A DYNAMIC MODEL FOR PENSION FUNDS MANAGEMENT

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ABSTRACT

The management of pension funds financial encompasses asset allocation and the control of the future flows of contributions. A high proportion of stocks in the portfolio has the benefit of a lower mean contribution level, but at the price of a higher time variation of contribution flows. This paper models the trade-off in a inter-temporal framework and uses of stochastic control to obtain an optimal asset allocation –between a risky asset and a riskless asset– and the contribution policy. The solution in the case of a defined benefit scheme shows that the proportion of the risky asset and the level of contribution are both proportional to the difference between the maximum wealth necessary to fund all pensions, and the actual wealth of the pension fund. Illustrative simulations for France, US and Japan for various periods show a decrease of the contribution level down to zero after some decades. The hypothesis made and some shortcomings of the model are discussed and further research is outlined.

I Introduction

The economic role of pension funds is considerable and well acknowledged even in countries where the pay-as-you-go system is dominant. Indeed, at their mature stage, which they are at in many Anglo-Saxon countries, pension fund assets represent a large percentage of stock and bond market capitalisation, in size comparable with the country's GNP, and the contribution flows to the pension fund account for a significant part of personal savings. Moreover, their final objective –to pay pensions to workers in their old age– is of up-most importance for the social and political stability of the wealthy economies. How to manage pension provisions adequately is thus a crucial question.

The principles underlying pension funds are quite simple, even if the variety of actual schemes from one country or one industry to another is vast and complex. Workers and corporations pay contributions to a pension fund, which invests them over a very long period of time and releases them when the workers retire, in the form of pensions. Obviously, the more the contribution, the higher should be the pension. Nevertheless, asset allocation also comes into play, in so far as that even a slight improvement of the asset portfolio mean return, say one or two percent, may result after thirty years of accumulation, in a sizeable increase –by 40% to 100%– of the pensions. On the other hand, too much exposure to stock market fluctuations could, in the absence of careful management of the asset portfolio, severely damage the asset value and impose an undesirable increase of the contributions.

In this context, portfolio management and the contributions scheme are clearly interdependant. Moreover, the decisions made over one year have certainly consequences in the future. Therefore multiple horizon optimization seems to be appropriate. Because stock returns are uncertain in efficient markets, stochastic control would help in finding the optimal investment policy, as well as the adequate level of contribution.

Up to our knowledge, the use of stochastic control for pension fund financial management has not been reported in the literature. However, Merton (1972) described the basic framework of intertemporal optimization and showed how the Bellman function can provide a solution to the asset allocation of an investor, given his objectives and risk tolerance. On the other hand, asset and

liability management has been invoked by several recent studies in order to determine the asset allocation of pension funds. Sharpe and Tint (1990) proposed to optimize $A-kL$, where A and L stand for asset and liability respectively and k is a positive constant less than one. Surprisingly, as remarked by Sharpe and Tint, investment policies of pension funds was hardly a subject of interest before the early 90's. A few authors, such as Tepper and Affleck (1974) and Black and Jones (1988) had made attempts to propose solutions to the asset allocation problem, given the liabilities of the pension fund. Nevertheless as asset and liability management techniques improve and are put into practice, an increasing number of papers have addressed the issue, now seen as important. Among them, Leibowitz et al. (1993) showed how to cope with a number of conflicting constraints and to come up with an appropriate and yet simple optimization. Griffin (1993) has also presented a methodology which he has applied to Dutch and English pension funds globally, in an attempt to explain why the Dutch invest less than 25% in stocks, whereas the British invest more than 80%. All these studies stick to the asset allocation problem in a one period framework. However Boender et al. (1993) have tried to investigate the financial management problem in a more general setting, making use of a scenario approach.

The outline of this paper is the following. The next section describes the financial framework and poses the optimization problem. The solution of the problem is presented in the following section, in the case of defined benefits. The last section discusses the results from a financial and economic point of view, and examines some possible generalizations and improvements of the method.

II Financial setting

1. Definition of variables

In the rest of the paper the following variables will be used :

p_t pension payments
c_t contributions
x_t portfolio market value
S_t market value of the risky asset
u_t investment in the risky asset, as a proportion of portfolio value.

