ON BINOMIAL MODELS OF THE TERM STRUCTURE OF INTEREST RATES

WERNER HURLIMANN
WINTERTHUR-LEBEN
ALLGEMEINE MATHEMATIK
.PAULSTRASSE 9
CH-8401 WINTERTHUR - SWITZERLAND
TELEPHONE : 41-52-261 58 61
AND
SCHÖNHOLZWEG 24
CH-8409 WINTERTHUR - SWITZERLAND

ABSTRACT

A general analysis of arbitrage-free binomial models of the term structure of interest rates is given. Several of the previous models encountered in the financial economics literature are included in the discussion. The proposed approach allows the construction of models which satisfy the following property. At each discrete time the current market forecast of the one-period bond price implied by the initial term structure is equal to the expected value of the one-period bond prices with respect to the risk-neutral probabilities. As a special case one obtains a simple (degenerate) diatomic binomial model with only two different bond price values at each time and for each maturity of the bond. Several properties of this new class of binomial models are discussed. It is possible to get a term structure with no negative and no arbitrarily large interest rates. This class of models is useful in modelling the bond price uncertainty in the immediate future and near bond maturity. Furthermore the relation between conditional yields of the bonds and conditional variances of the bond prices is analyzed and lower and upper bounds for these quantities are derived.

Keywords : term structure of interest rates, bond price modelling, arbitrage-free pricing, binomial model, discrete time model

SUR LES MODELES BINOMES DE LA STRUCTURE A TERME DES TAUX D'INTERET

WERNER HURLIMANN
WINTERTHUR-LEBEN
ALLGEMEINE MATHEMATIK
PAULSTRASSE 9
CH-8401 WINTERTHUR - SWITZERLAND
TELEPHONE : 41-52-261 58 61
AND
SCHÖNHOLZWEG 24
CH-8409 WINTERTHUR - SWITZERLAND

RESUME

Une analyse générale des modèles binôme et libre d'arbitrage de la structure à terme des taux d'intérêt est présentée. Plusieurs des modèles précédents de la littérature économique financière sont inclus dans la discussion. L'approche proposée permet la construction de modèles qui satisfont la propriété suivante. A chaque instant l'estimation actuelle de marché du prix des obligations zéro-coupon impliquée par la structure à terme initiale est égale à l'espérance mathématique des prix zéro-coupon prise par rapport aux probabilités d'arbitrage. Comme cas particulier on obtient un modèle binôme diatomique (dégénéré) simple ayant seulement deux valeurs distinctes de prix zéro-coupon à chaque instant et pour chaque échéance. Plusieurs propriétés de la nouvelle classe de modèles binôme sont discutées. Il est possible d'obtenir une structure à terme dont les taux d'intérêts sont ni négatifs ni arbitrairement larges. Cette classe de modèles est utile pour la modélisation de l'incertitude des prix zéro-coupon dans le futur immédiat et à l'échéance. De plus la relation entre rendements conditionnels et variances conditionnelles des prix zéro-coupon est analysée, et des bornes inférieures et supérieures de ces quantités sont dérivées.

0. Introduction.

The most widely used common language in term structure modelling is that of continuous-time stochastic calculus. Arbitrage-free consistent models are usually constructed starting from Markov diffusions, that represent the current term structure in a finite-dimensional state space (see Duffie and Kan(1994)). The most general methodology, of which all existing arbitrage pricing models are special cases, has been developed by Heath, Jarrow and Morton(1992).

Practitioners, which are often not aware of the higher mathematics underlying this abstract structure, need simple models whose construction is based on first principles. Some classes of discrete-time models share this requirement, especially binomial models.

Among the single-factor models, the first arbitrage-free binomial model of the term structure, which makes full use of all information available from the current term structure observed in the market, was developed by Ho and Lee(1986). However this model suffered from several shortcomings as negative interest rates, arbitrarily large interest rates and constant volatility of interest rates. In the subsequent analysis Pedersen and al.(1989) and Morgan and Neave(1991/93) have constructed models without these drawbacks. However the general class of all arbitrage-free binomial models has not yet been determined. The scientific interest in a description of the full class lies in a better understanding of the feasible movements of interest rates in an arbitrage-free economic environment. This knowledge would enable to test the range of an exhaustive class of models and the empirical validity against alternative arbitrage-free models.

As mentioned by Pedersen and al.(1989), p.28, their class of models cannot satisfy the property, denoted (P) in the following, that the current market forecast of the one-period bond price implied by the initial term structure is equal to the expected value of the one-period bond prices with respect to the risk-neutral probabilities. The main purpose of the present paper consists to develop a method, which overcomes this problem, and to show that resulting feasible models may follow interest rate behaviour observed in the modern economic world. A more detailed outline of the study follows.

In Section 1 the framework proposed by Pedersen and al.(1989) is developed in its full generality. A product representation of the feasible bond prices is displayed in Proposition 1.1, and a simpler statement for the one-period bond prices is found under formula (1.10). The models obtained previously by Ho and Lee(1986), Pedersen and al.(1989), and Morgan and Neave(1991/93) are recovered as special cases. In Section 2 a sufficient condition, which implies the validity of property (P), is derived. It is obtained by induction from the particular case of a three-period bond. The main result is the product representation Theorem 2.1. The existence of arbitrage-free binomial models with the property (P) is settled in Proposition 3.1. The proposed special class of models do possess state independent risk-neutral probabilities, and contains even a very simple (degenerate) diatomic model with only two distinct bond price values at each time and for each maturity of the bond. It is also shown how a subclass, whose interest rates fluctuate within a reasonable range, can be obtained by imposing quite simple parameter constraints. The final Section 4 is devoted to elementary properties of the new alternative binomial model. In particular the problem of "heteroskedasticity" (proper English spelling in McCulloch(1985)!), that is the variability of the interest rate volatility, can be solved as illustrated in Example 4.1.

1. Arbitrage-free binomial models of the term structure of interest rates.

In the arbitrage-free theory of pricing interest rate contingent claims one postulates according to the path-breaking paper by Ho and Lee(1986) that the term structure of interest rates at the initial time is *exogeneously* given, that is the present values $P_t(0,0)$ of default-free discount bonds maturing for the value of one unit at time t are those currently observed in the market. It follows that for each stream of fixed and certain cash flows, the price obtained from a model coincides with the market price.

A considerable amount of additional references on the subsequent development of the subject is found in Sercu(1991), Heath, Jarrow and Morton(1992), Duffie(1992) and Vetzal(1994).

In this paper we follow the approach proposed by Pedersen, Shiu and Thorlacius(1989) to construct arbitrage-free binomial models of the term structure of interest rates. In a discrete-time and discrete-space setting the basic model assumptions are the *perfect* capital market assumptions :

(A1) The market is *frictionless*. This means that there are no taxes, no transaction costs, and all securities are perfectly divisible. Information is available to all investors simultaneously and each investor acts rationally.
(A2) The market clears at *discrete* points in time, which are separated in

regular intervals. For simplicity one uses each period as unit of time.
(A3) The bond market is *complete*. There exist default-free discount bonds for all maturities t=1,2,... .
(A4) At each time n, there are *finitely* many states of nature. The equilibrium price of the discount bond of maturity t at time n and in state i is denoted by $P_t(n,i)$. One requires that for all non-negative integers t,n,i:

$$0 \le P_t(n,i) \le 1, \quad P_0(n,i)=1, \quad \lim_{t \to \infty} P_t(n,i)=0.$$

To describe the evolution of the term structure of interest rates, one considers the following *binomial lattice*. At the initial time 0 one has, by convention, the state 0 and the bond price $P_t(0,0)$ for a discount bond of maturity t. At time 1 there are only two states of nature, denoted by 0 and 1, and the bond prices are $P_t(1,0)$, $P_t(1,1)$. Proceeding by induction one assumes that at time n-1 there are n states, denoted 0, 1, ..., n-1, and the bond prices are $P_t(n-1, i)$, i = 0, 1, ..., n-1. Passing from time n-1 to time n, each state i gives rise either to an *upward movement* to state i+1 or to a *horizontal movement* to state i. Thus at time n there are n+1 states i=0,1,...,n, and the bond prices are $P_t(n,i)$, i=0,1,...,n. This construction defines a binomial lattice labelled by vertices (n,i).

