

## A COMPARATIVE EVALUATION OF ALTERNATIVE MODELS OF THE TERM STRUCTURE OF INTEREST RATES

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### ABSTRACT

In the paper alternative models of the term structure of interest rates are classified in two different approaches: the no-arbitrage and the general equilibrium approach. It is maintained that the general equilibrium approach is superior on a theoretical ground for two main reasons: first, relevant variables, such as the spot interest rate and the interest risk-premium, are endogenous; secondly, the relationship between the real and the financial side of the economy becomes a clear and important element in the understanding of the term structure. As regards the applications, however, the advantages of the general equilibrium over the no-arbitrage approach are not so clear: the major role in the empirical performance of alternative models is played by their ability to capture volatility. At the current state of the literature, there is no model that outperforms others, in particular on the empirical side.

**KEYWORDS:** arbitrage, general equilibrium, interest rates, term structure, volatility

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### **RESUME**

Dans cet article les modèles alternatifs de la structure à terme des taux d'intérêts sont classifiés en deux principes différents: l'absence d'arbitrage et l'équilibre général. On soutient que l'approche de l'équilibre général est supérieur, en théorie, pour deux raisons principales: des variables importantes, comme le taux d'intérêt et le prix à risque, sont endogènes; la relation entre l'économie réelle et l'économie financière devient un élément clair et important dans la compréhension de la structure à terme. Quant aux applications, les avantages des modèles d'équilibre général sur les modèles d'absence d'arbitrage ne sont pas clairs: le rôle le plus important dans la performance empirique des modèles alternatifs est joué par leur habilité à capturer la volatilité. Selon la littérature actuelle, il n'y a pas un modèle supérieur aux autres, particulièrement en ce qui concerne les applications.

## 1. INTRODUCTION

The first problem that one has to face in the analysis of the Term Structure of Interest Rates (TSIR) is the definition of the object of the investigation. The issue has already received great attention<sup>(1)</sup> and the current literature agrees on the following definition: "*The term structure of interest rates measures the relationship among yields on default-free securities that differ only in their term maturity.*" (Cox, Ingersoll and Ross, 1985b). Analytically it is the function that maps the time to maturity of a discount bond to its current price or yield to maturity.

The relevance of finding a "good" theory for the TSIR stems from the fact that the TSIR is not only interesting in itself, but enters also in the valuation of the growing number of Interest Rate Sensitive (IRS) claims.

Early theories (e.g. the expectation hypothesis, the liquidity preference, the market segmentation and the preferred habitat theory) are essentially based on a deterministic set-up.

The early seventies turbulence in the financial markets have stressed the need for embedding the TSIR analysis in a stochastic environment.<sup>(2)</sup> A natural approach would be that of extending asset pricing theories to IRS claim valuation, i.e. to use either the Intertemporal Capital Asset Pricing Model (ICAPM) or the Option Pricing Theory (OPT).

Yet, a TSIR theory cannot be mutated by the ICAPM because IRS claim risk cannot be diversified away in the same way as stock risk since IRS claims returns are highly correlated.

On the other hand, given the differences between IRS claims and claims contingent on stock prices, the Black and Scholes option valuation formula cannot be simply extended to IRS claims valuation.

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(1) Nuzzo(1993) contains a discussion of the issue.

(2) Ingersoll (1987) clearly explains under what conditions the above-mentioned deterministic theories can be embedded in a stochastic approach. De Felice and Moriconi (1991) make clear the distinction between semideterministic and stochastic models of the TSIR and stress the necessity of taking a stochastic approach.

Since the late seventies, stochastic models based on the no-arbitrage assumption combined with powerful martingale results<sup>(3)</sup> have attempted to explain the TSIR. Among articles within the stochastic approach it is worth mentioning Vasicek(1977), Dothan(1978), Cox, Ingersoll and Ross(1985a,b), Ho and Lee(1986), Heath, Jarrow and Morton(1992).

Most of empirical studies take a certain model without questioning much differences and similarities with others. Since the stochastic literature on the TSIR is growing very fast, there is a need of identifying, beyond the specific distinctive assumptions taken in each article, the main theoretical approaches.

In order to overcome this deficiency in the literature, this paper offers a survey on the stochastic approach to the TSIR and the related IRS claim valuation. To this end models are classified in two broad different approaches:

I) the Arbitrage Pricing Theory (APT)

II) the General Equilibrium Theory (GET).

The former is based on the assumption that the dynamic of discount bond prices is described by an Itô's differential equation and, imposing no arbitrage as an equilibrium condition, derives the equilibrium yield to maturity for different maturities, i.e. the TSIR, which, among other things turns out to depend on an exogenously specified market price for risk.

The latter is based on an intertemporal general equilibrium model where the market price for interest rate risk is determined endogenously.

This paper aims at highlighting distinctive features of the two approaches and stressing conditions under which the two approaches are actually equivalent. Plausible different assumptions within each approach are also discussed and an empirical evaluation of the various models is offered.

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<sup>(3)</sup> Duffie(1992) discusses the equivalence between no arbitrage, existence of state prices and existence of an equivalent martingale measure in a complete market setting. For a technical discussion of the issue, see Duffie(1992) and the original papers by Harrison and Kreps(1979) and Harrison and Pliska(1981).

The plan of the paper is as follows. In section 1 APT is presented and different versions of it are discussed. In section 2 the GET is analyzed and in section 3 a comparison of the two approaches both at a theoretical and at an empirical level is made. Conclusions and hints for future research follows.

