A LINEAR PROGRAMMING APPROACH TO MARKET EQUILIBRIUM WITH AN APPLICATION TO FINANCIAL MARKETS

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Abstract :

This paper attemps to give a rigorous definition of a market-rate-process that should be used to determine the market value of financial operations.

In the first section,the paradox inherent in the Black & Scholes -and related results is highlighted and analysed. Next,to solve this paradox, a linear programming approach to financial markets is developped.

First,a linear programming model for an ordinary cash market (one time period) is presented.

This model is then generalised to include time (several time periods) and uncertainty. The model yields a market-interest-rate-process as a set of optimal dual variable values.This process permits a consistent evaluation of financial operations.

Finally,the model is applied to a business environment to produce as a set of duals a cost-of-capital-process wich should be used to evaluate assets,liabilities and projects.

The paper concludes with some suggestions for future research.

Keywords :

Linear programming, dual variables, stochastic process, cost of capital, risk aversion coefficient.

1. INTRODUCTION

The research that eventually produced this paper was initially motivated by the desire for a rational justification for the value of the discount rate in present-value-calculations. Later on it became gradually clear that a justification of the entire valuation procedure of financial operations was within reach of the analytical methods applied to the original problem. In fact, the approach used turned out to provide rules for the calculation of market-equilibrium-valuations given the detailed preferences of all the market participants. This full market approach also resolves a paradox inherent in the commonly used formulae for option pricing.

In the first section this option pricing paradox is highlighted and analyzed. Next, a linear programming model for an ordinary cash market (one time period) is presented. This model is then generalized to include the future (several time periods) and uncertainty. This model yields a market-interest-rate-process as a set of dual variables. Using this process, consistent market-values of financial operations can be obtained. Finally, the model is applied to a business environment to produce - as a set of dual variables - a cost-of-capital-process which can be used to evaluate financial operations or investment processes within an existent business. To conclude some directions of future research are proposed.

2. THE OPTION PRICING PARADOX

Consider the Black and Scholes option pricing formula:

$$\begin{aligned} &\textit{put} \quad d=\frac{1}{\sigma\sqrt{T}}\cdot\left(\ln\frac{X_0}{K}+\left(r+\frac{1}{2}\sigma^2\right)\cdot T\right) \\ &\textit{then} \quad P_{BS}=X_0\cdot N(d)-K\cdot N(d-\sigma\sqrt{T}) \end{aligned} \tag{1}$$

where X_0 is the initial (stock) price, K the exercice price, N the standard normal distribution function, r the riskless interest rate, σ the (stock) price volatility and T the expiration time.

Assuming a drift μ in the (stock) price process and applying straightforward probability reasoning, the option price would have been:

$$P_{HB}=X_0\cdot N\left(d-\frac{\mu}{\sigma\sqrt{T}}\right)-K\cdot N\left(d-\frac{\mu}{\sigma\sqrt{T}}-\sigma\sqrt{T}\right) \tag{2}$$

where the subscript HB stands for Huygens-Bernoulli. The drift does not influence the Black and Scholes formula.

Clearly $P_{BS}\neq P_{HB}$ if $\mu\neq 0$. But, in the limit for σ approaching 0, the risk dissapears while still $P_{BS}\neq P_{HB}$. However,P_{HB} corresponds to the properly discounted value of the certain cash-flow at time T whereas P_{BS} entirely ignores the certain price rise of the stock. If the volatility σ is very small but not zero a large deviation from the Huygens-Bernoulli option price is unacceptable unless extreme risk aversion is assumed. The paradox can be avoided if the stock price evolution is included in the model: a drift in the expected stock price evolution is then impossible under equilibrium conditions and this immediately implies $P_{BS}=P_{HB}$ .This example illustrates that a segmented approach to the evaluation of financial operations is doomed and can at best produce suboptimal solutions. Similar paradoxes can be shown to exist in all the presently existing models of option pricing based on the no-arbitrage principle. An integrated market approach seems unavoidable if correct globally consistent values for different financial operations are needed and this paper intends to show how this can be achieved.

