.

(a) It is said that $(P_n)_{n\in I}$ is **cylindrically concentrated in set A to the number** ε if $P_n(\pi_n A) \geq 1-\varepsilon$ for every $n \in I$.

(b) It is **flatly concentrated in set A to the number** ε if $P_n(\pi_n A) \geq 1-\varepsilon$ for every hyperspace X_n.

(c) Let Σ be the family of sets of the Borel-σ-algebra $B(X)$. Cylindrical probability is **concentrated in family Σ (flatly)** if to each $\varepsilon > 0$ a set $A \in \Sigma$ corresponds such that $(P_n)_{n\in I}$ is (flatly) cylindrically concentrated in set A to the number ε.

The following proposition gives sufficient condition to the existence of the Radon measure by cylindrical concentration.

PROPOSITION 1.5. Let $(P_n)_{n\in I}$ be a cylindrical measure of reflexive Banach space X and cylindrically concentrated in the balls of space X. It then defines a Radon measure in space X.

Proof. See ([Schwartz], p.204).

However, this proposition is too weak; e.g. the most important cylindrical measure from which it can be derived, the Gaussian probabilities of all finite-dimensional subspaces of X, is not cylindrically concentrated in the balls of X, because for every $R > 0$

$$\left[\int_{-R}^{R} \exp(-\frac{1}{2}t^2)dt\right]^n \to 0 \tag{1.2}$$

when the dimension $n \to \infty$, and because every ball of X is contained in a cylinder of base $[-R,R]^n$. Despite this the cylindrical measure in question defines a Radon measure. From formula (1.2) it can be seen that the cylindrical measure is flatly concentrated in the balls and thus also in the compact sets, because in the condition of flat concentration it is sufficient to be $n = 1$ in formula (1.2). By choosing a sufficiently large value for R the integral can be arbitrarily chosen to be close to one.

The following proposition gives condition to the existence of a Radon probability in continuous-function space with the help of flat concentration after first defining the new concept of convergence.

DEFINITION 1.6. Let X be a totally regular space and (P_n) a net of finite Radon measures of space X. It is said that (P_n) converges **narrowly** on the finite Radon measure P if $P_n(F)$ converges on the number $P(F)$ for every continuous and bounded function F: $X \to R$. According to the definition **the narrow topology** is the coarsest topology where convergence of the nets of finite Radon measures ($M^1(X)$-space) is the pointwise convergence in continuous and bounded functions. Thus the neighbourhood base of the origin of space $M^1(X)$ is formed by the sets $U(F_1, ..., F_k; \varepsilon) = \{ \mu \mid \mu \in M^1(X), |\mu(F_j)| < \varepsilon, 1 \leq j \leq k \}$ where $\varepsilon > 0$ and $F_1, ..., F_k$ are continuous and bounded functions $X \to \mathbf{R}$.

PROPOSITION 1.7. Let K be a compact topological space, $C(K)$ the Banach space of continuous functions with respect to uniform convergence on K (formula (2.5)), $(P_n)_{n \in I}$ its cylindrical probability, A another directed set and $(p_a)_{a \in A}$ a net of projections of the space $C(K)$ onto closed subspaces which converges uniformly on every compact set of space $C(K)$ towards the identity map of the compact set in question. If $(P_n)_{n \in I}$ is flatly concentrated in

(a) the compact subsets with respect to uniform convergence or

(b) the convex and weakly compact subsets

$(P_n)_{n \in I}$ defines Radon measure P on space $C(K)$ if and only if the net of Radon probabilities $(p_a(P))_{a \in A}$ converges narrowly on a Radon measure Q, and thus $P = Q$.

Proof. P has been defined on Page 4 after Definition 1.2. Schwartz proved the theorem for general topological spaces ([Schwartz], p.253).

The previous proposition is not very useful in proving the existence of the Radon probability, but is so in calculating the integral when knowing from other facts that the Radon probability exists. Thus, the narrow convergence means that

$$\lim_{a \in A} \int_{X_a} F(f) d(p_a(P)(f)) = \int_X F(f) dP(f) \tag{1.3}$$

for every bounded and continuous F: $C(K) \to \mathbf{R}$. If the Radon probability exists, its integral can be calculated as the limit of the probabilities calculated in the finite-dimensional spaces. It is most useful when calculating the limit using the subsequence of the directed set.

2. GAUSSIAN MEASURE

2.1 Notations. We use the following notations:

$$\begin{aligned} & T \in \mathbf{R}^+, \\ & K = [\,0\ T\,], \\ & t_{ni} = iT / 2^n, n \in N \text{ and } i = 0, 1, \ldots, 2^n = N, \\ & \Omega_n = \left\{ t_{ni} \,\middle|\, i = 0, \ldots 2^n \right\} \text{and} \\ & C(K) = X = \text{the set of continuous real functions} \\ & \qquad\qquad \text{of interval } K. \end{aligned} \tag{2.1}$$

The meaning of real T is the moment from when the discounting is done to the present moment 0. The dividing moments Ω_n of interval K divide the subintervals defined by moments Ω_{n-1} in such a way that every subinterval is divided into two intervals of equal length. In the present study the rate of interest is considered to be a continuous function in interval K even though it may be very oscillatory as is Brownian motion. Another possibility is that the function of the rate of interest can be chosen to be a measurable function. Each of these choices can be sustained with respect to realization; both are approximations of reality when the rate is observed only in finite numbers.