## 2. The Arbitrage Pricing Theory

### 2.1 The basic model

The basic model aims at describing the term structure of interest rates which is, according to the most widespread definition, the relationship among yields on default-free securities that differ only in their term maturity. Hence, the TSIR can be described either in terms of yield to maturity  $h(t,T)$  or in terms of the price of a discount bond  $v(t,T)$ .

In describing the basic model we will follow Vasicek(1977)<sup>(4)</sup> and De Felice and Moriconi(1991). The former provides a crystal clear description of the TSIR based on an arbitrage argument similar to that of Black and Scholes(1973) for option pricing; the latter offers a detailed explanation of the same within the frame of stochastic immunization.

The model is based on the following assumptions.

#### Hp 1: assumptions on the market

The market is frictionless and competitive; the agents are price-takers; trading is continuous and consistent (i.e. no profitable riskless arbitrage is possible).

#### Hp 2: the basic variable

Let  $h(t,T)$  be the yield to maturity and  $\delta(t,T)$  be the instantaneous interest rate. The spot rate is the basic variable and is defined as:  $r(t) = h(t,t+dt)$  or equivalently as:

$$r(t) = \lim_{T \rightarrow t} \frac{1}{T-t} \int_t^T \delta(t,u) du$$

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<sup>(4)</sup> It should be noted that in Vasicek(1977) the dynamic of the spot rate is defined in a more specific way than in the model presented here. This point is discussed in section 2.3.

The spot rate is the risk-free rate and is the only source of uncertainty, i.e. the model is a single factor model (univariate) where  $r(t)$  is the state variable.

Hp 3: Stochastic process for the spot rate

The spot rate follows a Markov process, i.e. the probability distribution of the future values of  $\{r(t)\}$  is determined only by the current value of  $r(t)$ , and it is assumed to be continuous (i.e. no shocks in the bond market).

Hp 4: homogeneous expectations

Agents hold homogeneous expectations on the probability distribution of  $\{r(t)\}$ .

Assumptions 1 to 4 imply that the value function  $v(t)$  is univocally determined.

Based on the above assumptions the model is articulated in five main steps.

FIRST STEP

The dynamics of the spot rate is described by the following Itô's Partial Differential Equation (PDE):

$$(2.1) \quad dr(t) = f(r(t),t)dt + g(r(t),t)dZ(t)$$

where  $f(r(t),t)$  is the drift of the process;  $g(r(t),t)$  is the diffusion coefficient and  $Z(t)$  is a standard brownian motion with zero mean and unit variance (Wiener process with incremental variance  $dt$ ).

SECOND STEP

The dependence of the value function  $v(t,T)$  on the spot rate  $r(t)$  is taken into account

$$(2.2) \quad v(t,T) = v(r(t),t,T) \quad \text{with } T \geq t$$

$v(r(t),t,T)$  is assumed to be a monotone function of  $r(t)$  with partial first and second derivatives  $v_t, v_r, v_{rr}$  assumed to be continuous.

THIRD STEP

Based on the assumptions made in the previous step, Itô's lemma can be used to derive the dynamic of the rate of return:

$$(2.3) \quad \frac{dv(r(t),t,T)}{v(r(t),t,T)} = \mu(r(t),t,T)dt - \sigma(r(t),t,T)dZ(t)$$

where

$$\mu(r(t),t,T) = \frac{1}{v(r(t),t,T)} \left[ \frac{\partial}{\partial t} + f(r(t),t) \frac{\partial}{\partial r} + \frac{1}{2} g^2(r(t),t) \frac{\partial^2}{\partial r^2} \right] v(r(t),t,T)$$

$$\sigma(r(t),t,T) = - \frac{1}{v(r(t),t,T)} g(r(t),t) \frac{\partial}{\partial r} v(r(t),t,T)$$

(2.3) states that the rate of return can be decomposed in an anticipated change and an unanticipated change, due to the shocks represented by the Wiener process and whose magnitude depends on  $\sigma(r(t),t,T)$ .

From the expression for  $\mu(r(t),t,T)$  we get the following PDE:

$$(2.4) \quad v_t + f v_r + \frac{1}{2} g^2 v_{rr} - \mu v = 0$$

where we have omitted variable dependence.

FOURTH STEP

The drift  $f(r(t),t)$  and the variance  $g^2(r(t),t)$  of the spot rate process are normally assumed to be known or estimated. Thus, if the function  $\mu(\cdot)$  is known the PDE (2.4) can be solved for the functional  $v(r(t),t,T)$  so that the whole term structure is determined.

Yet, the form of the function  $\mu(\cdot)$  is unknown and cannot be deduced from the model. In order to say something on  $\mu(\cdot)$  an arbitrage argument must be used and the following result can be shown to hold for any  $t_1$  and  $t_2$  with  $t_2 > t_1 > t$ :

$$(2.5) \quad \frac{\mu(t,t_1) - r(t)}{\sigma(t,t_1)} = \frac{\mu(t,t_2) - r(t)}{\sigma(t,t_2)}$$

Each side of (2.5) can be interpreted as a risk-premium or better term premium, which is independent of the term. Hence it is a common market value which represents the market price for interest rate risk<sup>(5)</sup>:

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(5) It is in fact the compensation required in equilibrium by agents in order to take the risk connected with unanticipated changes in the spot rate.

$$(2.6) \quad q(r(t), t, T) = \frac{\mu(r(t), t, T) - r(t)}{\sigma(r(t), t, T)}$$

From (2.6) and recalling the expression for  $\sigma(r(t), t, T)$ , we obtain  $\mu(\cdot)$  as a function of the market price for risk:

$$(2.7) \quad \mu(r, t, T) = r(t) - q(r, t) g(r, t) v_r(r, t, T) / v(r, t, T)$$

Since the market price for interest rate risk is an equilibrium price, it depends on the risk preference of the agents in the market and so does the function  $\mu(\cdot)$ .