All of them are functions of time t, either stochastic or deterministic. On the other hand, the following variables are supposed to be constant :

r risk free rate
λ risk premium (positive)
σ volatility of the risky asset
α pension growth rate
β psychological discount rate
γ contribution growth rate

Finally, we shall refer to (c^*, u^*) as the optimal policy and to x_m as the maximum necessary wealth.

2. Main hypothesis.

We consider the financial management of the aggregated pension fund position and assume that either pension flows or contribution flows in the future are known. The first case corresponds to the defined benefit type of pension fund and the second to the defined contribution type. For sake of simplicity, their growth rates are taken as constant, α and γ respectively. This growth rate may account for a demographic trend, an inflation scenario, a purchasing power evolution or any kind of combination of these factors as

long as it leads to a deterministic growth rate. This last hypothesis is certainly an important limitation of the model which will be discussed further. Thus, in the defined contribution pension fund we have :

$$dp_t = \alpha p_t$$

and for the defined contribution pension fund :

$$dc_t = \gamma c_t$$

In each case, the contributions made are invested in a portfolio allocated in various financial assets. Although more general assumptions are clearly possible, we restrict this study to the case of two assets investigated by Merton (1972) :

−a riskless asset whose return is r, the risk-free rate, assumed constant ;
−a risky asset whose price S_t follows the standard geometric brownian motion

$$dS_t \ / \ S_t = (\lambda + r)dt + \sigma dW_t$$

where dW_t denotes the usual differential of brownian motion. The expected return of this risky asset, $r + \lambda$, is therefore higher than the riskfree rate, the difference being the constant risk premium λ. On the other hand the future returns of the risky asset are not known with certainty because of the volatility (assumed to be constant) and the stochastic process W.

In this simple setting the portfolio management consists in allocating a proportion u_t of the value x_t into the risky asset. Typically the portfolio is composed of stocks and bills. Again, generalization is possible and will be discussed later.

3. Optimization

As mentioned before there are basically two cases. In the defined benefit framework one should try to minimize the contributions, with the obligation to meet the liabilities of the pension fund. On the other hand, the defined

contribution pension fund manager aims at maximizing the pensions to be paid to the retirees, knowing the stream of contributions. In this study, we concentrate on the defined benefits case, which occurs, more frequently leaving the development of the defined contributions case to another study.

We suppose that contributors are reluctant to pay higher contributions either today or in the future, but that they have their own judgement as to the discount rate, which we have denominated the psychological discount rate β. For sake of mathematical tractability we have also assumed that their disutility is a power function of the contribution c. In the rest of the paper the exponent will be taken as 2, but generalization is possible. Under these circumstances, a rational pension fund manager would try to minimize :

$$V = \int_0^\infty \exp(-\beta s)c_s^2 \, ds$$

The optimum policy must satisfy the following constraints :

       –payment of the pension p_t
       –positive value for x_t

Therefore we seek the policy (c_t, u_t) which minimizes $E(V)$ under the two preceding constraints.

III Solution

c_t and p_t being respectively the contributions and the pensions paid per unit of time, continuously discounted, the evolution of the portfolio is described by the following equation :

$$dX_t = u_t X_t \frac{dS_t}{S_t} + (1 - u_t)X_t r dt + (c_t - p_t)dt$$

The first term on the right is the growth of wealth due to the part of the portfolio invested in the risky asset. The second term comes from the part invested in the riskless asset. The third term represents the flow due to the balance of subscriptions and payment of pensions. Making use of the model assumed for S_t the above equation can be rewritten as :

$$dX_t = \left[rX_t + \lambda u_t X_t + c_t - p_t \right] dt + u_t X_t \sigma dW_t$$

For technical reasons, we assume that

$$0 = inf\,(\,e^{-\beta t}c^2 + V'_t + V'_x\,dx + V'_p\,dp + \frac{1}{2}V''_{x,x}\langle dx\rangle^2\,)$$