Following Pedersen et al.(1989), to define a *binomial lattice model*, one needs to prescribe at each vertex (m,k) of the binomial lattice :

$p(m,k)$: a risk-neutral probability
$P_{m+1}(m,k)$: a one-period bond price (at time m in state k)

Arbitrages are *eliminated* if and only if (cf. (3.2) in Pedersen et al.(1989))

$$\begin{aligned} &P_t(t,k)=1, \quad k=0,\dots,t, \\ (1.1) \qquad &P_t(m,k)=P_{m+1}(m,k) \cdot \{p(m,k) \cdot P_t(m+1,k+1)+[1-p(m,k)] \cdot P_t(m+1,k)\}, \\ &0 \le m \le t-1, \quad k=0,\dots,m. \end{aligned}$$

We use the following notation

$$(1.2) \qquad c(m,k) = P_{m+1}(m,k+1)/P_{m+1}(m,k), \quad k=0,\dots,m-1.$$

Earlier authors make one of the following simplifying assumptions :

(A5) $p(m,k)=p$, $c(m,k)=c$, for all m,k (Ho and Lee(1986))

(A6) $p(m,k)=p(m)$, $c(m,k)=c(m)$, for all m, k (Pedersen et al.(1989))

In the general case it follows that

$$(1.3) \qquad P_{m+1}(m,k) = P_{m+1}(m,i) \prod_{j=0}^{k-i-1} c(m,i+j), \quad k=i,i+1,\dots,m.$$

As suggested by Pedersen et al.(1989), section IV, the idea is to derive a formula for $P_t(n,i)$, $n < t$, in terms of the one-period bond prices $P_{j+1}(j,i)$, $j=n, n+1, \dots, t-1$.

Proposition 1.1. For all $n=0,1,\dots,t-1$, $k=i,i+1,\dots,n$, one has

$$(1.4) \qquad P_t(n,k) = G(n,t-1,k) \prod_{j=n}^{t-1} \Big[P_{j+1}(j,i) \prod_{\ell=0}^{k-i-1} c(j,j+\ell)\Big],$$

where the function G is defined recursively as follows :

$$(1.5) \qquad G(s,s,k) = 1, \quad k=0,\dots,s$$

$$(1.6) \qquad G(j,s,k) = [1-p(j,k)] \cdot G(j+1,s,k) + p(j,k) \cdot \Big[\prod_{\ell=j+1}^{s} c(\ell,k)\Big] \cdot G(j+1,s,k+1),$$

$$j < s, \quad k=0,1,\dots,j.$$

Proof. This is shown by backward induction. For $n=t-1$ this is (1.3) above. By induction assumption assume (1.4) true for $n=m+1,m+2,\dots,t-1$ and show it for $n=m$. Consider (1.1) for $k=i,i+1,\dots,m$. From (1.3) and (1.4) one has the relations

$$P_{m+1}(m,k) = P_{m+1}(m,i) \prod_{j=0}^{k-i-1} c(m,i+j)$$

$$P_t(m+1,k+1) = G(m+1,t-1,k+1) \prod_{j=m+1}^{t-1} \Big[P_{j+1}(j,i) \prod_{\ell=0}^{k-i} c(j,i+\ell)\Big]$$

$$P_t(m+1,k) = G(m+1,t-1,k) \prod_{j=m+1}^{t-1} \Big[P_{j+1}(j,i) \prod_{\ell=0}^{k-i-1} c(j,i+\ell)\Big]$$

Inserting these expressions in (1.1) one obtains

$$P_t(m,k) = \{ \prod_{j=m}^{t-1} [P_{j+1}(j,i) \prod_{\ell=0}^{k-i-1} c(j,i+\ell)]\} \cdot \{[1-p(m,k)] \cdot G(m+1,t-1,k) +$$

$$+ p(m,k) \cdot G(m+1,t-1,k+1) \prod_{j=m+1}^{t-1} c(j,k)\},$$

which is (1.4) for n=m by taking (1.6) into account.

The special case of the structure (1.4) in Pedersen et al.(1989) is recovered as follows. Assume that the recursively defined function

$$(1.7) \qquad g(j,s) = 1 - p(j,k) + p(j,k) \prod_{\ell=j+1}^{s} c(\ell,k), \ \ j < s, \ \ g(s,s)=1,$$

is *independent* of k, then one has the nested product representation

$$(1.8) \qquad G(j,s,k) = \prod_{\ell=j}^{s} g(\ell,s).$$

It follows that

$$(1.9) \qquad P_t(n,i) = \prod_{j=n}^{t-1} [g(j,t-1)P_{j+1}(j,i)],$$

which is relation (4.5) in Pedersen et al.(1989).

To express the one-period bond prices in terms of market forecasts of one-period bond prices, let us apply (1.4) twice to get

$$P_{t+1}(0,0)/P_t(0,0) = [G(0,t,0)/G(0,t-1,0)] \cdot P_{t+1}(t,0).$$

On the other side rewrite (1.3) as

$$P_{n+1}(n,i) = P_{n+1}(n,0) \prod_{j=0}^{i-1} c(n,j).$$

By comparison one obtains the formula

$$(1.10) \qquad r_n P_{n+1}(n,i) = [G(0,n-1,0)/G(0,n,0)] \prod_{j=0}^{i-1} c(n,j),$$

where $r_n = P_n(0,0)/P_{n+1}(0,0)$ is the market forecast of the one-period accumulated rate of interest at time n. In other words the r_n's are the one-period forward interest rate factors, which describe the initial term structure of interest rates (each factor equals one plus the forward rate of interest).