Hence, preference-free pricing, which is normally considered a great advantage of asset pricing based on the no-arbitrage argument, does not hold when the TSIR is at issue.

#### FIFTH STEP

Substituting (2.7) in (2.4) we get a new PDE that does not depend on  $\mu(r(t), t, T)$ :

$$(2.8) \quad v_t + (f+qg)v_r + \frac{1}{2}g^2v_{rr} - rv = 0$$

Once the characteristics of the spot rate process  $f$  and  $g$  are known and the price of the interest rate risk  $q$  is specified, the PDE (2.8) can be solved with the following boundary condition:

$$(2.9) \quad v(r(T), T, T) = 1.$$

The solution is the value function  $v(r, t, T)$  from which it can be obtained the yield to maturity for each maturity, i.e. the whole term structure.

The solution can be also expressed in the integral form as:

$$(2.10) \quad v(r(t), t, T) = E_t \left[ e^{-S(t, T)} \right]$$

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$$(2.11) \quad S(t, T) = \int_t^T r(u) du + \frac{1}{2} \int_t^T q^2(r(u), u, d(u)) - \int_t^T q(r(u), u) dZ(u)$$

The model just described differs from other no-arbitrage models in that it assumes that the spot rate is the only source of uncertainty. This implies that returns on securities of different maturities are perfectly correlated. Other models that overcome this limitation are described in the next section.

As a final comment we stress a point which will be central in the comparison between NA and GET type of models. The basic model presented here requires the price of interest rate risk  $q$  to be exogenously specified in order for the model to be closed. Furthermore, this specification cannot be arbitrarily done: consistency with no arbitrage must be verified. We will discuss this point extensively in Section 4.

### 2.3 Other versions of the basic model

The basic model presented requires the market price for risk  $q$  to be specified in order for the model to be solved. Moreover it is a single factor model, since there is only a source of uncertainty in the model: the spot rate.

Since the end of the seventies, many versions of the basic no arbitrage model presented in the previous section have been proposed. For need of exposition, in this section we outline a few models which differ for the specification of the market price for risk and/or for bringing more than one factor into the analysis.

#### The Vasicek(1977) model

Vasicek assumes that the market price for risk and the spot rate process are specified respectively as:

$$(2.12) \quad q(r(t),t,T) = q$$

$$(2.13) \quad dr(t) = \alpha(\theta - r(t))dt + \beta dz(t) \quad \text{with } \alpha, \beta > 0 \text{ and } \theta \geq 0$$

i.e. the market price for risk is constant and the interest rate is an Ornstein-Uhlenbeck process, where:  $\theta$  is the long term rate,  $\alpha$  is the speed of adjustment to the long term rate.

If  $0 < \alpha < 1$ , then (2.13) describes a mean-reverting autoregressive process. Since  $\beta$  is constant, the spot rate fluctuates around its long term value  $\theta$ .

Unfortunately, this model might imply negative values for the interest rate. Yet, for high values of  $\alpha$  and  $\theta$  the probability of such a case is very low and hence the model has been widely applied.

The Dothan(1978) model

Dothan takes Vasicek's assumption over the market price for risk, but modifies the assumption over the interest rate process as follows:

$$(2.14) \quad dr(t) = \beta r(t) dz(t)$$

The value function  $v(r(t),t,T)$  are presented for the case of  $q=0$  and the result is the same as in the traditional pure expectation theory: the expected rate is equal to the spot rate. A good property of the solution is that the price of discount bonds remains above zero.

The Brennan and Schwartz(1979) two-factor model

The model presented so far imply that the price of discount bonds with different maturities are perfectly correlated. Brennan and Schwartz overcome this limitation by assuming that the term structure is determined by two factors: the spot rate  $r(t)$  and the long term rate  $l(t)$  which are governed by the following processes respectively:

$$(2.15) \quad dr(t) = \alpha_1(r,l,t)dt + \beta_1(r,l,t)dz_1(t)$$

$$(2.16) \quad dl(t) = \alpha_2(r,l,t)dt + \beta_2(r,l,t)dz_2(t) \quad \text{where:}$$

$dz_1(t)$  and  $dz_2(t)$  are Wiener processes with  $E[dz_1(t)]=E[dz_2(t)]=0$ ,  $dz_1^2(t)=dz_2^2(t)=dt$  and  $dz_1(t)dz_2(t)=\tau dt$ .

The idea of introducing the long term rate as a second factor<sup>(6)</sup> reflects the assumption which is the basis of both the traditional expectation literature and the liquidity premium theory.

As a consequence of the two-factor assumption, the PDE for the price of the default-free discount bonds contains two utility-dependent parameters: the market price for instantaneous and long-term risk. The latter is actually eliminated assuming that there exists a traded consol bond, which corresponds to the second state variable (the long-term interest rate).

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<sup>(6)</sup> Schaefer and Schwartz(1984) also set up a two-factor model, but they choose different factors, i.e. the long-term interest rate and the spread between the long term and the short term interest rate. Other two-factor models can be found in Richard (1976) and Cox, Ingersoll and Ross (1977).

## 2.4 Recent developments of the APT

In the present survey we have classified models according to the pricing methodology, i.e. arbitrage pricing or equilibrium pricing. Another dichotomy emerges in the literature between two approaches which differ in the starting point of their investigation.