For every subset $\Omega = \left\{ 0 = t_0, t_1, \ldots, t_{n-1}, t_n = T \middle| t_{i-1} < t_i, i = 1, \ldots, n \right\}$ of interval K the projection of space X is defined. Let $f \in X$. Projection $p_\Omega f$ is the broken line which joins the points of set $\left\{ (t, f(t)) \middle| t \in \Omega \right\}$.

$$(2.2) \qquad p_\Omega : f \mapsto p_\Omega f = \left[t \mapsto \frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}}(t - t_{i-1}) + f(t_{i-1}) \right],$$

when $t \in [t_{i-1}, t_i]$. We denote

$$(2.3) \qquad X_\Omega = \{ p_\Omega f \mid f \in X \},$$

which is finite-dimensional and $\dim X_\Omega = n$ when the initial value $f(0)$ is kept constant. We especially denote

$$(2.4) \qquad p_{\Omega_n} = p_n \quad \text{and} \quad X_{\Omega_n} = X_n .$$

$C(K) = X$ is a Banach space with respect to the norm

$$(2.5) \qquad \|f\|_\infty = \sup_{t \in K} |f(t)| .$$

The topology defined by this norm is called the topology of uniform convergence. Another topology used in space X is the weak topology $\sigma(X,X')$ or the coarsest topology with respect to all the functionals or all the maps

$$\mu' \in X' : f \mapsto \int_K f(t) d\mu(t)$$

are continuous where μ is a Radon measure of interval K (Riesz representation theorem). For the Radon measure each topology is equally used because with respect to both topologies the Radon measures, measurable sets and their measures are equal ([Schwartz], p.162).

2.2 Gaussian measure in infinite-dimensional space. We define map J by the formula

$$(2.6) \qquad J : L_2(K) \to X, g \to \left[t \mapsto \int_0^t g(u) du \right]$$

The map is well-defined and $J(g)$ is absolutely continuous [Rudin]. We denote

(2.7) $$H = J(L_2(K)).$$

Since $J(g)' = g$ almost everywhere (a.e.) with respect to the Lebesgue measure [Rudin], J is an isomorphism between spaces $L_2(K)$ and H. In space H the inner product can be defined as

(2.8) $$(f, g) = \int_K f'(t)g'(t)dt = (J^{-1}(f), J^{-1}(g)).$$

Since $L_2(K)$ is the Hilbert space it is thus also H.

In every finite-dimensional subspace H_0 of H, dim $H_0 = d$, the Gaussian measure can be defined as

(2.9) $$\exp - \pi |x|^2 dx$$

where $|.|$ is the Euclidean norm in space H_0 and dx the Lebesgue measure. These are dependent on the considered isomorphism $H_0 \to \mathbf{R}^d$. Schwartz proved ([Schwartz], p.364) that a Radon measure G exists in space X, the so-called **Wiener-Lévy measure**, whose projections onto the finite-dimensional subspaces of H are the Gaussian measures (2.9). The expected value of the one-dimensional density function (2.9) is zero and the variance is $1/2\pi$. Fixing the multiplier of the exponent to be π is not essential; it was done so that the multiplier would be independent of dimension d of the space.

2.3 Gaussian measure in finite-dimensional space. We can project function f and the measure onto the finite-dimensional spaces X_n (2.4). For this purpose we study the orthonormalized vectors of space $L_2(K)$. The maps

(2.10) $$e_{2ni} : K \to \mathbf{R}, t \mapsto \begin{cases} \sqrt{N/T}, & \text{when } t \in [t_{n,i-1}, t_{ni}] \\ 0, & \text{otherwise} \end{cases}$$

are such vectors because the integrals of their squares over the intervals $\left[t_{n,i-1}, t_{ni}\right]$ are $(N/T)(T/N) = 1$. With the integral mapping of J they are mapped onto space H to the functions

$$(2.11)\quad e_{\infty ni}(t) \equiv J(e_{2ni})(t) = \begin{cases} 0 & , t \in \left[0, t_{n,i-1}\right] \\ (t - t_{n,i-1}) \Big/ \sqrt{T/N} & , t \in \left[t_{n,i-1}, t_{ni}\right] \\ \sqrt{T/N} & , t \in \left[t_{ni}, T\right] \end{cases}$$

Let $f \in X$ and $f(t_{ni}) = f_{ni}$. According to formula (2.2)

$$(2.12)\qquad (p_n f)(t) = \frac{1}{\sqrt{T/N}} \sum_{i=1}^{N} (f_{ni} - f_{n,i-1}) e_{\infty ni}(t) .$$

We denote $t_n = t_{ni} - t_{ni-1}$. Due to formula (2.8) of the square of the norm and the orthonormality of the vectors (2.10) and the property $e'_{\infty ni} = e_{2ni}$ (a.e.) the formula

$$(2.13)\qquad \|p_n f\|^2 = \sum_{i=1}^{N} \left(f_{ni} - f_{n,i-1}\right)^2 / t_n .$$

is true. As mentioned after formula (2.9) the Lebesgue measure in space $\mathbf{R}^N$ depends on the isomorphism which is now between the spaces X_n and $\mathbf{R}^N$. Two isomorphisms are

$$F_1 : p_n f \mapsto \frac{1}{\sqrt{t_n}} (f_{n1}, f_{n2} - f_{n1}, \ldots, f_{nN} - f_{n,N-1})^T$$

$$F_2 : p_n f \mapsto (f_{n1}, f_{n2}, \ldots, f_{nN})^T$$

where ^{T} denotes the transpose. Between these isomorphisms is the connection $F_1(p_n f) = AF_2(p_n f)$ where $A = \left(\delta_{jk} - \delta_{j-1,k}\right) / \sqrt{t_n}$ and δ_{jk} is the Kronecker delta, $i,j = 1, \ldots, N$ and $\delta_{ok} = 0$. The Jacobian between the isomorphisms is $\det A = t_n^{-\frac{1}{2}N}$. Therefore the expected value of the

continuous and bounded function $F: X \to \mathbf{R}$ in Banach space X_n with the norm (2.13) is

$$\mathrm{E}G_n(F) = \tag{2.14}$$
$$t_n^{-\frac{1}{2}N} \int_{K^N} F(p_n f) \cdot \exp\left[-\pi \sum_{i=1}^{N} \left(f_{ni} - f_{n,i-1}\right)^2 / t_n\right] df_{n1} \ldots df_{nN}$$

PROPOSITION 2.4. Let $F \in C(C(K))$ be a continuous and bounded function of $C(K)$. Its expected value $\mathrm{E}G(F)$ with respect to the Gaussian Radon measure G can be calculated as the limit

$$\mathrm{E}G(F) = \lim_{n \to \infty} \mathrm{E}G_n(F) \tag{2.15}$$

The notations have been expressed in points 2.1 - 2.3.