Using the equations above we have :

$$2r - \beta - \lambda^2 / \sigma^2 > 0$$

In the case where $\beta = r$, this assumption becomes

$$r - \lambda^2 / \sigma^2 > 0$$

This inequality means that the risk premium is to be justified by a sufficient high volatility ($\sigma > \lambda / \sqrt{r}$).
If $V(t,x,p)$ denotes the value function of the problem, Bellman's equation is then :

$$0 = inf\left(e^{-\beta t}c^2 + V'_t + (rx + \lambda ux + c - p)V'_x + \alpha p V'_p + \frac{1}{2}V''_{x,x}\,x^2 u^2 \sigma^2 \right) \qquad (1)$$

under the constraints $x > 0$ et $u > 0$. The term in brackets is a polynomial function of u and c, therefore the optimal policy (u^*, c^*) satisfies

$$\lambda x V'_x + V''_{x,x}\,x^2 u^* \sigma^2 = 0$$

$$2e^{-\beta t}c^* + V'_x = 0$$

That is to say

$$u^* = -\frac{\lambda V'_x}{V''_{x,x}\, x\sigma^2} \qquad (2)$$

$$c^* = -\frac{V'_x}{2e^{-\beta t}} \qquad (3)$$

Substituting the expressions (2) and (3) in (1) leads to

$$-\frac{1}{4}e^{\beta t}V'^2_x + V'_t + (rx - p)V'_x + \alpha p V'_p - \frac{\lambda^2}{2\sigma^2}\frac{V'^2_x}{V''_{x,x}} = 0$$

Let us now search *a priori* for a solution of the type

$$V(t,x,p) = e^{-\beta t}F(x,p)$$

The differential equation satisfied by F is

$$-\frac{1}{4}F'^2_x - \beta F + (rx - p)F'_x + \alpha p F'_p - \frac{\lambda^2}{2\sigma^2}\frac{F'^2_x}{F''_{x,x}} = 0$$

Remark that this equation is homogenous for the variable $y = x/p$. Let us then set

$$F(x, p) = p^2 f(x/p)$$

The differential equation satisfied by f is now

$$-\frac{1}{4}f'^2 + (2\alpha - \beta)f + ((r - \alpha)y - 1)f' - \frac{\lambda^2}{2\sigma^2}\frac{f'^2}{f''} = 0$$

Once more, the solution is obtained by searching *a priori* for a solution of the form :

$$f(y) = Ay^2 + By + C$$

We identify A, B and C and find

$$f(y) = (2r - \beta - \frac{\lambda^2}{\sigma^2})(y - \frac{1}{r - \alpha})^2$$

Combining the equation for V, F and the last one, we find the final expression for the value function :

$$V(t, x, p) = e^{-\beta t}\left(2r - \beta - \lambda^2/\sigma^2\right)(x - p/(r - \alpha))^2 \quad (4)$$

Direct substitution in (1) shows that this function is effectively *the* solution of the problem in the domain $0 \leq x \leq x_m$, where $x_m = p/(r - \alpha)$. In the domain $x_m \leq x$, it is obvious that the optimal policy is zero-contribution and no risky asset in the portfolio.

Using (2), (3) and (4), the final expressions for the optimal policy are :

$$c^* = \begin{cases} \left(2r - \beta - \lambda^2/\sigma^2\right)(x_m - x) & \text{if } x \leq x_m \\ 0 & \text{if } x_m \leq x \end{cases}$$

$$u^* = \begin{cases} \dfrac{\lambda}{\sigma^2} \dfrac{x_m - x}{x} & \text{if } x \leq x(p) \\ 0 & \text{if } x_m \leq x \end{cases}$$

The condition $\sigma > \lambda\sqrt{r}$ ensures that the constraints on u and C are satisfied.

IV <u>Discussion</u>

1- Some comments from an economic point of view

The value x_m represents a critical threshold equal to the discounted value of all future flows due to pension payment, for an infinite horizon. Above it, the fund is entirely invested in riskless assets and receives no contribution. It is the situation of a *rent* : the interest generated by the principal is sufficient to pay the pensions. Below this threshold, the fund is partially invested in risky assets and receives contributions. The optimal contribution (c^*) and the amount invested in risky assets $(u^* x)$ are decreasing linear functions of the fund's value when it is under the critical threshold. The general idea is that the wealthier the fund, the less risk it needs to take, and the more cautious and risk-adverse it appears. As a consequence, the pension fund tends to buy more stocks when the market falls. Moreover as we have not contrained u , it can be greater than one, meaning that the fund will borrow in order to invest more in stocks. Although one can expect such a behaviour for a very long term investor it is certainly necessary to set some limits to prevent too large a debt. In addition, a minimum investment in bonds could be imposed in order to ensure the payment of the accumulated liabilities to those workers who have contributed.