The first takes as a starting point a plausible assumption on the short-term interest rate process and deduce from the model the current yield curve. Since the model used for deriving the yield curve can be either a APT or a GET model, we can include into this approach both APT models (e.g. Vasicek(1977), Dothan(1978) etc.) and GET models (e.g. Cox, Ingersoll and Ross(1985b), Longstaff and Schwartz(1992)).

The second approach takes the current term structure of interest rates as given and develops a no-arbitrage yield curve model so that the latter is perfectly consistent with the data, i.e. this second approach could be thought of as "a variation of the arbitrage approach" as Longstaff and Schwartz(1992) put it.

The models within this second approach represent the most recent development of the APT.<sup>(7)</sup>

The first paper that took this approach is that of Ho and Lee(1986). The authors take the initial bond prices and bond price processes as exogenously given, i.e. they take the current term structure as given. Within a discrete trading economy they impose that the movements of the term structure ensure the absence of arbitrage: specifically the bond prices are assumed to fluctuate randomly over time according to a binomial process. Once the arbitrage-free rates movement are determined, contingent claims can be priced relative to the observed term structure. Hence the model is a relative pricing model (i.e. relative to the observed term structure) whose main advantage is that of using the information content embedded in the current term structure to price contingent claims.

Heath, Jarrow and Morton(1992) generalize the model just described to a continuous time economy with multiple factors. In contrast with Ho and Lee, the authors take the initial forward rate curve as given. Then they describe its fluctuations by means of a continuous time stochastic process. They use Harrison and Kreps(1979) and Harrison and Pliska(1981) results to ensure

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<sup>(7)</sup> Within the APT, a different type of development is provided by Hull and White (1990). They extend Vasicek (1977) model assuming that the coefficient in the spot rate process are functions of time and choosing them to reflect the current and futures volatilities of the short-term interest rate.

absence of arbitrage and to price contingent claims respectively. The continuous time nature of the model also facilitates the estimation of the process parameters which may be problematic in Ho and Lee(1986).

The most valuable advantage with respect to the traditional APT models is that contingent claim valuations do not explicitly depend on the market price for risk, but only on observables and forward rate volatilities.

In our opinion, the main disadvantage of the approach taken by both Ho and Lee(1986) and Heath, Jarrow and Morton(1992) is that it does not explain the shape of the observed current term structure. Their approach represents mainly a theory for the evolution of the term structure and for pricing contingent claims relative to the observed term structure. Yet, it is not a theory of the term structure in the way we mean it in the present survey in that the above-mentioned models are not able to explain the shape of the observed structure taking it simply as exogenously given.

A nice comparative analysis of alternative approaches to constructing arbitrage-free yield curves models of the type we have briefly described in this section is given in Hull and White(1992).

### 3. THE GENERAL EQUILIBRIUM THEORY

#### 3.1 The framework

An alternative approach to describe the TSIR is the one based on general equilibrium. The theoretical framework for general equilibrium models of the TSIR is that of general intertemporal equilibrium of the asset market, the tools required for solving them are those of dynamic stochastic optimization. A pioneering work in this field is that of Merton(1970). Since Merton's model is aimed at pricing the capital structure of the firm, the TSIR is initially assumed to be flat or non-existent (i.e. no riskless rate assumption). The final extension of the model to the case of the interest rate stochastically varying over time allows the author: to *explain the existence of a term structure*; to *determine the shape of the term structure*; to be *consistent with other models*, e.g. no-arbitrage models.

The above-mentioned advantages implied by the general equilibrium setting remain as distinctive features of the famous Cox, Ingersoll and Ross(1985a,b) model. These papers move a step forward in that they : a) provide an intertemporal general equilibrium model where asset prices and their stochastic properties are endogenously determined since they depend on

variables of the underlying real economy; b) use the framework sub a) to study the TSIR so that many of the traditional factors influencing the term structure (anticipations, risk-aversion, investment alternatives and preferences) are introduced in a way which is consistent with agents' maximizing behaviour, equilibrium and Rational Expectations (RE).

Since Cox, Ingersoll and Ross(1985b) model - from now on CIR - is more complete<sup>(8)</sup>, has good empirical properties and has been widely used, we take it as the representative model of the general equilibrium approach.

### 3.2 The Cox-Ingersoll-Ross model (CIR)

In the mid-eighties Cox, Ingersoll and Ross have published two papers which represent milestones in the general theoretic approach to finance. The first, Cox, Ingersoll and Ross(1985a) "develops a continuous time general equilibrium model of a simple but complete economy and uses it to examine the behaviour of asset prices". The second, Cox, Ingersoll and Ross(1985b), uses the model developed in the previous paper to study the term structure of interest rates.

Given the technicalities involved by both of the papers, we will outline only features relevant for the comparison at issue.

#### 3.2.1 The basic model

Cox, Ingersoll and Ross(1985a) is a general equilibrium model which gives a complete intertemporal description of a continuous time competitive economy and gives equilibrium prices of any asset as solution of a partial differential equation which represents the fundamental valuation equation of the paper. In fact, the authors prove that the price of any contingent claim must satisfy the following differential equation :

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<sup>(8)</sup> The dynamics of the spot rate is still exogenous in Merton(1990) due to exogeneity in supply dynamics. By contrast it is endogenous in Cox, Ingersoll and Ross(1985b) since it is determined by the assumptions on the stochastic variable governing the real economy.

$$\begin{aligned}
 (3.1) \quad & 1/2(\text{Var}W)F_{ww} + \sum_{i=1}^k \text{cov}(W, Y_i)F_{wY_i} + 1/2 \sum_{i=1}^k \sum_{j=1}^k \text{cov}(Y_i, Y_j)F_{Y_i Y_j} + \\
 & + [r(W, Y, t)W - C^*(W, Y, t)]F_w + \\
 & + \sum_{i=1}^k F_{Y_i} \left[ \mu_i - \frac{-J_{ww}}{J_w} \text{cov}(W, Y_i) - \sum_{j=1}^k \frac{-J_{wY_j}}{J_w} \text{cov}(Y_i, Y_j) \right] + \\
 & F_t - r(W, Y, t)F + \delta(W, Y, t) = 0
 \end{aligned}$$

where:  $r(W, Y, t)$  is the equilibrium interest rate,  $W$  is wealth,  $F(\cdot)$  is the value of the claim,  $\delta$  is the payout and  $Y$  is the state of technology.