Proof. The proposition is the consequence of proposition 1.7 after stating the following facts:

(i) The cylindrical measure of Banach space $C(K)$ can be defined in Hilbert space H with the help of projections p_n because there is a one-to-one connection in the Hilbert space between projections p_n and the canonical surjections $X \to X/X_n^{\perp}$ of the finite-dimensional cospaces where $X_n^{\perp}$ is the orthogonal complement of the closed subspace X_n.

(ii) The cylindrical measure is flatly concentrated in the compact subsets of space $C(K)$. (See remark after formula (1.2).)

(iii) Let $F \subset C(K)$ be compact. We shall prove that the sequence of projection (p_n) converges uniformly to the identity map in set F and before this that F is **equicontinuous**, i.e. for each $\varepsilon > 0$ thereby corresponding to a $\delta > 0$ such that $\sup_{f \in F} |f(t) - f(u)| < \varepsilon$ for all t and $u \in$ K for $|t - u| < \delta$:

From the compactness of the set F it follows that there exist functions $f_1, \ldots, f_m \in F$ such that

$$F \subseteq \bigcup_{i=1}^{m} B(f_i, \varepsilon / 3)$$

where $B(f_i, \varepsilon / 3) = \left\{ f \in X \middle| \|f - f_i\|_\infty < \varepsilon / 3 \right\}$. On the other hand, since the functions of set X are uniformly continuous, the numbers $\delta_i, i = 1, \ldots, m$ exist such that $|f_i(t) - f_i(u)| < \varepsilon / 3$ when $|t - u| < \delta_i$. We $\delta = \min\{\delta_i | i = 1, \ldots, m\}$ and let $f \in F$ and i in such an index that $f \in B(f_i, \varepsilon / 3)$. Then

$$|f(t) - f(u)| \le |f(t) - f_i(t)| + |f_i(t) - f_i(u)| + |f_i(u) + f(u)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

if $|t - u| < \delta$. //

Let n be so large that $|t_{ni} - t_{n,i-1}| < \delta, t \in K$, and i such an index that $t \in [t_{n,i-1}, t_{ni}]$. Then

$$|p_n f(t) - f(t)| \le |p_n f(t) - f(t_{n,i-1})| + |f(t_{n,i-1}) + f(t)| \le$$
$$\left| \left[f(t_{ni}) - f(t_{n,i-1}) \right] \frac{(t - t_{n,i-1})}{(t_{ni} - t_{n,i-1})} \right| + \varepsilon \le 2\varepsilon$$

independently of function $f \in F$ and the argument $t \in K$. Thus

$$\left\| p_n | F - id | F \right\|_{\infty, F} \equiv \sup_{f \in F} \sup_{t \in K} |p_n f(t) - f(t)| < 2\varepsilon$$

when n is large enough as mentioned above, $\|.\|_{\infty, F}$ is the norm of the uniform convergence in set F and id denotes the identity map.

2.5 GAMMA MEASURE. In space X_Ω (formula (2.3)) the expectation value of function $F(p_\Omega f)$ is

$$\text{(2.16)} \qquad \mathrm{E}\Gamma_\Omega(F) = \int_{K^n} F(p_\Omega f) \prod_{i=1}^{n} \frac{\beta_i^{\alpha_i} f_i^{\alpha_i - 1}}{\Gamma(\alpha_i)} \exp(-\beta_i f_i) df_1 \ldots df_n$$

where α_i and β_i are parameters depending on the expectation values and the variances of arguments f_i. This case has not been considered in the studies listed in the bibliography. The Gaussian measure gives the possibility of negative mortality and thus the need exists for obtaining a measure giving only positive values. The above measure cannot be extended to an infinite-dimensional space as canonically as the Gaussian measure with formula (2.9) and the extension problem is such a difficult one that it does not belong to the area of the present study even though it would solve the negativity problem.

3. PRESENT VALUE

We will examine the results of the previous chapter with the choices $f = \delta$ = the continuous rate of interest in interval K and $F = v$ is defined with the formula

$$v(\delta)(T) = \exp - \int_0^T \delta(t)dt. \tag{3.1}$$

The map $\delta \to \int_0^T \delta(t)dt$ is a Borel function ([1] p.140). Since $f \to \exp - f, f \in X$, v is also a bounded Borel function and therefore can be integrated. The argument T in notation $v(\delta)(T)$ only reminds us of the upper bound of the integration; T is fixed and not a variable.

We denote the initial value and the variance of the rate of interest in the dividing points (2.1) as follows:

---

[1] Ylinen, K.: Mitat ja integrointi topologisissa avaruuksissa (Measures and integration in topological spaces), lecture series [in Finnish], Turku University, Department of Mathematics (1981). Also see Bourbaki N., Élements de mathématique, Livre VI, Intégration, Hermann (1965)

$$\delta_0 = \delta(t_{n0}) \quad \text{and}$$

(3.2)

$$\sigma_{ni} = \sigma(\delta(t_{ni})).$$

The expectation value in space X_n is

$$\mathrm{E}_n(v) = \int \exp\left[-\sum_{i=1}^{N} \delta(t_{ni}) \cdot (t_{ni} - t_{n,i-1})\right] dP(\delta(t_{ni})), \tag{3.3}$$

where the exponent of the integrand has been calculated from formula (2.2) with the semiparallelogram formula leaving a small error for simplifying the formula by subtracting $-\frac{1}{2}\delta_{n0}$ and adding $\frac{1}{2}\delta_{nN}$. The error can also be corrected by defining projection p_n in a slightly different way in formula (2.2) and by proving that the correctness of proposition 2.4 does not depend on this projection.