3. EQUILIBRIUM IN A CASH MARKET

The classical economic description of (cash) market equilibrium is well known. The equilibrium price of a commodity is the price which equalizes supply and demand . It is found graphically by looking for the intersection point of the supply- and demand curves. In this section the market equilibrium problem for a cash market will be reformulated using linear programming in order to justify the use of this technique for this type of problem. By discarding the complications of several time periods and of uncertainty, this section is mainly focussed on the specification of the objective function. Some other advantages of this approach will become apparent after the description of the technique.

First the information contained in the supply- and demand curves has to be rendered in a format suitable for a linear programming treatment. The supply curve is obtained by aggregating the supply conditions of individual potential suppliers. These conditions are composed of elementary conditions of the type: "from a price on, supply up to a quantity Q". The "return" function associated with this intention is: $(\pi\text{-}p)^{+}Q$, where π is the market price. The "return is in this case the income that exceeds the desired minimum income πQ . The shape of this "return" curve is identical to the return curve of a call option. Analogously, a buyer has a return curve in the shape of a put return curve corresponding to the "return" function $(p\text{-}\pi)^{+}Q$. Both curves are of the general form $(p\text{-}\pi\sigma(p))^{+}Q$ where $\sigma(\ )$ is the sign function, if we adopt the convention that supply prices are negative; This permits - after aggregation - to capture the information of both supply- and

$$\sum_{i} (p_i - \pi\cdot\sigma(p_i))^{+}Q_i \tag{3}$$

demand curves in the unique market "return" function:

It should be emphasized that this "return" is an expected return (whence the quotes) which only gets realized if every market participant can realize his intentions. Actually only the minimal point π^* of the market "return" curve is feasible because at every other π-value there is no equality of supply and demand. Indeed, the supply terms in the "return" function (3) have positive slopewhile the demand terms slope downwards with increasing π. Equality between supply and demand implies a horizontal tangent to the "return" curve which only exists at π^* (or at another minimal point if there is more than one)

To complete the picture something has to be said about the quantities traded because they do not appear explicitly in (3). This will be done by reformulating the minization of (3) as a linear programming problem. The

$$\max_x\{\sum_i (p_i-\pi\cdot\sigma(p_i))\cdot x_i : 0\leq x_i\leq Q_i\} \tag{4}$$

function (3) can be rewritten as:
So the problem of minimizing (3) can be written as:

$$\begin{array}{lll} \min\limits_{\pi}: & \max\limits_{x}\sum\limits_i (p_i-\pi\cdot\sigma(p_i))\cdot x_i & \\ \textit{subject to}: & 0\leq x_i\leq Q_i & \textit{for all } i \end{array} \tag{5}$$

This last problem is the Lagrangean dual of the following linear programming problem:

$$\begin{array}{lll} \max\limits_{x}: & \sum\limits_i p_i\cdot x_i & \\ \textit{subject to}: & \sum\limits_i \sigma(p_i)\cdot x_i=0 & \\ & 0\leq x_i\leq Q_i & \textit{for all } i \end{array} \tag{6}$$

This problem is known in operations research as a continuous knapsack problem and its solution is trivial. Its solution however yields the equilibrium market price together with all the amounts traded and this proves that a cash market can be modeled by a linear program - at least for the purpose of equilibrium computations.

As a result of this exercise we now know the nature of the objective function to be used in a linear programming model of a market. It is the total market satisfaction measured in an equilibrium situation as the difference between the total amount which the buyers would have been prepared to pay and the total amount the suppliers would have deemed acceptable. Note that the solution of (6) does not specify how this total market satisfaction is to be distributed among the market players participating in the equilibrium transactions. It seems plausible that at least this objective function nature will not be affected by the various possible generalizations of the linear model. Adding more commodities only changes the number of constraints - one per additional commodity plus one per additional variable. The same holds true for multiple time periods; the monetary flows in the first period determine the opportunities for transactions: as long as some combination of elementary transactions yields an additional positive satisfaction, the requirements of market transparency and efficiency dictate that it should be executed. Section 5. will focus more closely on these considerations.