Note that, logically, the higher the volatility and the lower the risk premium, the higher the contributions and the lower the part invested in the risky asset. In this simple framework there is no defined time horizon and thus the fund manager knows he can benefit from the higher return of stocks. His incentive to do so increases obviously with the risk premium. As it is also a decreasing function of σ^2 one could consider $1/\sigma^2$ as a characteristic time for the investor. The longer this time horizon the higher the propensity to invest in stocks.

2. Historical simulations

For sake of illustration a number of simulations have been performed, making the assumption that recent historical data fit out simple model. Making use of

early records of inflation, stock market prices (in form of a domestic market index with reinvested dividends), and interest rates (short term) for France, Japan and the USA, we have computed the parameter values shown in table 1 below.

	Period	r	λ	σ	α	$r-(\lambda/\sigma)^2$
France	1960-92	9	2.4	19.5	6.4	6.5
Japan	1973-93	7	2.9	17.5	5.1	4.3
USA	1970-93	8.9	4.3	15.7	5.9	1.5

Table 1
Parameter values (%) used in the simulation

Note that these data correspond to different time periods (see Column 2). Indeed the performance of the strategies will largely depend on the stock market returns over the first years of simulation. Figure 1 shows the stock index evolution for the markets considered.

Two pension funds were considered . PF1 has a comfortable surplus of 50% of its current accumulated liabilities, whereas PF2 has a limited surplus of 10%. This surplus is by definition the difference between the actual portfolio value minus the discounted liabilities at the rate r. For sake of simplicity we have assumed that these liabilities were equal to all the pensions to be paid over the next thirty years.

We have simulated yearly contributions and reallocations of the portfolio according to the optimal policy derived in section III. We have not studied the sensitivity of the result to the time interval between contributions and reallocation, but it would certainly have some influence.

The resulting evolution of the portfolio values, of the proportion (%) of stocks held in the portfolio and finally of the contributions are represented in Figures 2 a, b, c (respectively 3 and 4) in the case of France (respectively in the case of Japan and USA).

The long term benefits of the solution are visible in the French case. No more contributions are necessary after 23 years for PF1 and 32 years for PF2. The

decreasing exposure to the stock market is striking. However for both there is borrowing in the first year. This situation is also encountered in the case of the Japanese pension funds and to a much larger extent in the American case. Nevertheless in this latter case the upper bound (x_m) is reached within a short time period of 11 years for PF1, although it entails a very high leverage. On the other hand, none of the Japanese pension funds reach the point at which no more contributions are necessary. The impact of the Tokyo markets poor performance after 1990 is clear. The pension fund is obliged to raise the contribution and increase at the same time its exposure to the stock market.

3. Criticism and further research

This too simple model suffers from a number of shortcomings that we have already mentioned. Let us try to summarize them and to discuss the possible developments that could overcome these difficulties.

First, the investment in stock u was not bounded (to 1 or to some level depending on the liabilities). Adding such a constraint is obviously not a theoretical problem, because under this situation the solution would still exist. But it will no longer allow us to derive a closed form solution. Numerical methods must be used, the accuracy of which can fortunately be checked in the non-constrained case.

Secondly the condition $2r - \beta - (\lambda / \sigma)^2 > 0$ need not be met, as when the stock market does not achieve a reasonable return-to-risk ratio. Such a situation could force the pension fund to look for a better expected performance in foreign markets. If β remains still too big compared to the available foreign market performance, probably a pension funds system cannot be achieved, simply because the aversion to savings is too high.

Thirdly, the single risky asset could without theoretical difficulty be replaced by a much more realistic mix.

A finite-time horizon corresponding to known mortality tables (in average of course) could also be included. In this case the derivation of the close form solution will be cumbersome, though possible.

Inflation was considered to be of constant rate, which is an unrealistic assumption, especially in European countries, where inflation over a long period of time happens to have been devastating. A stochastic description of inflation would have to consider stochastic real returns of various asset classes, with the possible problem of having serial correlations to take into account.