3.1. NORMALLY DISTRIBUTED RATE OF INTEREST

We denote $t = t_{ni} - t_{n,i-1}$ and assume that

$$\delta(t_{ni}) \div \mathrm{N}(\delta(t_{n,i-1}), \sigma_{ni}\sqrt{t})(\delta) =$$

(3.4)

$$= \frac{1}{\sqrt{2\pi\sigma_{ni}^2 t}} \int_{-\infty}^{\delta} \exp\left[-\frac{(\delta(t_{ni}) - \delta(t_{n,i-1}))^2}{2\sigma_{ni}^2 t}\right] d\delta(t_{ni})$$

Although this is not the distribution N(0,1/π), proposition 2.4 can be applied because the results are independent of whether the variables $\delta(t_{ni})$ or $\dfrac{(\delta(t_{ni}) - \delta(t_{n,i-1}))^2}{2\pi\sigma_{ni}^2 t}$ are studied. Now

$$\mathrm{E}_n(v) =$$

$$(3.5) \qquad \int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_{ni}^2 t}} \exp\left[-\left(\delta(t_{ni})t + \frac{(\delta(t_{ni}) - \delta(t_{n,i-1}))^2}{2\sigma_{ni}^2 t}\right)\right] \prod_{k=1}^{N} d\delta(t_{nk})$$

We translate $z_i = \delta(t_{ni}) - \delta(t_{n,i-1})$ whose Jacobian is one. Therefore,

$$\mathrm{E}_n(v) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_{ni}^2 t}} \exp\left[-\left(t(\sum_{j=1}^{i} z_j + \delta_0) + \frac{z_j^2}{2\sigma_{ni}^2 t}\right)\right] \prod_{k=1}^{N} d\delta(t_{nk})$$

We can integrate with respect to the measure $d\delta(t_{nk})$ in the order $k = N, N\text{-}1, \ldots, 2, 1$. Then

$$\mathrm{E}_n(v) = \prod_{i=1}^{N} \frac{\sqrt{2\pi\sigma_{ni}^2 t}}{\sqrt{2\pi\sigma_{ni}^2 t}} e^{-\delta_0 T} \exp\left(t^2 2\sigma_N^2 \frac{t}{4}\right) \exp\left((2t)^2 2\sigma_{N-1}^2 \frac{t}{4}\right) \ldots \exp\left((Nt)^2 2\sigma_1^2 \frac{t}{4}\right) =$$

$$= \exp\left[-\delta_0 T + \frac{t^3}{2} \sum_{i=1}^{N} (\sigma_i (N+1-i))^2\right]$$

If it is assumed that $\sigma_i = \sigma$ for every index i, then

$$(3.6) \qquad \mathrm{E}_n(v) = \exp\left[-\delta_0 T + \frac{\sigma^2 T^3}{2N^3} \frac{N(N+1)(2N+1)}{6}\right]$$

and according to proposition 2.4 the expected value calculated with respect to the Wiener-Lévy measure is

$$(3.7) \qquad \mathrm{E}(v) = \lim_{n \to \infty} \mathrm{E}_n(v) = \exp\left[-\delta_0 T + \frac{\sigma^2 T^3}{6}\right]$$

In deterministic theory $\sigma = 0$ and $\mathrm{E}(v) = \exp[-\delta_0 T]$;thus $\exp\frac{\sigma^2 T^3}{6}$ is caused by the stochasticity of the rate of interest.

EXAMPLE 3.1. Let $T = 10$ and the variance in the whole interval to be $\sigma^2 T = 4\%^2$ (e.g. from the formula (3.4) it can be seen that the dimension of $\sigma^2 T$ equals that of square δ^2). The deviation from the present value calculated with deterministic theory is

$$\exp\left(\tfrac{1}{6}\cdot 16\cdot 10^{-4}\cdot 10^{2}\right) - 1 = 2{,}70\%.$$

EXAMPLE 3.2. Let $T = 30$ and the variance in the whole interval to be $\sigma^2 T = 6\%^2$. The deviation from the present value calculated with deterministic theory is

$$\exp\left(\tfrac{1}{6}\cdot 36\cdot 10^{-4}\cdot 9\cdot 10^{2}\right) - 1 = 71{,}6\%.$$

It can be seen from the examples that the increase of reserve compared with the reserve calculated with deterministic theory may be very large if one takes into the account the variance of interest. The more long-term the insurance the larger is the increase of reserve.

The insurance is riskier the larger its variance. The skewness and kurtosis are also relevant in assessing the distribution of the present value. For these reasons we will calculate the moments of the present value, which are simply

$$\mathrm{E}(v(m\delta)) = \exp\left[-m\delta_0 T + \frac{m^3\sigma^2 T^3}{6}\right] \tag{3.8}$$

Therefore the variance is

$$\mathrm{Var}(v) = \left(\exp\sigma^2 T^3 - 1\right)\cdot\exp\left[-2\delta_0 T + \frac{\sigma^2 T^3}{3}\right] \tag{3.9}$$

EXAMPLE 3.2 CONTINUES. With the assumptions of the example and with the additional assumption $\delta_0 = 6\%$

$$\mathrm{Var}(v) = 1.974.$$

3.2. GAMMA DISTRIBUTION

We assume that

$$\delta(t_{ni}) \div \frac{\beta_{ni}^{\alpha_{ni}} \delta(t_{ni})^{\alpha_{ni}-1}}{\Gamma(\alpha_{ni})} \exp(-\beta_{ni}\delta(t_{ni})) \tag{3.10}$$

where α_{ni} and β_{ni} are reals. It is known that

$$\mathrm{E}(\delta(t_{ni})) = \frac{\alpha_{ni}}{\beta_{ni}} \quad \text{and}$$

(3.11)

$$\sigma_{ni}^2 \equiv \mathrm{Var}(\delta(t_{ni})) = \frac{\alpha_{ni}}{\beta_{ni}^2} = \frac{\mathrm{E}(\delta(t_{ni}))}{\beta_{ni}} = \frac{\mathrm{E}(\delta(t_{ni}))^2}{\alpha_{ni}}$$

If the random variables $\delta(t_{ni})$ are mutually independent, the formula (3.3) can be factorized and thus it is enough to calculate one factor which is in the abbreviated form

$$\frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{-\infty}^{\infty} e^{-\delta t} \delta^{\alpha-1} e^{-\beta\delta} \, d\delta \tag{3.12}$$

This is the moment-generating function of Γ-distribution, equal to $\left(\frac{\beta}{\beta + t}\right)^{\alpha}$.