__Example 1.1__. In the case considered by Pedersen et al.(1989) one has $p(n,i)=p(n)$, $c(n,i)=c(n)$, hence (1.7) and (1.8) hold. One gets the formula (see (4.7) in Pedersen et al.(1989)) :

$$(1.11) \qquad r_n P_{n+1}(n,i) = c(n)^i \prod_{j=0}^{n-2} g(j,n-1) / \prod_{j=0}^{n-1} g(j,n)$$

__Example 1.2__. The model of Ho and Lee(1986) is obtained setting $p(n)=p$, $c(n)=c$. One shows that $g(j,s)=1 - p + p \cdot c^{s-j}$ and (1.11) reads

$$(1.12) \qquad r_n P_{n+1}(n,i) = c^i/(1 - p + p \cdot c^n).$$

To determine now the risk-neutral probabilities $p(n,i)$ consistent with the bond prices

$$P_{n+2}(0,0) = [\prod_{k=0}^{n+1} r_k]^{-1}, \quad n=0,1,2,...,$$

data which is known at time 0, consider again the formula (1.4) :

$$(1.13) \qquad P_{n+2}(0,0) = G(0,n+1,0) \prod_{j=0}^{n+1} P_{j+1}(j,0).$$

By comparison and using that $r_0 P_1(0,0)=1$ one gets immediately the conditions

$$(1.14) \qquad G(0,n+1,0) \prod_{j=1}^{n+1} r_j P_{j+1}(j,0) = 1, \quad n=0,1,2,...$$

__Example 1.3__. In the case $p(n,i)=p(n)$, $c(n,i)=c(n)$, one obtains using the multiplicative representation (1.8) the relation

$$(1.15) \qquad [\prod_{k=0}^{n} g(k,n+1)] \cdot [\prod_{k=1}^{n+1} r_k P_{k+1}(k,0)] = 1, \quad n=0,1,2,..., \text{ where}$$

$$(1.16) \qquad g(k,n+1) = 1 - p(k) + p(k) \cdot [\prod_{j=k+1}^{n+1} c(j)].$$

It is possible to solve for $p(n)$, $n=0,1,2,...$, using a backward induction procedure. One gets the recursive formula

$$(1.17)\quad p(n)=\{1-\prod_{k=0}^{n-1}g(k,n+1)\prod_{k=1}^{n+1}r_kP_{k+1}(k,0)\}/(c(n+1)-1)\prod_{k=0}^{n-1}g(k,n+1)\prod_{k=1}^{n+1}r_kP_{k+1}(k,0)$$

This formula determines the risk-neutral probabilities $p(n)$ provided $c(k)$ and $r_kP_{k+1}(k,0)$ are known, $k=1,2,...$

__Example 1.4__. Morgan and Neave(1991) proposes to model the TSIR by considering a discrete time multiplicative binomial model of the future forward interest rate factor, which moves stochastically around the initial TSIR described by r_n, $n=0,1,2,...$ If $R_n(i)$ is the random value the future forward factors may take at time n in state i, then they assume that

$$(1.18)\quad R_n(i) = u^{d(n,i)}\cdot r_n,\ i=0,...,n,\ \text{with } d(n,i) = (2\cdot i/n - 1)\cdot S(n),$$

where $S(n)$ may formally be any sequence. One has $P_{n+1}(n,i)=R_n(i)^{-1}$ and

$$(1.19)\quad c(k) = u^{-(2/k)S(k)},\ k=1,2,...,\ r_kP_{k+1}(k,0) = u^{S(k)},\ k=0,1,2,...,$$

One obtains

$$(1.20)\quad p(0) = u^{S(1)}/(1 + u^{S(1)}),\ p(1) = u^{S(1)+S(2)}/(1 + u^{S(1)+S(2)}),$$

$$(1.21)\quad p(n)=\{1-u^{S(1)+...+S(n+1)}\prod_{k=0}^{n-1}g(k,n+1)\}/\{(u^{-2S(n+1)/(n+1)}-1)u^{S(1)+...+S(n+1)}\prod_{k=0}^{n-1}g(k,n+1)\}$$

where

$$(1.22)\quad g(k,n+1)=1 - p(k) + p(k)\cdot u^{-2\{S(k+1)/(k+1)+...+S(n+1)/(n+1)\}},\ k=0,...,n-1.$$

To imply reversion to the mean of both the future forward factor and the underlying term structure, Morgan and Neave(1991) assume that the series $\Sigma\ S(k)$ is a convergent one, for example $S(k)=2^{-k}$. In Morgan and Neave(1993) the divergent case $S(k)=k$ is considered. In this quite simple situation one gets $c(k)=c=u^{-2}$, $p(k)=u^{2k+1}/(1+u^{2k+1})$, $k=0,1,2,...$

2. Sufficient conditions for a new class of binomial models.

In their general analysis of binomial models, Pedersen et al.(1989), p. 28, have considered the condition that the current market forecast $r_n^{-1}=P_{n+1}(0,0)/P_n(0,0)$ of the one-period bond price at time n is equal to the expected value of the bond prices $P_{n+1}(n,i)$, $0\leq i\leq n$, with respect to the risk-neutral probabilities p(n,i). This condition is expressed by the equation

$$(2.1)\qquad P_{n+1}(0,0)/P_n(0,0) = \sum_{i=0}^{n} P_{n+1}(n,i)\Pr(n,i), \quad n=1,2,...,$$

where Pr(n,i) is the probability to be in state (n,i). As shown by Pedersen et al.(1989), provided that p(n,i)=p(n) and c(n,i)=c(n) are independent of i, the condition (2.1) cannot be fulfilled if n=2, except for the trivial degenerate case c(n,i)=1. However if the restriction on c(n,i) is relaxed, it is possible to fulfill the desirable property (2.1). In particular we construct in Section 3 a binomial model for which p(n,i)=p(n) and (2.1) holds.

Using (1.10) the condition (2.1) is equivalent to the formula

$$(2.2)\qquad G(0,n,0) = G(0,n-1,0) \sum_{i=0}^{n} [\Pr(n,i) \prod_{j=0}^{i-1} c(n,j)], \quad n=1,2,...$$

To motivate the sufficient conditions given below in (2.4) under which the condition (2.2) is always fulfilled, let us first analyze the non-trivial case n=2 of a three-period bond, and then pass to the general case $n\geq 2$. Observe that for a two-period bond n=1 the condition (2.2) reads $G(0,1,0)=1-p(0)+p(0)c(1,0)$ and is always satisfied by definition of G.

2.1 Particular case n=2.

By the definition of G in Proposition 1.1 the left-hand side of (2.2) equals $G(0,2,0) = (1-p(0))G(1,2,0) + p(0)c(1,0)c(2,0)G(1,2,1)$. For the right-hand side one gets

$$G(0,1,0)\cdot[(1-p(0))(1-p(1,0))+\{(1-p(0))p(1,0)+(1-p(1,1))p(0)\}c(2,0)$$
$$+\ p(0)p(1,1)c(2,0)c(2,1)]$$
$$=$$
$$G(0,1,0)\cdot[(1-p(0))\{1+p(1,0)(c(2,0)-1)\}$$
$$+\ c(2,0)p(0)\{1+p(1,1)(c(2,1)-1)\}]$$
$$=$$
$$G(0,1,0)\cdot[(1-p(0))G(1,2,0) + c(2,0)p(0)G(1,2,1)].$$

It follows that (2.2) is equivalent with the condition

$$(1-p(0))G(1,2,0)\{G(0,1,0)-1\} + c(2,0)p(0)G(1,2,1)\{G(0,1,0)-c(1,0)\}=0.$$

But one has $G(0,1,0)-1=p(0)(c(1,0)-1)$, $G(0,1,0)-c(1,0)=-(1-p(0))(c(1,0)-1)$.

Therefore (2.2) is equivalent with the condition

$$p(0)(1-p(0))\{c(1,0)-1\}\{G(1,2,0) - c(2,0)G(1,2,1)\} = 0.$$

In the following let us *assume* that $p(0)(1-p(0))(c(1,0)-1) \neq 0$. Then (2.2) is equivalent with the condition $G(1,2,0) = c(2,0)G(1,2,1)$, which is the inductive step n=2 of the sufficient conditions (2.4) given in the general analysis of Subsection 2.2. Using that

$$G(1,2,0) = 1 + p(1,0)(c(2,0)-1),\ G(1,2,1) = 1 + p(1,1)(c(2,1)-1),$$

this condition is further equivalent to the relation

$$(1-p(1,0))(c(2,0)-1) = - p(1,1)c(2,0)(c(2,1)-1).$$

Let us illustrate its resolution for two special cases.