4. UNCERTAINTY

This section tackles almost exclusively the modeling of uncertainty in order to make the solution to this problem stand out more visibly. Therefore we will consider a market of gamblers at a snail race. The example of a snail race is used because the author is totally ignorant about actual betting procedures at horse races or cock fights while anyone can safely pretend to be an autority on snail races. Assume a number of snails contesting the grand national one meter race and assume that only betting on winners is admitted but to keep the analysis interesting distributed bets are possible. Assume furthermore that there are enough addicted gamblers present to allow an interesting analysis. For each snail, that snail winning, constitutes a relevant elementary outcome in this gambling market. A relevant elementary outcome will be denoted by ω and the set of all the relevant elementary outcomes will be denoted by Ω. Obviously, irrelevant events such as Bill Clinton getting a second term as president or the Fidji islands acquiring a nuclear bomb need not worry us in this context. To avoid any arguments about the possibility of a market developing assume that there are a number of bookmakers who - being nice well-behaving middle-class family men - do not usually gamble themselves. Each gambler is prepared to stake a certain amount of money on one or more outcomes against particular odds each. The bookmakers accept bets on their conditions and balance their books as well as they can in this way helping to reach the equilibrium. Finally assume that they are particularly good at doing this (they are all actuaries) so that they end up retaining a minimal residual risk. Nevertheless the bookmakers remain in the model because they collect the stakes and pay out the gains (and pocket the difference).

Because the gain of any gambler is the loss of one or more others independent of the outcome we can write down a continuity constraint (money does not vanish) for every outcome. Although the outcome is stochastic, the constraints are certain. The constraints are different in general because both amounts paid and beneficiaries depend on the actual outcome. The objective function is again a total market satisfaction based on the intended amounts at stake. Again we are not interested to know who thinks he (or she) got a deal i.e. in what way the market satisfaction is distributed. The exposition in the previous section now suggests the flollowing model:

$$\begin{aligned} &\max_{x}: \quad \sum_{i} p_i \cdot x_i \\ &\textit{subject to:} \quad \sum_{i} g_{i\,\omega} \cdot \sigma(p_i) \cdot x_i = 0 \qquad \textit{for all } \omega \in \Omega \\ &\qquad\qquad 0 \le x_i \le L_i \qquad \textit{for all } i \end{aligned} \tag{7}$$

Each i in (7) corresponds to one side of a single (in general distributed) bet which can be wagered up to the limit L_i , costs p_i per unit and yields a gain $g_{i\,\omega}$ should ω happen. The p_i are negative for the bookmakers and positive for the gamblers.The function $\sigma(\ )$ is again the sign function. The remainder of the interpretation should be obvious.

The Lagrangean dual of (7) is:

$$\begin{aligned} &\min_{\pi}: \quad \max_{x} \sum_{i} \sum_{\omega} (p_i - \pi_\omega \cdot g_{i\,\omega} \cdot \sigma(p_i)) \cdot x_i \\ &\textit{subject to:} \qquad 0 \le x_i \le L_i \qquad \textit{for all } i \end{aligned} \tag{8}$$

and of course the market satisfaction function is:

$$\max_{x} \{ \sum_{i} (p_i - \sigma(p_i) \sum_{\omega} \pi_\omega \cdot g_{i\,\omega}) \cdot x_i : \ 0 \le x_i \le L_i \ \textit{for all } i \} \tag{9}$$

$$\sum_{i} (p_i - \sigma(p_i) \sum_{\omega} \pi_\omega \cdot g_{i\,\omega})^{+} L_i \tag{10}$$

or:

The ordinary dual problem of (7) is:

$$\min_{\pi}: \quad \sum_i L_i \cdot v_i$$

$$\textit{subject to:} \quad v_i + \sigma(p_i) \sum_{\omega} \pi_{\omega} \cdot g_{i\omega} \geq p_i \quad \textit{for all } i \qquad (11)$$

$$v_i \geq 0 \quad \textit{for all } i$$

Because - as a consequence of the absense of irrelevant outcomes - elementary events can be constructed by set-theoretic operations on the distributed events corresponding to bets available in the market:

$\forall \omega \, \exists \, i : [\, g_{i\,\omega} = 1$ and $[\, \forall \, j \neq i : g_{j\,\omega} = 0 \,]\,]$

i.e. without loss of generality all the elementary events are in the market.