While the valuation equation (3.1) holds for any claim, the form of the payout  $\delta(W, Y, T)$  and the appropriate boundary conditions depend on the contractual conditions of the particular claim.

### 3.2.2 The model for the term structure

Cox, Ingersoll and Ross(1985b) obtain their model for the term structure by restricting and specializing some of the assumptions characterizing their general equilibrium model for asset pricing described in the previous section. In this section we will present the most basic model for the term structure described by the authors. The model is based on the following assumptions:

Assumption 1. The utility function, assumed to exhibit constant relative risk-aversion (CRRA) and to be independent of the state variable  $Y$ , is the following:

$$(3.2) \quad U(C(s), s) = e^{-is} \ln[C(s)]$$

This assumption on the preference structure implies that neither the risk-free rate nor the factor risk premium  $\Phi_Y$  depend on wealth. In addition the securities valued in the model have contractual terms which do not explicitly depend on wealth (i.e.  $F_w = F_{ww} = F_{wY} = 0$ ). Hence, the fundamental valuation equation becomes:

$$(3.3) \quad \frac{1}{2} \text{tr}(SS'F_{YY}) + [\mu' - a^*GS']F_Y + F_t + \delta - rF = 0$$

where  $SS'$  and  $GS'$  represent

Assumption 2. Technology is represented by a single sufficient statistic.

Assumption 3. The means and the variances of the rates of return on the production process are proportional to  $Y$ .

Assumption 4. The evolution of the state variable is specified as follows:

$$(3.4) \quad dY(t) = [y_1 Y + y_2] dt + v\sqrt{Y} dw(t)$$

where  $y_1$  and  $y_2$  are constants,  $y_2 \geq 0$  and  $v$  is a  $1 \times (n+k)$  vector of constant components  $v_0$ .

Under these assumptions we can specialize the results obtained in the previous section. In order to do that, we follow the authors in introducing three constants according to the following definitions:

$$\underline{\alpha} \equiv \alpha / Y ; \quad \Omega \equiv GG' / Y ; \quad \Sigma \equiv GS' / Y ;$$

The equilibrium interest rate can now be written as follows:

$$(3.5) \quad r(Y) = \frac{1' \Omega^{-1} \underline{\alpha} - 1}{1' \Omega^{-1} \mathbf{1}} Y$$

where  $\mathbf{1}$  is, as before, the unit vector. It is useful to rewrite (3.5) as:

$$(3.5') \quad r(Y) = \left[ \frac{1' \Omega^{-1} \underline{\alpha}}{1' \Omega^{-1} \mathbf{1}} - \frac{1}{1' \Omega^{-1} \mathbf{1}} \right] Y$$

It is easy to verify that the equilibrium interest rate is a linear function of the expected return on production and the variance of production return. The interest rate follows also a diffusion process whose drift and variance can be obtained using Ito's formula:

$$(3.6) \quad \text{drift} \quad r(Y) = \frac{1' \Omega^{-1} \underline{\alpha} - 1}{1' \Omega^{-1} \mathbf{1}} Y (y_1 Y + y_2) \equiv k(\theta - r)$$

$$\text{var} \quad r(Y) = \left[ \frac{1' \Omega^{-1} \underline{\alpha} - 1}{1' \Omega^{-1} \mathbf{1}} \right]^2 v v' Y \equiv \sigma^2 r$$

where  $k$ ,  $\theta$  and  $\sigma^2$  are constants with  $k\theta \geq 0$  and  $\sigma^2 > 0$ . The risk-free rate dynamics can be expressed as follows:

$$(3.7) \quad dr = k(\theta - r)dt + \sigma\sqrt{r} dz_1$$

where  $z_1(t)$  is a one-dimensional Wiener process such that:

$$(3.8) \quad \sigma\sqrt{r}dz_1(t) \equiv v\sqrt{Y}dw(t)$$

An essential characteristic of the present model is the endogenous nature of the interest rate process which is a continuous time first autoregressive mean-reverting process whose long term value is  $\theta$ .

The stochastic process for the risk-free rate is endogenous in that it depends on the basic assumptions made about the real economy (e.g. the stochastic process for technology and the preference structure) and it is derived as the process governing the evolution of the equilibrium interest rate. Hence different assumptions on technology, production and the preference structure would lead to a different form for  $dr$ .

The assumptions taken by Cox, Ingersoll and Ross are useful in that they imply the following desirable empirical properties for the interest rate: the mean reverting process ensures that the interest rate will never become negative; zero is an inaccessible boundary for the spot rate if  $2k\theta \geq \sigma^2$ ; the absolute variance is directly proportional to the interest rate itself; there is a steady state distribution for the interest rate itself.

At this stage we can face the problem of valuing default-free discount bonds promising to pay one unit at time  $T$ , i.e. we are able to determine the term structure.