Case 1 : $c(2,0)=c(2,1)=c(2) \neq 1$
The derived condition implies that $1-p(1,0)=-p(1,1)c(2)<0$, which is impossible. In the special case $p(1,0)=p(1,1)=p(1)$ this fact has already been shown by Pedersen et al.(1989), p.28.

Case 2 : $c(2,0) \neq c(2,1)$
If one assumes that $p(1,0)=p(1,1)=p(1)$, then this risk-neutral probability is given by $1/p(1) = 1 - c(2,0) \cdot [c(2,1)-1]/[c(2,0)-1]$. If $p(1,0)\neq p(1,1)$, then one has the relation $1-p(1,0) = -p(1,1)c(2,0) \cdot [c(2,1)-1]/[c(2,0)-1]$.
In both subcases one has necessarily $[c(2,1)-1]/[c(2,0)-1] < 0$.

Note that the present relatively simple analysis is the basic inductive step for the construction of our new class of binomial models in Section 3 (see the relations (3.2) and (3.3)).

2.2. General analysis.

Under appropriate assumptions (2.2) simplifies considerably.

Lemma 2.1. For each n=1,2,..., one has the identity

$$\sum_{i=0}^{n} \left[\Pr(n,i) \prod_{j=0}^{i-1} c(n,j)\right] = \sum_{i=0}^{n-1} \left[\Pr(n-1,i) \cdot G(n-1,n,i) \prod_{j=0}^{i-1} c(n,j)\right] \tag{2.3}$$

Proof. Start with the right-hand side and reorder it using that $G(n-1,n,i) = 1 - p(n-1,i) + p(n-1,i) \cdot c(n,i)$. One has

$$\sum_{i=0}^{n-1} \Pr(n-1,i)[1-p(n-1,i)] \prod_{j=0}^{i-1} c(n,j) + \sum_{i=0}^{n-1} \Pr(n-1,i)p(n-1,i) \prod_{j=0}^{i} c(n,j)$$

$$= \Pr(n-1,0)[1-p(n-1,0)] + \sum_{i=1}^{n-1} \Pr(n-1,i)[1-p(n-1,i)] \prod_{j=0}^{i-1} c(n,j)$$

$$+ \sum_{i=0}^{n-2} \Pr(n-1,i)p(n-1,i) \prod_{j=0}^{i} c(n,j) + \Pr(n-1,n-1)p(n-1,n-1) \prod_{j=0}^{n-1} c(n,j)$$

Change the index of summation in the second sum to get

$$\Pr(n,0) + \sum_{i=1}^{n-1} \{\Pr(n-1,i)[1-p(n-1,i)] + \Pr(n-1,i-1)p(n-1,i-1)\} \prod_{j=0}^{i-1} c(n,j)$$

$$+ \Pr(n,n) \prod_{j=0}^{n-1} c(n,j)$$

Observing that the state probabilities satisfy the recursive relations

$$\Pr(n-1,i)[1-p(n-1,i)] + \Pr(n-1,i-1)p(n-1,i-1) = \Pr(n,i), \quad i=1,...,n-1,$$

the result follows immediately.

In the subsequent discussion *assume* the following relations hold :

(2.4) $G(n-1,n,i-1) = c(n,i-1) \cdot G(n-1,n,i)$, $i=1,...,n-1$, or equivalently

(2.5) $$G(n-1,n,0) = G(n-1,n,i) \prod_{j=0}^{i-1} c(n,j), \quad i=1,\dots,n-1.$$

Inserted in (2.2) using (2.3) the equation to solve is

(2.6) $$G(0,n,0) = G(0,n-1,0) \cdot G(n-1,n,0).$$

We show below that under the assumption (2.4) the equation (2.6) is always fulfilled. For this one needs the following intermediate result.

__Lemma 2.2__. For each $n=2,3,\dots$ assume that

(2.7) $$G(k,k+1,i-1) = c(k+1,i-1) \cdot G(k,k+1,i), \quad i=1,\dots,k, \quad k=0,1,\dots,n.$$

Then one has

(2.8) $$G(i,n+1,k-1) = G(i,n+1,k) \prod_{j=i+1}^{n+1} c(j,k-1), \quad i=1,\dots,n-1, \; k=1,\dots,i.$$

__Proof__. This is shown using induction on the indices n and i. As induction step let n=2. In this case (2.8) reads

(2.9) $$G(1,3,0) = c(2,0) \cdot c(3,0) \cdot G(1,3,1).$$

Using the recursive definition (1.6) of G the formula (2.9) is equivalent to

(2.10) $$\begin{aligned}&(1-p(1,0)) \cdot G(2,3,0) + p(1,0)c(2,0)c(3,0) \cdot G(2,3,1)\\ &=\\ &c(2,0)c(3,0) \cdot \{(1-p(1,1)) \cdot G(2,3,1)\\ &+ p(1,1)c(2,1)c(3,1) \cdot G(2,3,2)\}.\end{aligned}$$

From the assumption (2.7) one has

(2.11) $$G(2,3,0) = c(3,0) \cdot G(2,3,1),$$
(2.12) $$G(2,3,1) = c(3,1) \cdot G(2,3,2).$$

Inserting in (2.10) using again (1.6) one gets

$$(2.13)\qquad G(1,2,0)\cdot G(2,3,0) = c(2,0)\cdot G(1,2,1)\cdot c(3,0)\cdot G(2,3,1).$$

Using (2.11) this is equivalent to

$$(2.14)\qquad G(1,2,0) = c(2,0)\cdot G(1,2,1),$$

which is satisfied by the assumption (2.7). Hence (2.8) is shown for n=2. By induction assumption assume now the result is true for the indices 2,3,...,n-1, and show it for the index n. This is shown by backward induction on the index i.

<u>Step 1</u> : i=n-1. Using (1.6) the formula (2.8) for i=n-1 is equivalent to

$$\begin{aligned}&(1-p(n-1,k-1))\cdot G(n,n+1,k-1) + p(n-1,k-1)c(n,k-1)c(n+1,k-1)\cdot G(n,n+1,k)\\ &=\\ (2.15)\qquad &c(n,k-1)c(n+1,k-1)\cdot\{(1-p(n-1,k))\cdot G(n,n+1,k)\\ &+ p(n1,k)c(n,k)c(n+1,k)\cdot G(n,n+1,k+1)\}\end{aligned}$$

Using assumption (2.7) and (1.6) this is equivalent to

$$\begin{aligned}(2.16)\qquad &G(n-1,n,k-1)\cdot G(n,n+1,k-1)\\ &=\\ &c(n,k-1)\cdot G(n-1,n,k)\cdot c(n+1,k-1)\cdot G(n,n+1,k).\end{aligned}$$

Using again (2.7) this is equivalent to

$$(2.17)\qquad G(n-1,n,k) = c(n,k-1)\cdot G(n-1,n,k).$$

But this relation is fulfilled by assumption (2.7).