Hence: $\forall \omega \, \exists \, i : \; v_i + \sigma(\, p_i \,)\, \pi_{\omega} \geq p_i$

By the complementary slackness theorem from linear programming, if $x_i > 0$ then $v_i = 0$ hence $\sigma(\, p_i \,)\, \pi_{\omega} \geq p_i$. If on the contrary $x_i = 0$ then $v_i > 0$ and $v_i + \sigma(\, p_i \,)\, \pi_{\omega} = p_i$ hence $\sigma(\, p_i \,)\, \pi_{\omega} \leq p_i$. It follows that for any ω there exists an elementary bet with e.g. gambler side i and bookmaker side j or conversely so that: $|\, p_i \,| \leq \pi_{\omega} \leq |\, p_j \,|$ whence $\forall \omega : \pi_{\omega} \geq 0$. From (11) it can be inferred that a gambler wagers a bet i if : $\sum_{\omega} \pi_{\omega} \cdot g_{i\,\omega} \geq p_i$ whereas a bookmaker agrees if: $\sum_{\omega} \pi_{\omega} \cdot g_{i\,\omega} \leq p_i$

It is tempting to interpret the π_{ω} as probabilities but in general $\sum_{\omega} \pi_{\omega} \neq 1$. An interpretation in terms of probabilities is perhaps still possible if the concept of risk preference (or aversion) is introduced. This issue will be more detailedly discussed in section 6. For now it is important to understand that in equilibrium conditions decisions should be made by comparing the price of a bet with its value which is a sort of modified expected value of the possible gains.

5. FINANCIAL MARKETS

As things stand, it will not be necessary to discuss financial markets under certainty. A linear programming model can be written down immediately. Market satisfaction is measured by the intended cash flows in the first period because the action is concentrated there: only the present is available to express your (present) preferences (abouy present *and* future) . Again, we do not care about the actual ditribution of gains: it is a byproduct of the bargaining process which leads to market equilibrium but it does not figure in the model. Uncertainty is handled as in the previous section but the distinction between gamblers and bookmakers - which was introduced for the sake of the exposition - dissapears. Every market participant and even every individual operation available on the market is treated on an equal footing. Bearing all this in mind the model for a financial market under uncertainty is:

$$\max_x: \quad \sum_i C_{i00} \cdot x_i$$

$$\text{subject to:} \quad \sum_i C_{it\omega} \cdot x_i = 0 \quad \text{for all } \omega \in \Omega, t \in H \qquad (12)$$

$$0 \leq x_i \leq L_i \quad \text{for all } i$$

where $C_{it\omega}$ is the cash-flow of operation i at time t in the scenario ω. Ω is now a set of scenarios rather than events in the usual sense of this word because of the time dimension. Besides Ω also consider the sets Ω_t of scenarios truncated at time t i.e. each element of Ω_t is a sequence of events from time 0 up to and including time t . Let furthermore $\Omega_H = \cup_t \Omega_t$, then an order relation can be defined on Ω_H by setting $\omega_s \prec \omega_t$ iff s < t and the sequence of events ω_s is an initial subsequence of ω_t . Finally define for $\omega_{t-1} \in \Omega_{t-1}$ the set $\Omega_t(\omega_{t-1}) = \{\omega \in \Omega_t : \omega_{t-1} \prec \omega_t\} \subseteq \Omega_t$. This description of Ω is prefered here to the usual one in terms of a filtered σ-algebra on Ω because it is better adapted to the analysis to be performed below . The sign function does not appear in the model anymore, the proper signs are attached to the coefficients. Note however that, should we change the signs in an entire equation, this would only change the formal appearence of the model because the duals take care of this modification by adapting their signs also.

The ordinary dual of this model is:

$$\min_{\pi}: \quad \sum_{i} L_i \cdot v_i$$

$$\text{subject to:} \quad v_i + \sum_{\omega}\sum_{t} \pi_{t\omega} \cdot C_{it\omega} \geq C_{i00} \text{ for all } i \tag{13}$$

$$v_i \geq 0 \quad \text{for all } i$$

and the Lagrangean dual is:

$$\min_{\pi}: \quad \sum_{i} (C_{i00} - \sum_{t}\sum_{\omega} \pi_{t\omega} \cdot C_{it\omega})^{+} L_i \tag{14}$$

By analogy with the previous examples each term in (14) can be interpreted as a sort of modified expected net present value of an intended financial operation so the market satisfaction function to be minimized is the total intended risk-adjusted expected net present value.