Finally, the economic criteria choosen for the optimization should be reconsidered. Indeed the variation in the contributions is probably unbearable for economic agents who would prefer a smoother pattern even at the expense of a higher mean contribution rate. In addition, no risk aversion was considered in the portfolio, which is also unrealistic. Again, general criteria would lead either to no solution or to a numerical solution with some real computing difficulties.

Results pertaining to the defined contribution case have already been obtained and will be pesented in another paper. The framework remains simplistic in the interest of obtaining a closed form solution.

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APPENDIX

Figure 1
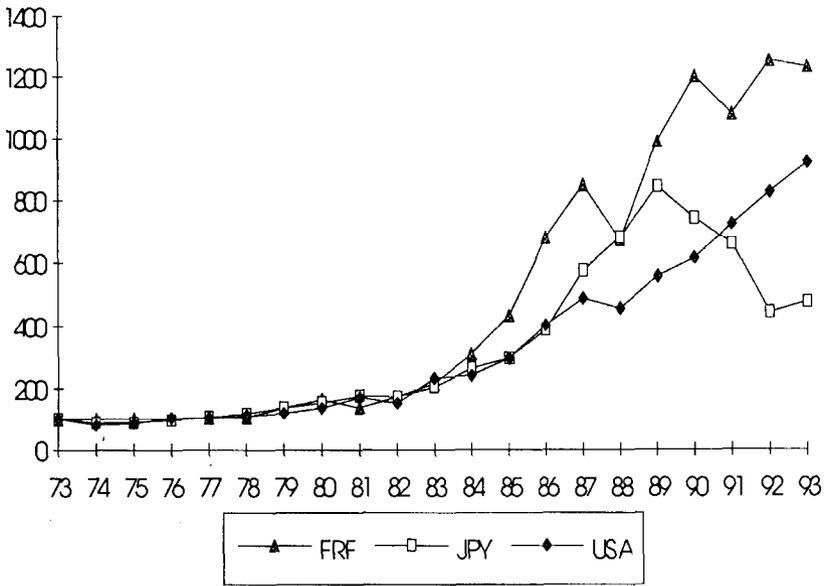
Illustration for a French fund

Figure 2a : The value of the fund

Figure 2a : The value of the fund

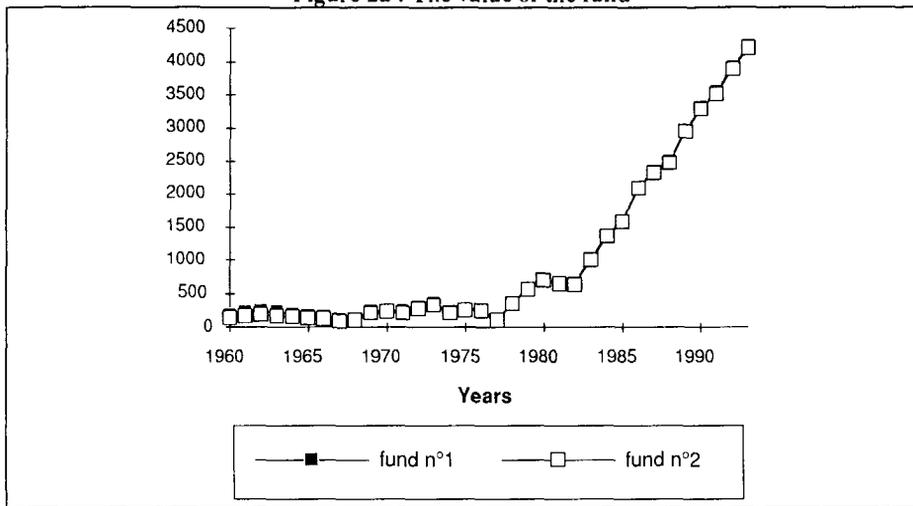### Figure 2b : Percentage invested in risky assets