<u>Step 2</u>. Assume the relation (2.8) is valid for the indices i=r+1,...,n-1, k=1,...,i, and show it for the index i=r. This is similar to step 1. One has

$$G(r,n+1,k-1) = G(r,n+1,k)\prod_{j=r+1}^{n+1} c(j,k-1)$$

$$\Leftrightarrow\quad (1-p(r,k-1))\cdot G(r+1,n+1,k-1) + p(r,k-1)\cdot G(r+1,n+1,k)\prod_{j=r+1}^{n+1} c(j,k-1)$$

$$=\Big[\prod_{j=r+1}^{n+1} c(j,k-1)\Big]\cdot\Big\{[1-p(r,k)]\cdot G(r+1,n+1,k) + p(r,k)\cdot G(r+1,n+1,k)\prod_{j=r+1}^{n+1} c(j,k)\Big\}$$

$$\Leftrightarrow \quad G(r,r+1,k-1) \cdot G(r+1,n+1,k-1)$$
$$= c(r+1,k-1) \cdot G(r,r+1,k) \cdot G(r+1,n+1,k) \prod_{j=r+2}^{n+1} c(j,k-1)$$
$$\Leftrightarrow \quad G(r,r+1,k-1) = c(r+1,k-1) \cdot G(r,r+1,k).$$

Since the last relation is fulfilled by assumption (2.7) the result follows.

Let us show that under the validity of (2.4) for all $n=2,3,\ldots$, the relation (2.6) is always satisfied. This follows immediately from the special case $k=i=0$ of the following main result.

Theorem 2.1. (*product representation* of the function G) For all $n=2,3,\ldots$ assume that the relations (2.4) are satisfied. Then one has

$$G(k,n,i) = \prod_{j=k}^{n-1} G(j,j+1,i), \quad k=0,\ldots,n-2, \quad i=0,\ldots,k. \tag{2.18}$$

Proof. For $n=2$, $k=i=0$, one has

$$G(0,2,0) = (1-p(0,0)) \cdot G(1,2,0) + p(0,0)c(1,0)c(2,0) \cdot G(1,2,1).$$

Since $c(2,0)G(1,2,1)=G(1,2,0)$ by (2.4) one gets

$$G(0,2,0) = [1 - p(0,0) + p(0,0)c(1,0)] \cdot G(1,2,0) = G(0,1,0)G(1,2,0).$$

Let now $n \geq 3$. By assumption the formulas (2.7) are fulfilled. From Lemma 2.2 one has

$$G(s,n,i) = G(s,n,i+1) \prod_{j=s+1}^{n} c(j,i), \quad s=1,\ldots,n-2, \; i=1,\ldots,s-1. \tag{2.19}$$

We show by induction on r that

$$G(k,n,i) = G(r,n,i) \prod_{j=k}^{r-1} G(j,j+1,i), \quad r=k+1,\ldots,n. \tag{2.20}$$

First of all one has

$$G(k,n,i) = [1-p(k,i)] \cdot G(k+1,n,i) + p(k,i) \cdot G(k+1,n,i+1) \prod_{j=k+1}^{n} c(j,i).$$

Using (2.19) one gets

$$G(k,n,i)=[1-p(k,i)+p(k,i)c(k+1,i)]\cdot G(k+1,n,i)=G(k,k+1,i)\cdot G(k+1,n,i),$$

which is (2.20) for $r=k+1$. By induction assumption assume now that (2.20) is true for r and show it for $r+1$. One has

$$G(r,n,i) = [1-p(r,i)]\cdot G(r+1,n,i) + p(r,i)\cdot G(r+1,n,i+1) \prod_{j=r+1}^{n} c(j,i).$$

Using (2.19) it follows that

$$G(r,n,i) = [1-p(r,i) + p(r,i)c(r+1,i)]\cdot G(r+1,n,i) = G(r,r+1,i)\cdot G(r+1,n,i).$$

From the induction assumption one obtains now

$$G(k,n,i) = G(r+1,n,i) \prod_{j=k}^{r} G(j,j+1,i).$$

Hence (2.20) is shown and (2.18) follows setting $r=n$.

3. Alternative binomial models.

We have shown that binomial models satisfying the property (2.1) are obtained if one solves the relations (2.4). Let us construct such models for which additionally the risk-neutral probabilities $p(n,i)=p(n)$ are independent of the state i. Written out the relations (2.4) are equal to the system of equations

(3.1) $$1 + p(n-1)\cdot(c(n,i-1)-1) = c(n,i-1)\cdot\{1 + p(n-1)(c(n,i)-1)\},$$
$n=2,3,\dots,\ i=1,\dots,n-1.$

Let us search for *nondegenerate* binomial models, that is assume that $c(n,i)\neq 1$, $i=0,\dots,n-1$. Then one has

(3.2) $$1/p(n-1)=1 - c(n,i-1)\cdot[c(n,i) - 1]/[c(n,i-1) - 1],\ i=1,\dots,n-1.$$

To obtain a value $0 < p(n-1) < 1$, let us choose

(3.3) $$u(n,0) = -[c(n,1) - 1]/[c(n,0) - 1] > 0.$$

Defining further

(3.4) $$u(n,k) = -[c(n,k+1) - 1]/[c(n,k) - 1],\ k=1,\dots,n-2,$$

one sees that (3.2) is fulfilled if

$$c(n,k)u(n,k) = c(n,k-1)u(n,k-1), \quad k=1,\dots,n-2. \tag{3.5}$$

Combining (3.4) and (3.5) one sees that $c(n,k)$, $k=2,\dots,n-1$, can be evaluated by recursion. The obtained result is summarized as follows.

Proposition 3.1. Given is a nondegenerate binomial model such that $p(n,i)=p(n)$, $i=0,\dots,n$. Then for all $n=2,3,\dots$ there exists $c(n,0)\neq 1$ and $u(n,0)>0$ such that

$$p(n-1) = 1/(1 + c(n,0)u(n,0)), \tag{3.6}$$

$$c(n,i) = 1 - (c(n,i-1) - 1) \cdot u(n,i-1), \quad i=1,\dots,n-1, \tag{3.7}$$

$$u(n,i) = u(n,i-1) \cdot c(n,i-1)/c(n,i), \quad i=1,\dots,n-1. \tag{3.8}$$

Having shown the existence of binomial models satisfying (2.1), let us derive some consequences following from such a model. Given is a nondegenerate binomial model as in the preceding result. From the product representation (2.18), (2.5) and (1.10) one has the formula

$$r_n P_{n+1}(n,i) = 1/G(n-1,n,i) = \Big[\prod_{j=0}^{i-1} c(n,j)\Big]/G(n-1,n,0), \tag{3.9}$$
$$n=1,2,\dots, \quad i=0,\dots,n.$$

In particular for $i=0$ one has

$$r_n P_{n+1}(n,0) = 1/(1 - p(n-1) + p(n-1) \cdot c(n,0)), \quad n=1,2,\dots \tag{3.10}$$

Using (3.6) this implies that

$$r_n P_{n+1}(n,0) = (1 + c(n,0)u(n,0))/c(n,0) \cdot (1 + u(n,0)), \quad n=2,3,\dots \tag{3.11}$$

From (1.4) and (3.8) one gets the *bond price formula* :

$$P_t(n,i) = G(n,t-1,i)\prod_{j=n}^{t-1} P_{j+1}(j,i) = P_{n+1}(n,i)\Big[\prod_{j=n+1}^{t-1} G(j-1,j,i)P_{j+1}(j,i)\Big].$$

Using (3.9) one obtains herewith

$$(3.12)\qquad P_t(n,i) = P_{n+1}(n,i)\Big[\prod_{j=n+1}^{t-1} r_j\Big]^{-1}.$$

Applying (3.9) and (3.11) this expression transfoms to

$$(3.13)\quad P_t(n,i) = \Big[\prod_{j=n}^{t-1} r_j\Big]^{-1} \cdot \prod_{j=0}^{i-1} c(n,j) \cdot \{1 + c(n,0)u(n,0)\}/\{c(n,0)[1 + u(n,0)]\},$$

$n=2,3,\ldots$

This formula means that the bond prices of arbitrary maturity date at time n depend only on the one-period bond prices at time n, that is on $c(n,0)$, $u(n,0)$, and on the market forecast of future interest rates.