To determine the different components of this valuation procedure the duals have to be dismembered. This can be done recursively. First, the constraints of (12) have to be appropriately regrouped and summarized to avoid repetition i.e. we use Ω_H instead of Ω as index set. Then we first take Ω_0 and set $w_{0\,\omega} = \pi_{0\,\omega}$ for $\omega \in \Omega_0$, next we move to Ω_1 and set $w_{1\,\omega} = \pi_{1\,\omega} / w_{0\,\varpi}$ for $\omega \in \Omega_1$, $\varpi \in \Omega_0$ and $\varpi \prec \omega$, next we move one period further and set $w_{2\,\omega} = \pi_{2\,\omega} / w_{1\,\varpi}$ for $\omega \in \Omega_2$, $\varpi \in \Omega_1$ and $\varpi \prec \omega$; and we continue in this fashion until Ω_H is exhausted. Then the following relations hold: $\pi_{t\,\omega} = \Pi \{ w_{t\,\psi} : \psi \in \Omega_H \text{ and } \psi \prec \omega \}$. The variables $w_{t\,\omega}$ are a sort of one-period-ahead discounting factors and contain information on interest rates one-period-ahead conditional probabilities and risk-preference coefficients. The next section will describe in outline how these variables could be further analized.

7. ANALIZING DUALS

This section will give a description of the further anlysis of the $w_{t\,\omega}$ variables. It will be explained how particular probabilities and risk preferences could enter the picture. However, the only purpose of this explanation will be the clarification of the puzzling fact that the straightforwardly calculated probabilities can not always be made to add up to 1. An exhaustive analysis would lead us to far.

A straighyforward decomposition of $w_{t\,\omega}$ would proceed as follows. First define for $\omega \in \Omega_t$, the precursor $\omega^- \in \Omega_{t-1}$ by imposing $\omega^- \prec \omega$. This prescription uniquely defines the precursor of any partial scenario in Ω_H and we can compute $v_t(\omega^-) = \Sigma_\omega \{w_{t\,\omega} : \omega \in \Omega_t(\omega^-)\}$. This value is certain at time t-1, so we could define probabilities $P_{t\,\omega} = w_{t\,\omega} / v_t(\omega^-)$ which would then satisfy $\Sigma_\omega \{P_{t\,\omega} : \omega \in \Omega_t(\omega^-)\} = 1$. Note that $P_{t\,\omega}$ denotes the probability of arriving in ω from ω^-. The expression $v_t(\omega^-)$ can be interpreted as a dicounting factor from t back to $t-1$ and so it seems that the model (13) yields a full description of a market interest rate process. This would be so if the $P_{0\,\omega}$ would add up to 1 but this need not be. An easy way out would be to prohibit gambling and perhaps the government would be glad to have a mathematical reason to do so, but gambling does occur and we would like to be able to model it. Moreover, the neat solution for the other time periods is in some sense too nice to be true. Indeed, there seems to be no way to distinguish probabilities from the interest rate process and probabilities attached to the cash-flows.

Take the last problem first. If cash-flow events are classified according to their origin to obtain classes corresponding to stochastic cash-flows, we can sum the probabilities for each stochastic cash-flow. For every stochastic cash-flow, this sum should be 1 . So perhaps the interest rate probabilities can now be devided out? This might be so but again, their might be other adders lurking in the undergrowth. In any case, a very detailed analysis requires detailed data and possibly unwieldy models. Now, for the other problem it was suggested in section 4. that an introduction of risk-preference might explain the observed anomaly. As a matter of fact, the pricing of the operations by the market participants i.e. the $C_{i\,0\,0}$ affects the value of the duals. More specifically, the pricings of the critical operations in the optimal basis completely determine the duals and so do the risk attitudes underlying those pricings. A careful analysis of this relation between risk attitudes and duals could separate the contributions of probability and risk-preference but it is clear from the outset that this relation could be very intricate. A former draft of this paper contained some material on this issue but it was abandoned due to as yet unsurmountable analytical difficulties. As a consequence the results already obtained seemed somewhat beside the point and so they were dropped.