Thus,

$$\text{(3.13)} \qquad E_n(v) = \prod_{i=1}^{N} \left(\frac{\beta_{ni}}{\beta_{ni} + t_{ni} - t_{n,i-1}} \right)^{\alpha_{ni}}$$

The mutual independence proposition is not necessarily correct, but neither is it too far from the truth. Without this proposition, the problem would become too complex. In Britain, the rates of interest observed over about 50 successive years have almost no correlation, whereas rates of interest at two-year intervals have clear correlations [Waters].

If the $\{\delta(t_{ni}) | i = 1, \ldots N\}$ are distributed so that their expectation values and variances are identical in each subinterval $\{[t_{n,i-1}, t_{ni}] \, i = 1, \ldots N\}$, the index i can be omitted ($\alpha_{ni} = \alpha_n$ and $\beta_{ni} = \beta_n$). The parameters of interval K can also be denoted $\alpha \equiv \alpha_1$, $\beta \equiv \beta_1$, $\sigma = \frac{\alpha}{\beta^2}$ and $\delta_0 = \frac{\alpha}{\beta} = \frac{\alpha_n}{\beta_n}$. If we assume that the rates of interest follow the distribution [Dufresne]

$$\text{(3.14)} \qquad \delta(t_{ni}) \div \delta_0 + \sqrt{N}(t_{11} - \delta_0),$$

then

$$\text{(3.15)} \qquad \sigma_{ni}^2 = N\sigma^2$$

and

$$\text{(3.16)} \qquad \sqrt{\sum_{i=1}^{N} \sigma_{ni}^2} = N\sigma .$$

Thus, according to formula (3.11),

$$\alpha = \left(\frac{\delta_0}{\sigma} \right)^2$$

$$\text{(3.17)} \qquad \beta = \frac{\delta_0}{\sigma^2} = \frac{n \cdot \alpha_n / \beta_n}{\alpha_n / \beta_n^2} = n\beta_n$$

$$\alpha = \delta_0 \beta = \delta_0 n \beta_n = n\alpha_n$$

Further, we find that

$$\mathrm{E}_n(v) = \left(\frac{\beta_n}{\beta_n + T/n}\right)^{n\alpha_n} = \left(\frac{\beta}{\beta + T}\right)^{\alpha} = \left(1 + \frac{T}{\beta}\right)^{-\alpha} = \tag{3.18}$$

$$= \left(1 + \frac{\sigma^2 T}{\delta_0}\right)^{-\left(\frac{\delta_0}{\sigma}\right)^2}$$

We denote the inverse of the square of the relative deviation as $h \equiv \left(\frac{\delta_0}{\sigma}\right)^2$, in which case

$$\mathrm{E}_n(v) = \left(1 + \frac{\delta_0 T}{h}\right)^{-h} \tag{3.19}$$

This is in fact independent of index n. When the deviation approaches zero, i.e. when h approaches infinity, this model approaches the deterministic model and

$$\mathrm{E}_n(v) \to e^{-\delta_0 T} \tag{3.20}$$

which is the result of the deterministic model.

The present study cannot prove that formula (3.19) is the expectation value for any measure defined in space $C(K)$, but it is an expectation value in finite-dimensional spaces. Since the actual observations are made at a finite number of times, formula (3.19) is an approximation of the real situation. In quantum physics, the Feynman path integral quantization makes use of similar integrals calculated in finite-dimensional spaces, allowing the dimensions to increase without limit. The limit value produces the wave function, which can also be obtained with other methods. Furthermore, the path integral method produces new ways of approximating the physical system. However, it has not been shown that this limit, i.e. the wave function,

derives from any measure of space $C(K)$ [2], yet despite this the method does depict reality. A more general question arises from these observations: is it even necessary to prove that the limit of the expectation value calculated in finite-dimensional spaces is the expectation value of a particular measure in space $C(K)$?

With the Γ-measure, the expectation value (3.19) cannot be decomposed into factors depending on the mean on one hand and on the variance on the other, as with the normal distribution (3.7); thus, the deviation of this model from the present value of the deterministic model is also dependent on the mean δ_0. This deviation is

$$\left(1+\frac{\delta_0 T}{h}\right)^{-h} e^{\delta_0 T}. \tag{3.21}$$

EXAMPLE 3.3. Let δ = 10%, T = 30 and σ = 3%; then (3.21) returns the deviation from the deterministic formula as 1.411.

Moments can be calculated for other statistical quantities:

$$\mathrm{E}_n(e^{-m\delta t}) = \left(1+m\frac{\delta_0 T}{h}\right)^{-h}. \tag{3.21}$$

Thus, for example,

$$\mathrm{Var}(v) = \left(1+2\frac{\delta_0 T}{h}\right)^{-h} - \left(1+2\frac{\delta_0 T}{h}\right)^{-2h} = o(h^{-4}) \tag{3.22}$$

[2] Ylinen, K.: Topologiset vektoriavaruudet (Topological vector spaces), lecture series [in Finnish], Turku University, Department of Mathematics (1980). Also see Bourbaki N., Élements de mathématique, Livre V, Espaces vectoriels topologiques, Hermann (1966)

4. LIFE ANNUITY

In the previous chapter, discounting was examined taking only rates of interest into account, and is thus also applicable to other sectors of business in addition to life insurance. In life insurance, another essential factor in discounting is the mortality of the insured.