Using the relations (3.7) and (3.8) it is possible to express $P_t(n,i)$ explicitely as a function of the parameters $c_n:=c(n,0)$ and $u_n:=u(n,0)$. Setting further $x_n:=u_nc_n$ one observes that

$$(3.14)\quad c(n,m)c(n,m-1) = (1 - x_n)c(n,m-1) + x_n,\quad n=2,3,\ldots,\ m=1,\ldots,n.$$

This follows from the following calculation :

$$\begin{aligned} c(n,m)c(n,m-1) &= \{1 - u(n,m-1)[c(n,m-1) - 1]\}c(n,m-1) \\ &= \{1 - u(n,0)c(n,0) + u(n,m-1)\}c(n,m-1) \\ &= \{1 - x_n\}c(n,m-1) + x_n. \end{aligned}$$

Lemma 3.1. For all $n=2,3,\ldots$, $m=1,\ldots,n$, one has

$$\begin{aligned} (3.15)\qquad \prod_{j=0}^{m} c(n,j) &= (c_n + x_n) \sum_{j=0}^{m-1} (-1)^j x_n^{\,j} + (-1)^m x_n^{\,m} \\ &= (c_n + x_n) \cdot (1 + (-1)^{m-1}x_n^{\,m})/(1 + x_n) + (-1)^m c_n x_n^{\,m}. \end{aligned}$$

Proof. This is shown by induction on m. If m=1 one has from (3.14) $c(n,1)c(n,0)=(c_n+x_n)-c_nx_n$, which is (3.15) for m=1. Assume the result true for $m=1,2,\ldots,r$, $r \geq 1$, and show it for $m=r+1$. From (3.14) one has $c(n,r+1)c(n,r) = (1 - x_n)c(n,r) + x_n$. Multiplying this relation with

$$\prod_{j=0}^{r-1} c(n,j)$$

and using twice the induction assumption the result follows by straightforward algebra. The details of the verification are left to the reader.

With the formula (3.15) the bond prices at time $n=2,3,\ldots$ read

$$P_t(n,0) = \left[\prod_{j=n}^{t-1} r_j\right]^{-1} \cdot (1 + x_n)/(c_n + x_n),$$

$$(3.16) \qquad P_t(n,1) = \left[\prod_{j=n}^{t-1} r_j\right]^{-1} \cdot c_n(1 + x_n)/(c_n + x_n),$$

$$P_t(n,i) = \left[\prod_{j=n}^{t-1} r_j\right]^{-1} \cdot \{1+(-1)^{i-2}x_n^{\,i-1}+(-1)^{i-1}c_n x_n^{\,i-1}(1+x_n)/(c_n+x_n)\}.$$

Simplifying further one sees that

$$(3.17) \qquad P_t(n,i) = \left[\prod_{j=n}^{t-1} r_j\right]^{-1} \cdot [1 + (-1)^i \alpha_n x_n^{\,i}],$$

$$n=2,3,\ldots,\ i=0,\ldots,n, \text{ with } \alpha_n = (1 - c_n)/(c_n + x_n).$$

The class of alternative binomial models contains a *degenerate diatomic model* of the term structure of interest rates obtained by setting $x_n=1$, or $p(n-1)=\frac{1}{2}$, for $n=2,3,\ldots$ From (3.15) one derives immediately that

$$(3.18) \qquad c(n,2k)=c_n,\ c(n,2k+1)=1/c_n,\ k=0,1,2,\ldots$$

(degenerate binomial lattice)

Furthermore for $n=2,3,\ldots$ one has

$$P_t(n,i) = \left[\prod_{j=n}^{t-1} r_j\right]^{-1} \cdot 2/(1 + c_n), \text{ if } i \text{ is even},$$

$$(3.19) \qquad P_t(n,i) = \left[\prod_{j=n}^{t-1} r_j\right]^{-1} \cdot 2c_n/(1 + c_n), \text{ if } i \text{ is odd}.$$

If one puts further $p(0)=\frac{1}{2}$, then (3.19) is also valid for $n=1$. In this special case the bond prices of a given maturity date t take only two different values at time n.

It is natural to put further restrictions on the bond prices. One usually requires at least the following fundamental properties :

(3.20) no negative interest rates, and no arbitrarily large interest rates.

As pointed out by Pedersen et al.(1989), the first binomial model by Ho and Lee(1986) does not fulfill this condition. On the other side the multiplicative binomial model by Morgan and Neave(1991) satisfies this property. However in the literature on binomial models it is not clarified if there exist models satisfying additionally the condition (2.1).

To satisfy (3.20) with formula (3.17), one has to choose the model numbers α_n and x_n such that

(3.21) $$r_n/r_{max} \le r_n P_{n+1}(n,i)=1 + (-1)^i \alpha_n x_n^i \le r_n, \quad i=0,\dots,n.$$

With the upper bound one avoids negative interest rates and with the lower bound one avoids accumulated interest rates higher than r_{max}. Assume that $\varepsilon_n=r_n/r_{max}<1$. It is easy to check that (3.21) is fulfilled if one assumes that

(3.22) $$0 < \alpha_n \le i_n=r_n - 1, \quad 0 < x_n \le \min\{(1-\varepsilon_n)/i_n, 1\}.$$

If one assumes further that $r_{max} \ge (1+i_n)/(1-i_n)$ for all n, then (3.22) simplifies to the quite simple parameter constraints $0<\alpha_n\le i_n$, $0<x_n\le 1$, retained in the next Section.

4. Some elementary properties of the new alternative binomial model.

Consider the constructed alternative binomial bond price model

(4.1) $$r_n P_{n+1}(n,i) = 1 + (-1)^i \alpha_n x_n^i, \quad n=1,2,\dots, \; i=0,\dots,n,$$
$$\text{with } 0 \le \alpha_n \le i_n, \quad 0 \le x_n \le 1.$$

First of all note that the mean and the variance of the random variable $P_{n+1}(n, \cdot)$, representing the one-year bond prices at time n, can be obtained in an elementary way. One needs the following identity.