Let  $P(r,t,T)$  be the price in  $t$  of a discount bond maturing at  $T$ , then the fundamental valuation equation becomes:

$$(3.9) \quad 1/2\sigma^2 r P_{rr} + k(\theta - r) P_r + P_t - r q' P_r - r P = 0$$

with  $P(r,T,T) = 1$

where  $q'$  is the factor risk premium, determined as explained below.

(3.9) can be obtained from the fundamental valuation equation (3.4) by substituting in (3.6) and taking into account that, under the assumptions taken in this section, the factor risk-premium is defined as follows:

$$(3.10) \quad \left[ \underline{\alpha} \Omega^{-1} \Sigma + \frac{1 - 1' \Omega^{-1} \underline{\alpha}}{1' \Omega^{-1} 1} 1' \Omega^{-1} \Sigma \right] Y = q' Y$$

Recalling Ito's formula the authors show that the expected rate of return on the bond is:  $r + (q' r P_r)/P$  and is proportional to its interest rate elasticity, where  $(q'r)$  represents the covariance of changes in the interest rate with percentage changes in optimally invested wealth.

A few comments are here in order. Cox, Ingersoll and Ross(1985b) do not provide any intuition for this result which is actually not easy to clarify within their model. A clear explanation can be found in Sun(1992). The author obtains a similar result for a two-period bond within a discrete-time model and maintains that the covariance measures the ability of the bond to hedge against unfavourable states. To see this, suppose that the covariance is negative, the bond is a bad hedge because has a low value when the marginal utility of wealth is high. Recalling from the Capital Asset Pricing Model that if a financial asset is a bad hedge it will be priced so to yield a positive premium, then we can conclude as follows: if the covariance between optimally invested wealth and the interest rate is negative, then the bond is a bad hedge and this implies a positive risk-premium and viceversa (recalling that  $P_r$  is always negative).

Second, bond prices depend in this model only on one random variable: the spot rate, i.e. the model is a one-factor model. The single-factor nature of the model depends on the initial assumption on technology. Hence, bond prices ultimately depend on the characteristics of the real economy.

By solving (3.10) taking the above-mentioned risk-adjusted expectations, it is possible to obtain bond prices in the following form:

$$(3.11) \quad P(r,t,T) = A(t,T) \exp\{-B(t,T) r\}$$

where  $A(\cdot)$  and  $B(\cdot)$  are non-linear functions of the factor risk premia  $q'$  and of the parameters of the model. (3.11) is consistent with common intuition on the behaviour of bond prices: they are a decreasing convex function of the interest rate, an increasing function of the time and a decreasing function of maturity.

Through the interest rate, the bond price is also sensitive to the parameters characterizing the interest rate process.<sup>(9)</sup> Dependence of the bond price on the market risk parameter  $q$  should also be stressed: as before, high values of  $q'$  imply a great covariance of the interest rate with wealth, thus bond prices will be higher when wealth is low and has a greater marginal utility.

By applying Ito's formula to (3.11) we can work out the dynamics of bond prices:

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<sup>(9)</sup> Sun(1992) stresses that the authors "incorrectly claim that their bond price is an increasing (decreasing) function of the speed of adjustment parameter if the current interest rate is greater (less) than the long-term mean" and gives a counter-example sustaining his critique.

$$(3.12) \quad dP = r[1-q'B(t,T)]Pdt - B(t,T)P\sigma\sqrt{rdz}_1$$

i.e. the returns on bonds are perfectly negatively correlated with changes in the interest rate, which is a typical feature related to the single-factor nature of the model.

The variability of return is lower when the interest rate is low and tends to disappear when the interest rate tends to zero. Moreover, variability decreases as the bond is closer to maturity time.

All the results obtained on bond prices can be restated in terms of yields to maturity:

$$(3.13) \quad R(r,t,T) = [rB(t,T) - \log A(t,T)]/(T-t)$$

which implies that as maturity nears the yield to maturity approaches the interest rate independently of the parameters. Solving the CIR model for the yield to maturity for all maturities, it is possible to obtain the whole term structure of interest rates.

Concluding on CIR, we can state that its fortune essentially hinges on four positive features of the models itself: i) it is parsimonious in the number of state variables (one); ii) it is parsimonious in the number of parameters to estimate (four); iii) the interest rate is a positive stationary process; iv) prices of discount bonds can be obtained in closed form.<sup>(10)</sup>

Moreover, as the authors suggest, the model presented in this section can also be extended in different ways, e.g. allowing the drift term in the interest rate process to be time dependent.<sup>(11)</sup> Another extension proposed by the authors stems from the need of overcoming the single factor nature of the model which makes return on bonds of all maturities perfectly correlated.

A third possible extension outlined by the authors implies the introduction of money and inflation by assuming that one state variable is the price level and that some contracts have payoff which depend on the price level.

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<sup>(10)</sup> The same is true for the prices of European options on bonds.

<sup>(11)</sup> Results obtained in such a case can be usefully compared with the traditional expectation assumption.

### 3.3 Other general equilibrium models

The CIR model has been widely used for empirical purposes, yet only a few extension of the same general equilibrium approach have been proposed since 1985.

The reasons for this relatively limited literature are to be found in the CIR model itself. In fact, it is sufficiently simple to have analytic tractability and enough sophisticated to make any deviation from the original assumptions too cumbersome to manage. Yet, the CIR model suffers from three main shortcomings:

- 1) the term premium of a given maturity discount bond displays the same sign for all states of nature;
- 2) the term premium of different maturities discount bonds has the same sign for any maturity;
- 3) the implied yield curve can attain only three shapes: monotone rising, monotone declining and humped.