Let $\mu(x)$ be the mortality of the insured at age x. The discounting function is then

$$D(\delta, \mu(x))(t) = v(\delta)(t) \cdot l(\mu(x))(t), \tag{4.1}$$

where

$$u(\mu(x))(t) = \exp - \int_0^t \mu(x+y)dy = \tag{4.2}$$

= the probability that a person x years of age will still be alive at the age of $x+t$ years.

Let μ be a random function whose expectation value and standard deviation are known. Since μ and δ are mutually independent, D can be decomposed into a product of two functions dependent on these two factors respectively. One of these functions, v, was discussed in the previous chapter; thus we need only examine function u here.

We divide the interval $K = [0,T]$ into $2^n = N$ subintervals according to formula (2.1). Using the same observations as with formula (3.3), we now find that

$$\mathrm{E}_n(u) = \prod_{i=1}^{N} \int \exp\left[-\mu(x+t_{ni}) \cdot (t_{ni} - t_{n,i-1})\right] dP(\mu(x+t_{ni})), \tag{4.3}$$

where P is also introduced to indicate the distribution of mortality.

With rates of interest, distribution P was determined by a model according to which the transition $\delta(t_{ni}) - \delta(t_{n,i-1})$ follows a normal distribution. This model does not depict mortality in successive years, since

$\mu(x+t_{ni})$ is independent of mortality $\mu(x+t_{n,i-1})$ in a large population of insured persons. If a dependence does exist, it can be found in small populations and is such that if in any period a greater number of people have died than normal, it is probable that in the next period a smaller number of people will die than normal, since the age-group will then be smaller than normal. With mortality, the essential variable is not the transition but the deviation from normal value, i.e. $\mu(x+t_{ni})-\mu_{ni}$, where μ_{ni} is the expectation value of mortality $\mu(x+t_{ni})$.

Therefore, if $\mu(x+t_{ni}) \div N(\mu_{ni}, s_{ni})$, the expectation value (4.3) in X_n space is

$$\mathrm{E}_n(u) =$$

$$(4.4)\quad \int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi s_{ni}^2}} \exp\left[-\left(\mu(t_{ni})(t_{ni}-t_{n,i-1}) + \frac{(\mu(t_{ni})-\mu_{ni}))^2}{2s_{ni}^2}\right)\right] d\mu(t_{ni})$$

.

We examine one of these factors, discarding the indices for the sake of simplicity. Let the mean $\mu_{ni}=\mu_0$ and $t=t_{ni}-t_{n,i-1}$. We translate $\mu(t_{ni})=z+\mu_0$, making one of the factors

$$\frac{1}{\sqrt{2\pi s_{ni}^2}} e^{-\mu_0 t} \int_{-\infty}^{\infty} \exp\left[-\left(zt+\frac{z^2}{2s_{ni}^2}\right)\right] dz .$$

Thus,

$$(4.5)\qquad \mathrm{E}_n(u) = \exp - \sum_{i=1}^{N} \mu_{ni}(t_{ni}-t_{n,i-1}) \exp \frac{1}{2} \sum_{i=1}^{N} s_{ni}^2 (t_{ni}-t_{n,i-1})^2$$

In this formula, we cannot assume that the expectation μ_{ni} and the standard deviations s_{ni} are equal; $t_{ni}-t_{n,i-1}$ is also in practice one to five years, thus estimating the limit $n \to \infty$ is neither easy nor necessary.

The variance can be calculated on the basis of the same statistics as the mean and is

$$\mathrm{Var}(u) = \mathrm{E}(u^2) - \mathrm{E}(u)^2 =$$

(4.6)
$$\exp - \frac{2T}{n}\sum_{i=1}^{N} s_{ni} \times \left\{ \exp 2\left(\frac{T}{n}\right)^2 \sum_{i=1}^{N} s_{ni}^2 - \exp\left(\frac{T}{n}\right)^2 \sum_{i=1}^{N} s_{ni}^2 \right\}$$

because $t_{ni} - t_{n,i-1} = \frac{T}{n}$.

If, on the other hand,

(4.7)
$$\mu(t_{ni}) \div \Gamma(\alpha_{ni}, \beta_{ni}, \mu(t_{ni}))$$

(4.8)
$$\mathrm{E}_n(l) = \prod_{i=1}^{N} \left(\frac{\beta_{ni}}{\beta_{ni} + t_{ni} - t_{n,i-1}} \right)^{\alpha_{ni}}$$

derived in the same way as formula (3.13).

4.1 DISCOUNTING WITH THE FINNISH BASIS OF CALCULATION FOR MORTALITY

As a calculation example, we will examine the numerical value of the expectation value (4.5) when the data are taken from the basis of calculation used in the Finnish life insurance business [Koskinen]. In the statistics used, mortality is observed for age-groups spanning five years: 15-19, 20-24, ... , 65-69, i.e. $T/n = 5$. Observations exist for persons aged over 70 years, but are not included in the present discussion. The observations cover the 13 fiscal years 1972-1984. The mean age of each subinterval is used as the observation age, i.e. 17.5, 22.5, ... , 67.5.

The means and standard deviations obtained from the statistics are shown in the table below. The standard deviation was calculated with the formula $\sum_i [m_i - \mathrm{E}(\mu)]^2$, where m_i is the annual mortality in the statistics, and the decreasing trend in mortality has not been taken into account. It would be more correct to replace $\mathrm{E}(\mu)$ with a mean that takes the decreasing trend into account. Since the present author is not aware of how the decreasing

trend is calculated, and since it does not affect the conclusion presented below, the standard deviation given in the table will be used.