Lemma 4.1. Assume a binomial lattice such that p(n-1)=p(n-1,i), n=1,2,..., i=0,...,n-1, is independent of i. Then one has

(4.2) $$\sum_{i=0}^{n} (-1)^i [(1-p(n-1))/p(n-1)]^i \Pr(n,i) = 0, \quad n=1,2,\dots$$

Proof. This is shown immediately using the recursive relations for the transition probabilities Pr(n,i), namely

(4.3) $$\begin{aligned} &\Pr(0,0)=1, \\ &\Pr(n,0)=\Pr(n-1,0)(1-p(n-1)), \quad \Pr(n,n)=\Pr(n-1,n-1)p(n-1), \\ &\Pr(n,i)=\Pr(n-1,i)(1-p(n-1)) + \Pr(n-1,i-1)p(n-1), \; i=1,\dots,n-1. \end{aligned}$$

Proposition 4.1. For the binomial bond price model (4.1) one has

$$E[r_n P_{n+1}(n,\cdot)] = 1, \tag{4.4}$$

$$\mathrm{Var}[r_n P_{n+1}(n,\cdot)] = \alpha_n^2[(1-p(n-1))/p(n-1)] \cdot \prod_{k=0}^{n-2} \{1 - p(k) + p(k)[(1-p(n-1))/p(n-1)]^2\}, \quad n=1,2,\ldots . \tag{4.5}$$

Let $p(k)=p$ for $k=0,\ldots,n-1$, then one has the simpler formula

$$\mathrm{Var}[r_n P_{n+1}(n,\cdot)] = \alpha_n^2[(1-p)/p]^2. \tag{4.6}$$

Proof. Observing that $x_n=(1-p(n-1))/p(n-1)$ by (3.6), the property (4.4), which is nothing else than the condition (2.1), follows directly from the identity (4.2). Using the same result one obtains the formula

$$\mathrm{Var}[r_n P_{n+1}(n,\cdot)] = \alpha_n^2 \sum_{i=0}^{n} [(1-p(n-1))/p(n-1)]^{2i} \cdot \Pr(n,i). \tag{4.7}$$

Let us denote by S_n the sum on the right-hand side of (4.7). One has

$$S_1 = \Pr(1,0) + [(1-p(0))/p(0)]^2 \Pr(1,1) = (1-p(0))/p(0). \tag{4.8}$$

Assume $n\geq 1$ and let us compute S_{n+1} using the recursion (4.3). One has

$$S_{n+1} = \sum_{i=0}^{n+1} [(1-p(n))/p(n)]^{2i}\Pr(n+1,i)$$

$$= \Pr(n,0)(1-p(n)) + \sum_{i=1}^{n} [(1-p(n))/p(n)]^{2i}(1-p(n))\Pr(n,i)$$

$$+ \sum_{i=0}^{n-1} [(1-p(n))/p(n)]^{2i+1}(1-p(n))\Pr(n,i) + [(1-p(n))/p(n)]^{2n+1}(1-p(n))\Pr(n,n)$$

$$= \Pr(n,0)(1-p(n))[1 + (1-p(n))/p(n)]$$

$$+ \sum_{i=1}^{n} \Pr(n,i)[(1-p(n))/p(n)]^{2i}(1-p(n))[1 + (1-p(n))/p(n)]$$

$$= (1-p(n))/p(n) \sum_{i=0}^{n} [(1-p(n))/p(n)]^{2i}\Pr(n,i).$$

Proceeding similarly using (4.3) and induction one gets

$$S_{n+1} = [(1-p(n))/p(n)] - \{1 - p(n-1) + p(n-1)[(1-p(n))/p(n)]^2\} \sum_{i=0}^{n-1} [(1-p(n))/p(n)]^{2i} \Pr(n-1,i).$$

$$= [(1-p(n))/p(n)] \prod_{k=0}^{n-2} \{1 - p(k) + p(k)[(1-p(n))/p(n)]^2\},$$

which shows (4.5). The formula (4.6) follows immediately from (4.5).

Example 4.1. This result is useful in modelling the bond price uncertainty. A first desirable property of bond pricing can already been fulfilled for the simple case (4.6). As pointed out by Ho and Lee(1986), p. 1016, the bond price uncertainty should be small at the two extreme points, namely for the time horizon in the immediate future and near bond maturity. If the implied term structure of interest rates is such that r_n first increases and then decreases this desirable property (with the variance as measure of uncertainty) can be fulfilled setting $\alpha_n = i_n$ in (4.6).

As next step we analyze how conditional variances and yields of the bonds are related and in which bounds they can actually move. Relatively simple bounds are obtained in the conditional case. The one-year yield of the bond at time n is described by the random variable R_n :

(4.9) $$R_n(i) = 1/P_{n+1}(n,i) = r_n/(1 + (-1)^i \alpha_n x_n^{\ i}), \quad i=0,\dots,n.$$

Consider the conditional means and variances of the accumulated yield at time n+1

(4.10) $$\mu_{n+1,i} = E[R_{n+1}/R_n=R_n(i)], \quad i=0,\dots,n,$$
(4.11) $$(\sigma^R_{n+1,i})^2 = Var[R_{n+1}/R_n=R_n(i)], \quad i=0,\dots,n,$$

and the conditional variances of the bond prices

(4.12) $$(\sigma^P_{n+1,i})^2 = Var[P_{n+2}(n+1, \cdot)/P_{n+1}(n,i)], \quad i=0,\dots,n+1.$$

In the following let us use the abbreviations

(4.13) $$\alpha=\alpha_{n+1}, \quad x=x_{n+1}, \quad c_{n+1,i}=r_{n+1} \cdot \sigma^P_{n+1,i}.$$

Since p(n)=1/(1+x) one gets from (4.9) that

(4.14) $\mu_{n+1,i}/r_{n+1} = \{x/[1 + (-1)^i\alpha x^i] + 1/[1 + (-1)^{i+1}\alpha x^{i+1}]\}/\{1 + x\}$
$= \{1 + (-1)^i\alpha x^i(1-x)\}/\{[1 + (-1)^i\alpha x^i][1 + (-1)^{i+1}\alpha x^{i+1}]\}$

A further calculation shows that

$$(4.15)\qquad (\mu_{n+1,i} - r_{n+1})/\mu_{n+1,i} = \alpha^2 x^{2i+1}/[1 + (-1)^i\alpha x^i(1-x)].$$

On the other side one gets similarly

$$(4.16)\qquad c^2_{n+1,i} = \{[1+(-1)^i\alpha x^i]^2 x + [1+(-1)^{i+1}\alpha x^{i+1}]^2\}/\{1+x\} - 1 = \alpha^2 x^{2i+1}$$

In this simple conditional situation let us now apply the mean/variance criterion of portfolio theory. For a given fixed conditional variance $c^2_{n+1,i}=c^2$ try to maximize the conditional accumulated yield $\mu_{n+1,i}$ of the bond. Equivalently one can maximize the relative conditional excess yield given by (4.15). Making the change of variable $\beta=\alpha x^i$ one has to solve the optimization problem

(4.17) $(\mu_{n+1,i} - r_{n+1})/\mu_{n+1,i} = c^2/[1 + (-1)^i(\beta-c^2/\beta)] = \max!$
under the constraints $x\beta^2=c^2$, $0 < \beta \le i_{n+1}$, $0 < x \le 1$.

For this it suffices to minimize the function $f(\beta) = 1 + (-1)^i(\beta - c^2/\beta)$. Since the derivative $f'(\beta) = 1 + (-1)^i[1 + (c/\beta)]^2$ is either positive or negative, two cases must be considered.

Case 1 : i even

Since $f'(\beta)=2+(c/\beta)^2>0$ the function $f(\beta)$ is monotone increasing. From the constraint $x=(c/\beta)^2\le 1$ one deduces that $c^2\le\beta^2$. It follows that $f(\beta)=\min!$ if $\beta=c$, where $0<c\le i_{n+1}$. Hence $x=1$, thus $p(n)=½$, which leads to the degenerate diatomic binomial model.