The general equilibrium models we briefly review in this section have been set up in the attempt to overcome the above-mentioned drawback of the CIR paper.

In the following we will outline four papers which, as far as we know, are the only general equilibrium models appeared after CIR's.<sup>(12)</sup> The models will be exposed following the chronological order of their publication.

#### The Longstaff(1989) model

The author drops the assumption of linear production possibilities by allowing technological change to affect production returns in a non linear way. He does so by deriving what he calls a double square root process (DSR) for the interest rate dynamics, in contrast with the square root process (SR) used in CIR:

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<sup>(12)</sup> Sun(1992) paper is not described in this survey since it is just a discrete -time version of CIR. The main contribution of the paper is that of having "elucidated essential ideas behind the model" (i.e. CIR model) and of having investigated the relationship between real and nominal interest rates in a partial equilibrium setting in which an exogenous price level is correlated with the real economy. Yet, the latter issue is not our primary concern here since most available data are on nominal bond prices. Hence a term structure model can be estimated even without a maintained hypothesis on the statistical relation between real variable and the price level.

The assumption taken in the paper implies "a richer set of yield curves and term premium shapes without introducing additional state variables or parameters". In particular the model offers four nice results: i) the yield curve can display both humps and troughs as well as monotone or humped patterns of the term premiums; ii) discount bond prices and interest rates are not always inversely related; iii) discount bond riskiness need not be a monotone increasing function of maturity or duration; iv) the local expectation hypothesis can hold simultaneously for some bonds and not for others. Thus the paper overcome the main shortcomings of the CIR model. Unfortunately, as Costantinides(1992) points out "Longstaff's bond pricing equation is the wrong solution of the problem that he sets up."<sup>(13)</sup> Hence the paper has not had much follow up.

#### **The Hull and White(1990) model**

Hull and White(1990) present the same type of extension already proposed for Vasicek(1977) paper, i.e. they assume that the parameters  $k$ ,  $\theta$  and  $\sigma$  describing  $dr(t)$  in CIR are also functions of time.

The authors show that such an extension of the CIR basic model makes it consistent with both the current term structure and the current volatilities. In this sense the model should better be considered a no arbitrage model: we have already discussed this point (see section 2.4).

Unfortunately, the CIR extended model does not show the same analytic tractability of the original model and of the analogously extended Vasicek(1977) model.

#### **Costantinides (1992) model**

Costantinides sets up a model that overcomes all of the three main restrictive properties of the CIR original model. He does so by taking an approach that which is quite different from CIR and is based on Harrison and Kreps(1979) paper.

As the author clearly explains, "The starting point is the specification of a positive nominal state-price density process or pricing kernel guaranteed to exist by the absence of arbitrage in a frictionless market.". The pricing kernel  $M(t)$  is modelled as a squared autoregressive AR(1) independent process. The nominal price at time  $t$  of claim with nominal payoff  $X(T)$  at maturity  $T$  is given then by:

$$P(t) = E_t[X(T)M(T)]/M(t)$$

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<sup>(13)</sup> Costantinides' argument is that the solution does not satisfy the appropriate boundary condition: see Costantinides (1992), footnote (4).

where  $E_t$  is the expectation operator conditional on information available at time  $t$ .

Using such a pricing equation to price a default-free bond that pays one unit at  $T$ , the author derives his squared-autoregressive-independent-variable nominal term structure model referred to as SAINTS. The model share the same positive features of CIR, but not the three mentioned restrictive properties. In other words, the model outcomes imply that: i) the term premium may change sign depending on the state; ii) at a given state, the sign of the term premium may be different for different bonds; iii) the yield curve may have an inverted hump along with the more common shapes.

The author states also that results based on the pricing kernel can be interpreted in terms of a representative consumer economy, but he stresses that such a representation is unnecessary in his model.

Yet, we think that this feature is at the same time a shortcoming of the model. In order to go beyond CIR's results the author takes an approach which is analytically very elegant but little intuitive and thus he loses an important positive feature of CIR, i.e. the dependence of the term structure on the underlying real economy.

### **The Longstaff and Schwartz(1992) model**

The authors develop a model along the lines already pursued in the no arbitrage framework, i.e. they introduce an additional state variable into the analysis in order to avoid perfect correlation among returns on bonds of different maturities.

The model thus becomes a two-factor model with the two-factor being the short term interest rate and the volatility of the short term interest rate which is also "intuitively appealing since volatility is a key variable in pricing contingent claims".<sup>(14)</sup>

The model is developed analogously to CIR, the main difference being that the evolution of return on production depends now on two stochastic processes.

Using CIR's results (see section 3.2.2), they can obtain the equilibrium interest rate  $r(t)$  and its volatility  $V(t)$  and, applying Ito's Lemma, they can derive the corresponding dynamics.

As CIR, Longstaff and Schwartz model allows for closed form expressions for discount bond prices. In this case the price of a riskless unit discount bond

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(14) Fornari and Mele(1994) also set up a two-factor model which is comparable with Longstaff and Schwartz(1992), but the main purpose of the authors is to illustrate most recent econometric techniques for the term structure.

with  $T$  period until maturity is a function of three variables ( $r(t)$ ,  $V(t)$  and  $T$ ), depends on six parameters and implies the following nice properties for the term structure: i) both  $r(t)$  and  $V(t)$  have stationary distributions; ii) bond prices can be either positively or negatively related with maturity; iii) the effect of  $V(t)$  on bond prices is also indeterminate; iv) for a given maturity, the yield on a bond is a linear function of  $r(t)$  and  $V(t)$ ; v) the yield curve can show a greater variety of shapes, also those precluded by the single factor model.