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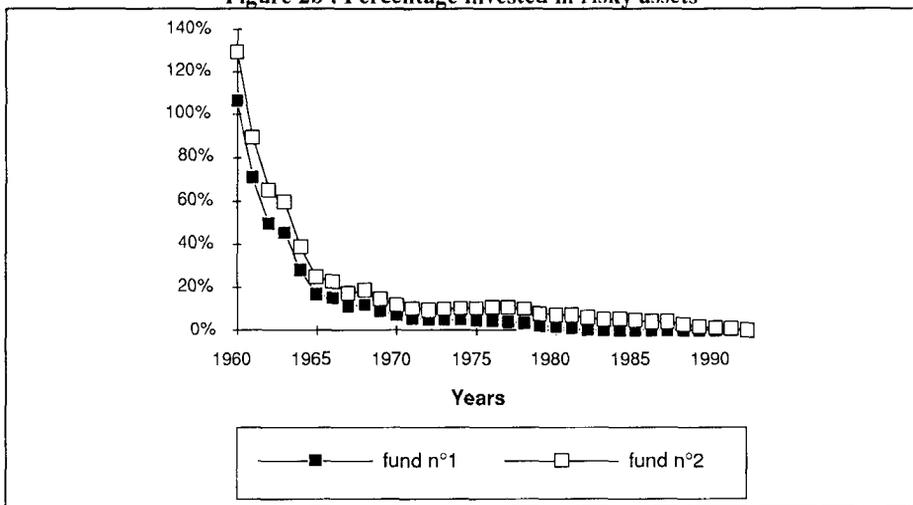
Figure 2c : Contributions
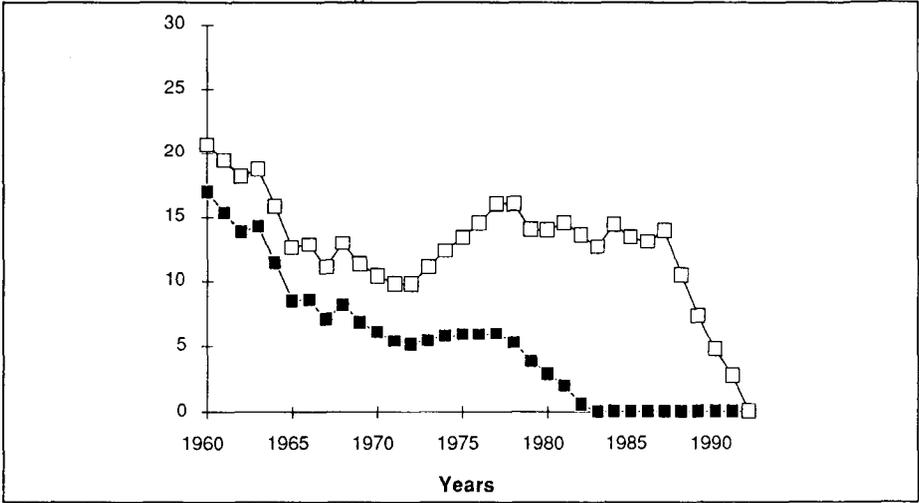
Illustration for a Japanese fund