Age	$E(\mu)\ 10^{-3}$	$\sigma\ 10^{-3}$	$\Sigma\ 10^{6}$	$\mu\ 10^{-3}$
15-19	0.7477	0.1607	0.32	0.5382
20-24	0.9785	0.1786	0.40	0.5896
25-29	0.8892	0.2068	0.53	0.6865
30-34	0.9877	0.1157	0.17	0.8690
35-39	1.3823	0.2483	0.77	0.2128
40-44	2.5315	0.8523	9.80	1.8603
45-49	4.4085	1.1653	16.97	3.0801
50-54	7.1269	1.4426	26.02	5.3777
55-59	12.2123	2.0668	53.40	9.7057
60-64	21.6046	4.0700	207.06	17.8580
65-69	49.9085	9.8029	1201.22	33.2140

TABLE 4.1. The table shows the means ($E(\mu)$) and standard deviation (σ) of the annual mortalities in the statistics for the years 1972-1984; the relative deviation minus one from deterministic theory caused by the standard deviation (Σ); and the mortality used as the basis of calculation derived from the statistics without a safety loading based on the mean age of each subinterval (μ). The mortality used for the basis of calculation takes into account the decreasing trend shown by the statistics [Koskinen].

The relative deviation minus one values are

$$\Sigma_i = \exp\frac{1}{2}\sigma_i^2 5^2 - 1 \tag{4.9}$$

and are extremely small. The largest relative deviation from the deterministic theory is found in the subinterval 40-70, and is

$$\sum_{i=42,5}^{67,5}(1+\Sigma_i) = 1 + 0{,}15\% \ . \tag{4.10}$$

Thus, we may observe that the need for increasing the reserve due to the variance of mortality is negligible.

5. TERM ASSURANCE

The above formulae for discount factor D ((3.1), (3.3), (4.1) and (4.3)) constitute the single premium liability of pure endowment assurance. In this chapter, we will derive the formula for the single premium liability of the term assurance as an example and application. The single premium liability for other types of assurance can also be derived, but are outside the scope of the present study.

A policy is taken out at the age of x years, and the heirs of the insured person will be paid 1 unit in monetary compensation if the insured person dies before the age of w years. The single premium liability is then

$$A(K,x,w) = \mathrm{E}_t \mathrm{E}(v)\,, \tag{5.1}$$

where E_t is the taking expectation value of the moment of death t over the age interval $[x,w]$. Let $p(x,x+t)$ be the distribution function of the remaining life of an insured person aged x years. Then, according to formula (3.7),

$$A(K,x,w) = \int_0^{w-x} \exp\left(-\delta_0 t + \frac{1}{6}\sigma^2 t^3\right) dp(x,x+t)\,. \tag{5.2}$$

If $\delta(t_{ni}) \div \mathrm{N}(\delta(t_{n,i-1}), \sigma_{ni}\sqrt{t})(\delta)$ (formula (3.4)) and $\mu(x+t_{ni}) \div N(\mu_{ni}, s_{ni})$, then according to formula (4.5)

$$p(x,x+t) = 1 - \exp\left[-\frac{t}{N}\sum_{i=1}^{N}\mu_{ni} + \frac{1}{2}\left(\frac{t}{N}\right)^2 \sum_{i=1}^{N} s_{ni}^2\right] \equiv 1 - l(x,x+t). \tag{5.3}$$

Therefore

(5.4)

$$A(K,x,w) = -\int_0^{w-x} d\left\{\exp\left(-\delta_0 t + \frac{1}{6}\sigma^2 t^3\right)\cdot l(x,x+t)\right\}dt$$

$$-\int_0^{w-x}\left(-\delta_0 + \frac{1}{2}\sigma^2 t^2\right)\cdot \exp\left(-\delta_0 t + \frac{1}{6}\sigma^2 t^3\right)\cdot l(x,x+t)dt.$$

We define

(5.5)

$$D(x,x+t) = \exp\left(-\delta_0 t + \frac{1}{6}\sigma^2 t^3\right)\cdot l(x,x+t)$$

$$\equiv D_0(x,x+t)\exp\left[\frac{1}{6}\sigma^2 t^3 + \frac{1}{2}\left(\frac{t}{N}\right)^2 \sum_{i=1}^{N} s_{ni}^2\right]$$

where D_O is the discounting factor of deterministic theory, and life annuity at

(5.6)

$$a(x,w) = \int_0^{w-x} D(x,x+t)dt$$

In this case

(5.7)

$$A(K,x,w) = 1 - D(x,w) - \delta_0 a(x,w) + \frac{1}{2}\sigma^2 \int_0^{w-x} t^2 D(x,x+t)dt\,.$$

If, in this case, $\sigma = s = 0$, the remaining single premium liability is that of deterministic theory. The term $\delta_0 a(x,w)$ is the discounted expectation value of the interest rate.

6. PROLONGATION

In the deterministic rate of interest model, prolongation is easily obtained by changing the sign $-\delta_0 \to \delta_0$ and the prolongation factor is the inverse of the discounting factor. In stochastic theory, the prolongation factor is

(6.1) $$\mathrm{E}(v(-\delta))(T) = \mathrm{E}(v(\delta))(-T)$$

i.e. the prolongation factor is obtained by performing a time inversion on the discounting factor, $-T \to T$. (This is also possible in deterministic theory.) Then, according to formula (3.7), the prolongation factor of the rate of interest with normal distribution is ([Dufresne])

(6.2) $$\mathrm{E}(v(-\delta))(T) = \exp\left(\delta_0 T + \frac{1}{6}\sigma^2 T^3\right).$$

EXAMPLE 6.1. If, after a period of time T, the unit payment is prepared for with the rate of interest $\delta \div N(\delta_d, \sigma_d\sqrt{T})$,the reserve must be $\exp(-\delta_d T + \frac{1}{2}\sigma_d^2 T^2)$ .If the development of this reserve is prognosed with the rate of interest $\delta \div N(\delta_p, \sigma_p\sqrt{T})$, the expectation value for the increased reserve is

(6.3) $$\exp\left[(\delta_p - \delta_d)T + \frac{1}{6}(\sigma_p^2 + \sigma_d^2)T^3\right].$$

If the prognosed rate of interest $\delta_d = \delta_p$, the expectation value (of the reserve) will be greater than one.