Case 2 : i odd

Since $f'(\beta)=-(c/\beta)^2<0$ the function $f(\beta)$ is monotone decreasing. Hence $f(\beta)$ takes its minimum when β is the maximum possible value. Since $x=(c/\beta)^2$ one gets $\alpha=\beta/x^i=\beta^{2i+1}/c^{2i}$. From $\alpha\le i_{n+1}$ one sees that β is maximum if $\beta^{2i+1}=c^{2i}i_{n+1}$. It follows that $x=(c/i_{n+1})^{2/(2i+1)}$, where $0<c\le i_{n+1}$.

To summarize the discussion we have shown the following result.

Proposition 4.2. At time n+1 the conditional accumulated yields of the bonds and the conditional variances of the bond prices satisfy the relations:

Case 1 : i even
One has the inequalities

(4.18) $1 + i_{n+1} \le \mu_{n+1,j} \le 1/(1 - i_{n+1})$,
(4.19) $(\mu_{n+1,j} - r_{n+1})/\mu_{n+1,j} \le (r_{n+1}\sigma^P_{n+1,j})^2 \le i^2_{n+1}$.

Furthermore one has equalities in (4.19) only in the degenerate case $p(n)=½$, $x=1$, $\alpha=\{(\mu_{n+1,j}-r_{n+1})/\mu_{n+1,j}\}^{½}$, and then one has

(4.20) $(\sigma^R_{n+1,j})^2 = \mu_{n+1,j}(\mu_{n+1,j} - r_{n+1})$

Case 2 : i odd
Define the function of two variables

(4.21) $F(\alpha,x) = \alpha^2 x^{2i+1}/[1 + (-1)^i \alpha x^i(1-x)]$.

Let $0 \le c_{n+1} \le i_{n+1}$ and set $\omega^2_{n+1}=F(i_{n+1},(c/i_{n+1})^{2/(2i+1)})$. Assume that $0 \le \alpha \le i_{n+1}$, $0 \le x \le 1$, are such that $F(\alpha,x) \le \omega^2_{n+1}$ and $\alpha^2 x^{2i+1} \le c^2$. Then one has the inequalities

(4.22) $0 \le \mu_{n+1,j} = r_{n+1}/[1-F(\alpha,x)] \le r_{n+1}/[1-\omega^2_{n+1}] \le 1/(1-i_{n+1})$,
(4.23) $0 \le (r_{n+1}\sigma^P_{n+1,j})^2 = \alpha^2 x^{2i+1} \le c^2 \le i^2_{n+1}$.

Furthermore one has equalities of the middle terms only if

(4.24) $\alpha = i_{n+1}, \; x = (c/i_{n+1})^{2/(2i+1)}$.

Equalities of the three upper terms hold only in the degenerate case $p(n)=½$, $x=1$, $\alpha=c=i_{n+1}$, and then one has

(4.25) $(\sigma^R_{n+1,j})^2 = [i_n/(1-i_{n+1})]^2$.

Proof. Up to the computation of the conditional variance $(\sigma^R_{n+1,j})^2$, this follows from the discussion preceding Proposition 4.2. The formula (4.20) is verified as follows :

$$\begin{aligned}(\sigma^R_{n+1,j})^2 &= ½(r_{n+1})^2[1/(1+\alpha)^2 + 1/(1-\alpha)^2] - (\mu_{n+1,j})^2 \\ &= (r_{n+1})^2(1+\alpha^2)/(1-\alpha^2)^2 - (\mu_{n+1,j})^2 \\ &= (\mu_{n+1,j})^2(1+\alpha^2) - (\mu_{n+1,j})^2 = (\alpha\mu_{n+1,j})^2 = \mu_{n+1,j}(\mu_{n+1,j}-r_{n+1}).\end{aligned}$$

The formula (4.25) follows from (4.20) setting $\mu_{n+1,j}=1/(1-i_{n+1})$.

5. Conclusion.

Further steps towards the determination of the complete class of all exogenously given arbitrage-free binomial models of the term structure of interest rates have been undertaken. A new parametric binomial model with the following properties has been constructed :

- property (P) (condition (2.1), or (2.2), or (4.4))
- no negative interest rates and no arbitrarily high interest rates
- flexible volatility structure of interest rates : long term volatility smaller than short term one, high volatility by high interest rates, etc.
- flexible yield curve by given initial term structure : variety of realistic shapes including flat, upward and downward sloped and humped shaped
- simple parametrization (end of Section 3)
- simple understanding from first principles (follows from (4.4))

A remaining drawback is the fact that short and long term interest rates are perfectly correlated while in the real world they are not, but often mover together. Since this property is shared by all single-factor models, only the more general framework of multi-factor models can resolve this inconvenience (consult e.g. Duffie and Kan(1994)).

Finally let us mention another more methodological problem, but quite important from the point of view of the interaction between academics and practitioners, which concerns the proper passage from discrete-time to continuous-time models and vice-versa. Early in Theoretical Physics one has shown that a discrete random walk converges to Brownian motion and that an Ehrenfest urn model converges to the Ornstein-Uhlenbeck process, as sketched among others by Kac(1947). To which continuous-model does the new parametric binomial model converge ? Does the most important limit theorem in the theory of stochastic processes by Donsker(1951) and Prohorov(1956) (see also Glynn(1990)) help solve this question ?

References.

Donsker, M.D. (1951). An invariance principle for certain probability limit theorems. Memoirs of the American Mathematical Society 6.

Duffie, D. (1992). Dynamic Asset Pricing Theory. Princeton Univ. Press.

Duffie, D., Kan, R. (1994). Multi-factor term structure models. Phil. Transactions Royal Society of London, Series A, 577-86.

Glynn, P.W. (1990). Diffusion approximations. In Heyman, D.P., Sobel, M.J. (eds.) Stochastic Models, Handbooks in Operations Research and Management Science, vol. 2, chap. 4, 145-98.

Heath, D., Jarrow, R., Morton, A. (1992). Bond pricing and the term structure of interest rates : a new methodology for contingent claims valuation. Econometrica 60, 77-105.

Ho, T.S.Y., Lee, S. (1986). Term structure movements and pricing interest rate contingent claims. Journal of Finance 41, 1011-1029.

Kac, M. (1947). Random walk and the theory of Brownian motion. American Mathematical Monthly 54, 369-91.

McCulloch, J.H. (1985). On heteroscedasticity. Econometrica 53, 483.

Morgan, I.G., Neave, E. (1991). A mean reverting process for pricing treasury bills and futures contracts. Proceedings of the 2nd AFIR Colloquium, Brighton, vol. 1, 237-266.

Morgan, I.G., Neave, E. (1993). A discrete time model for pricing treasury bills, forward, and futures contracts. ASTIN Bulletin 23, 3-22.

Pedersen, H.W., Shiu, E.W.S., Thorlacius, A.E. (1989). Arbitrage-free pricing of interest-rate contingent claims. Transactions of the Society of Actuaries 41, 231-65. Discussion, 267-79.

Prohorov, Y. (1956). Convergence of random processes and limit theorems in probability theory. Theory of Prob. and Applications 1, 157-214.

Sercu, P. (1991). Bond options and bond portfolio insurance. Insurance : Mathematics and Economics 10, 203-30.

Vetzal, K.R. (1994). A survey of stochastic continuous time models of the term structure of interest rates. Insurance : Mathematics and Economics 14, 139-61.