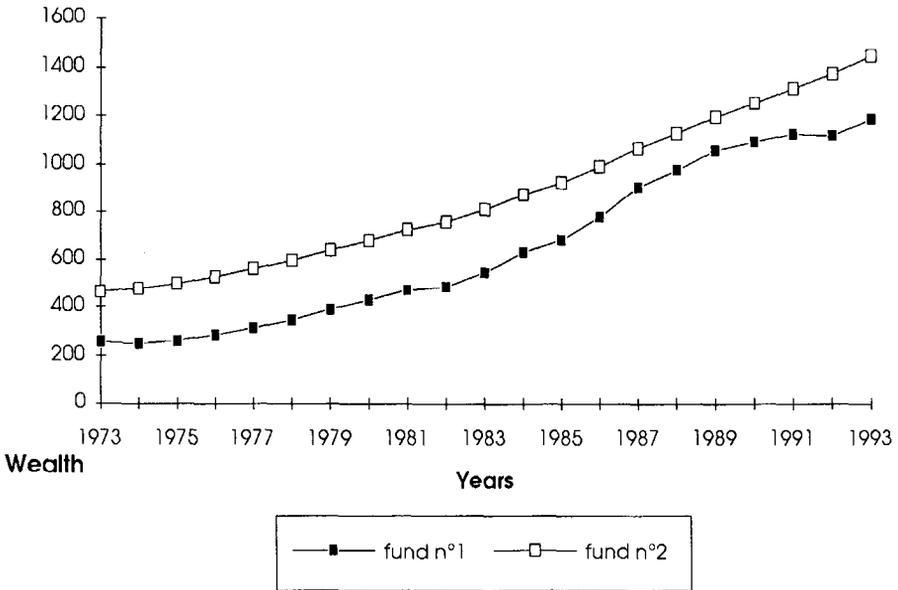**Figure 3a : The value of the fund**

Wealth

Years

fund n°1   fund n°2

Figure 3b : Percentage invested in risky assets

Proportion of risky asset
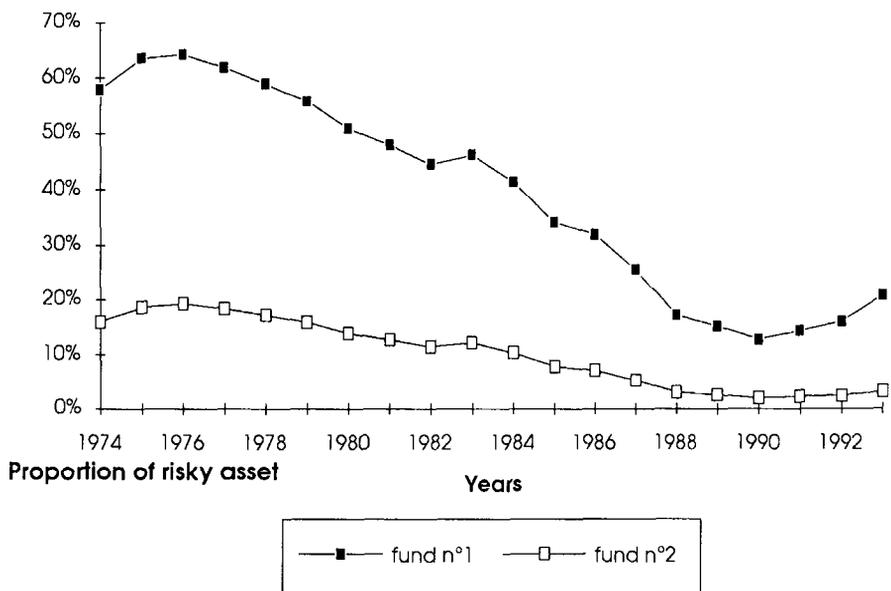
Figure 3c : Contributions
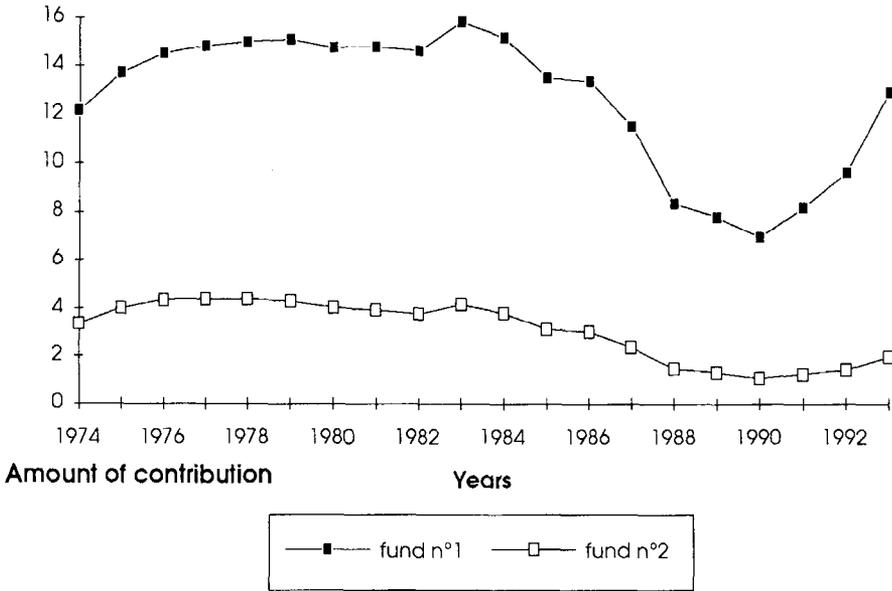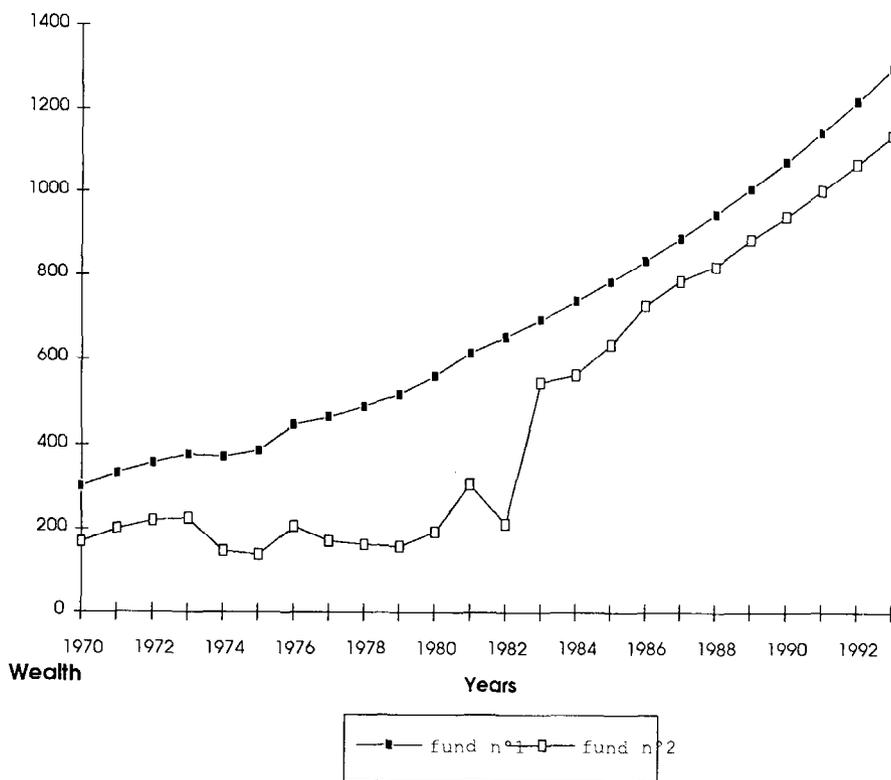