One thing should be stressed: the difficulties encountered above are caused by attitudes entering the model via the intended prices in the objective function. Without this inclusion of a priori preferences however, an equilibrium cannot be computed: we cannot compute it and neither can the market. This someway proves - if a proof was needed - that market participants have to declare their preferences, they have to take positions, in order for the market to be able to work. If everybody follows the market, the market can go anywhere. On the other hand: preferences can be irrational up to the point of seriously distorting market operations, in other words: markets can be wrong. After all, the prices and probabilities calculated by the model are shadow prices and shadow probabilities instead of real ones. So, it is possible to beat the market given more accurate information than commonly available. However, if the market is efficient, it will give you a hard time when you attempt to do it. What's more, markets are evidently not capable of performing instant global optimizations. All these considerations overwhelmingly demonstrate that we can neither trust the market nor any particular partial model of it. So it is not surprizing to encounter sometimes a paradox such as the one described in section 2.

7. A BUSINESS ENVIRONMENT

This section explores whether the model could be used in a business environment and discusses the modifications in both formalization and interpretation needed for this particular application. Whithin a business each operation no longer has a couterpart. Furthermore the optimization problem is not a market equilibrium problem: a business is not an arena where different preferences enter into competition with each other in order to acquire some prominence ...?? Well, in any case, it ought not be. Finally, a business has a past: it carries commitments ensuing from previous decisions dictating cash outflows to be met, as well as expectations of future cash inflows from existing contracts. But first things first: what could be a sensible objective function to evaluate operations in a business environment? Bearing in mind the warnings at the end of the previous section we might wager to propose the total market value. Now, what about the constraints? If deviations are not too large, commitments can always be met by borrowing, so, barring bancruptcy or other catastrophes we can assume that the constraints are not stochastic. The stochastic element in the commitments is taken care of by defining a constraint per scenario. Summarizing, the following model looks acceptable:

$$\max_x : \quad \sum_i V_i \cdot x_i$$

$$\textit{subject to}: \quad \sum_i C_{i\,t\,\omega} \cdot x_i = H_{t\,\omega} \quad \textit{for all } \omega \in \Omega,\, t \in H \qquad (15)$$

$$0 \le x_i \le L_i \quad \textit{for all } i$$

Herein V_i is the market value of operation i . Time 0 is included in the set H . The right hand sides $H_{t\,\omega}$ are the formalization of the previous commitments i.e. the financial structure of the business. The last set of constraints also has an interpretation slightly different from that in (12): the limits are not voluntary but imposed from outside, they limit the opportunities.

The model is similar to (12) but the differences are not negligible. They are more readily discerned after the presentation of the dual model:

$$\min_{\pi}: \quad \sum_{i} L_i \cdot v_i + \sum_{\omega}\sum_{t} \pi_{t\omega} \cdot H_{t\omega}$$

$$\textit{subject to:} \quad v_i + \sum_{\omega}\sum_{t} \pi_{t\omega} \cdot C_{it\omega} \geq V_i \ \textit{for all } i \qquad (16)$$

$$v_i \geq 0 \quad \textit{for all } i$$

The duals $\pi_{t\omega}$ are not necessarily equal to the market duals, consequently the interest rate process from this model differs from the market rate process. It could be called a cost-of-capital process. Under the assumption that the market value is the right choice for the objective function, it is this cost-of-capital process that should be used to evaluate operations. Of course, this cost-of-capital process is a byproduct of the optimization of the model and it changes with every modification of the latter but in practice it is fairly stable and modifications can in any case be effectuated by parametric programming. Using complementary slackness we deduce that an operation is accepted if its internal value based on the cost-of-capital is at least equal to the market value. Otherwise it is rejected. Decisions based on this prescription are optimal and although the collection and processing of the necessary data is all but easy, even rather crude approximations could be better than the common practice of discounting at the "riskless rate" plus risk supplement or at the return on equity. If this paper succeeds in sowing some doubt in the minds of a sizable fraction of common practicioners, it will largely have been worth the trouble.

8. FINAL COMMENTS

Hopefully this paper has been able to show the applicability of linear programming to both market equilibrium problems and business decisions, despite the fact that this exposition still contains many gaps. Some confidence that the gaps will eventually be closed seems justified. The full potential of the approach is however far from exhausted. The financial market model (12) could for instance be supplemented with variables for other commodities. It could in principle describe a whole economy. On the technical side, the approach could be generalized to continuous-time models and perhaps to geographic models with operations localized at points in a continuous space. The necessary analytical techniques already exist in the optimal control theory with the Pontryagin maximum principle as duality theorem.

Foremost however, this paper was intended to be a plea in favor of a more seriously scientific handling of business problems.

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