If, on the other hand, $\delta \div \Gamma(\alpha, \beta)$, then according to formula (3.19)

(6.4) $$\mathrm{E}(v(-\delta))(T) = \left(1 - \frac{T}{\beta}\right)^{\alpha}.$$

As in Chapter 2 above, the discussion of discounting is based on the assumption that the size of the reserve made against future payments is calculated with the discounting factor in the Thiele differential equation, giving the result that the reserve in the stochastic model must be larger than that in the deterministic model by a factor $\exp(\frac{1}{6}\sigma^2 t^3)$. If, on the other hand, the size of the reserve is calculated based on the demand that the reserve

must attain a certain level with a certain rate of interest, then according to formulae (6.2) and (6.4), a reserve smaller than that in the deterministic model is sufficient. This is a strange result in that one would think that introducing stochastic theory to prolongation would increase the reserve demand: the yield of successive years is larger if the annual yields are equal (e.g. 10% and 10%) than if they vary so that the average remains the same (e.g. 0% and 20%). Thus, the reserve obtained with discounting corresponds to the preconception of the size of the reserve.

SUMMARY

The present study has examined the effect of random variation in rates of interest and mortality change and the formulae for discounting, prolongation and term insurance compared with those of deterministic theory. The significance of randomness in mortality is found to be negligible in the Finnish life insurance business. Rates of interest, on the other hand, are of great importance, particularly over longer periods of time. Introducing random factors into discounting increases the average discounting and prolongation factors in proportion to the size of the variance and the length of the discounting interval.

The expectation values for interest rate used here were derived by assuming that the interest rate forms a continuous function over the period examined and by integrating all possible interest rates in an infinite-dimensional Banach space relative to the Gaussian measure. The measure of the space represents a process in which the rate of interest is at some point normally distributed, with the rate of interest of the previous point in time as its average, when a finite number of points in time is examined. The number of points in time is allowed to increase without limit, producing the desired expectation value. The derivation of the formula for the expectation value is based on the results of functional analysis. Integration relative to the Γ-measure was also examined in finite-dimensional spaces. While it is impossible to prove that the limits of the formulae obtained in these spaces could be derived through any measures in infinite-dimensional spaces, a satisfactory interpretation for the results can be produced.

Mortality is a process that differs from rate of interest, and cannot be integrated relative to the Gaussian measure. The expectation values of mortality are actually easier to derive than those of rates of interest, but the formulae used are more complex.

APPENDIX

FUNCTIONAL ANALYSIS TERMINOLOGY

This appendix is a short summary of the concepts of functional analysis used in the present study and is intended as a reminder for readers already familiar with functional analysis.

A **Banach space** is a normed space that is complete with respect to the metrics generated by its norm.

A **Hilbert space** is an inner product space that is complete with respect to the metrics generated by its norm.

A topological space is **locally compact** if its every point has a compact neighbourhood. A normed space is locally compact if and only if it is finite-dimensional.

A topological space X is **totally regular** if for its every point x and the neighbourhood U of point x there exists a continuous function $f: X \to [0,1]$ such that $f(x) = 0$ and $f(X - U) = \{1\}$.

Subset C of a vector space is **convex** if for all elements $x, y \in C$ and $0 \le t \le 1, tx + (1-t)y \in C$.. A vector space is **locally convex** if its every point has a convex neighbourhood.

A Banach space is **reflexive** if it is algebra-isomorphic with its bidual. A bidual is a dual of a dual.

The **weak topology** $\sigma(X,X')$ of a Banach space X is the coarsest topology relative to which all its functionals are continuous.

Let A and B be sets. The function set { f: A → B } **separates the points** of set A if for every $(x, y) \in A^2$ a function f of the set exists so that $f(x) = f(y)$.

$(I,\leq)$ is a **directed set** if $\leq$ is a binary relation of set I so that it is (i) transitive, (ii) reflexive and (iii) for each element m and n of set I an element p exists so that $m \leq p$ and $n \leq p$.

If A is a set, I is a directed set and $f: I \to A$ a function f is called a **net** and the notation $(f_n)_{n\in I}$ is used.

ACKNOWLEDGEMENTS

I would like to express my thanks to Assistant Professor **Kari Ylinen** for his great help in exploring the theoretical background to Chapters 1 and 2.

This study was prompted by a training session called Stochastic Models in Finance and Life Insurance, organized by the Actuarial Association of Finland and the Rolf Nevanlinna Foundation in Hyvinkää in August 1991.

REFERENCES

Berberian, S.K.: Lectures in Functional Analysis and Operator Theory, Springer Verlag (1973)

Dufresne,D., Ins.Math. and Econ. 8, 187-201 (1989)

Koskinen,S., private comminication, Summary of the mortality of the insured persons of the Finnish life insurance companies in the years 1972-1982 (1991)

Ledoux, M. and Talagrand, M.: Probability in Banach Spaces, Springer Verlag (1991)

Norberg, R.:, Scand. Act. J. 1990, nos. 1-2, 14-33 (1990)

Rao, M.M.: Stochastic Processes and Integration, Sijthoff and Noordhoff (1979)

Rudin, W.: Real and Complex Analysis, McGraw-Hill (1974)

Schwartz, L.: Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures, Oxford Univ. Press (1973)

Skorohod, A.V.: Integration in Hilbert Space, Springer Verlag (1974)

Waters,H.R., private comminication with connection to his representation Sickness Insurance, Some Simple Stochastic Interest Rate Models and the Present Value of the Profit on a Portfolio of Sickness/Life Insurance Policies, training session called Stochastic Models in Finance and Life Insurance, Hyvinkää (1991)