Amount of contribution   **Years**

Illustration for a US fund

Figure 4a : The value of the funds

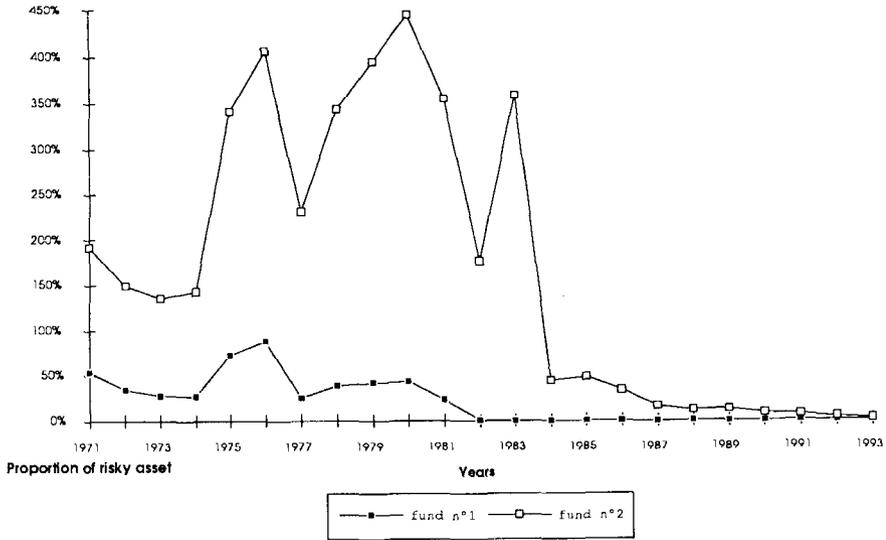**Figure 4b : Percentage invested in risky assets**

Proportion of risky asset

Years

fund n°1 — fund n°2

Figure 4c : Contributions