All of these features can be explained by the presence of both the interest rate and the interest rate volatility as factors determining the term structure and can also be intuitively understood in relation to the real underlying economy. Hence Longstaff and Schwartz model overcomes CIR's drawbacks within the same general equilibrium setting, thus highlighting the interesting dependence of the term structure on the characteristics of the underlying real economy.

#### **4. A comparative evaluation of the two approaches**

In this section we compare the APT and the GET for the term structure of interest rates and we are immediately faced with a general question: when is a model a good model?

Even if it is impossible to provide an easy answer to this question we will try to do it in order to make a comparative evaluation of the two approaches at issue.

We think that a model for the TSIR should be judged on the basis of its tractability and its realism.

By tractability we mean that the model should be: a) analytically easy to manage; b) easily fitted to real data; c) easily applied to pricing IRS claims.

By realism we mean that the model should: d) show closeness to real behaviour; e) have explanatory power even on a theoretical ground.

The issue sub c) is mainly related to pricing options or futures on interest rates, a feature that we have not explicitly dealt with in this survey. Hence a comparative evaluation at that level goes beyond the purposes of the present paper.

The issue sub a) and e) are discussed in section 4.1, while the other issues are treated in section 4.2..

All the opinions we express are referred to the basic models of the APT and the GET, which have been respectively described in section 2.2 and 3.2, but

take also into account the developments of the basic models presented in the other sections.

#### 4.1 A theoretical evaluation

The first question here is what to compare in order to evaluate the TSIR theories stemming from the APT and the GET.

The TSIR is essentially given by the value function  $v(r,t,T)$  solving the PDE (2.8) for the APT and the zero coupon bond price  $P(r,t,T)$  solving the PDE (3.9) for the GET: both, considered as functions of maturity, give the whole term structure.

In fact, the value function  $v(r,t,T)$  represents also the price of a discount bond. We have used on purpose different symbols in the two different approaches in order to stress the theoretical differences in their derivation and to make the comparison more straightforward.

Recalling the derivation of the two mentioned functions, it is clear that both depend on the process governing the instantaneous riskless rate  $r(t)$ . Yet, precisely at this starting point comes the first main difference between the two approaches: in the APT such a process is *assumed* while in the GET it is *derived* from the assumption made on technology, i.e. on an underlying real variable.

This point has been often missed in the literature<sup>(15)</sup> and it is common to read that Cox, Ingersoll and Ross "assume" the process for  $r(t)$  while they actually derive it as the dynamic of an equilibrium rate.

Hence the PDE for  $r(t)$  is exogenous in APT and endogenous in GET. This feature stems essentially from the different equilibrium concept used: in contrast with GET, in the APT the no arbitrage condition corresponds to a partial equilibrium.

The equilibrium characterization used is also at the basis of the second main difference between the two theories. Both the PDE (2.24) and (3.20) display the interest rate risk-premium,  $q$  and  $q'$  respectively, as one of the coefficients. Yet, in the APT the interest rate risk premium  $q$  must be exogenously specified in order to close the model: different specifications of  $q$  give origin to different conclusions on the shape of the TSIR. Such an exogenous specification is undesirable because it may not be consistent with the absence of arbitrage. In other words arbitrage opportunities precluded by the no arbitrage condition could be reintroduced via an improper specification of  $q$ .

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(15) Among the few exceptions Longstaff and Schwartz(1992).

Hence, the assumption on  $q$  must be done carefully, but the APT does not provide a general criteria for doing that.

Moreover  $q$  is a market price for interest rate risk and naturally depends on the preference structure of the agents. Hence, what is a great advantage of APT for option pricing, i.e. a pricing mechanism which is independent of preferences, is lost when APT is used as a theory of the TSIR.

In contrast, the GET identifies the form of the risk premium endogenously in relation to the characteristics of the underlying economy: expected return on production, variance of production, covariance between production and technology (see eq.(3.21) and explanatory comments).

It should be said that the cost to be paid for this desirable result are the strong assumptions made in CIR and other GET paper about preferences and dynamics of state variables.

Moreover, recent GET models (see section 3.3) allow for a variety of shapes in the term structure which are also close to observed behaviour of interest rates.

At the level of analytic tractability, the GET models are definitely more difficult to manage but are able to offer closed form solution for the zero coupon bond prices.

They have higher explanatory power than the APT models because they describe the TSIR in relation to the assumptions made on the underlying economy. It would be interesting to further investigate how different assumptions about technology, production and preferences influence the result on the TSIR.

As a final comment, we compare GET models with recent models of the APT (see section 2.4). If we want a theory that explains the shape that we observe for the current term structure, then it is necessary to endogenize it. Models of the Ho-Lee type are unable to explain the current TSIR and can only provide a good theory for the evolution of the structure which exploits all the information contained in the observed structure and which is consistent with the initial interest rate curve.

## 4.2 An empirical evaluation

In this section we will try to compare the empirical performance of the two classes of models outlined in the previous sections.<sup>(16)</sup>

### 4.2.1 A comparison of alternative models

The theoretical literature on term structure models has developed very fast, and many applications of different models have appeared in the empirical literature. However, very little is known about the comparative performance of different models, in particular whether the theoretical advantages that we have identified for GET over APT models are retained in empirical applications.

A useful framework for nesting and comparing different models is provided by Chang, Karolyi, Longstaff and Sanders (CKLS) (1992). We start our comparative evaluation by summarizing some of their main results. We focus on four of the eight models used by CKLS in their comparison, which we report in table 1. Two of these models belong to the APT category, Vasicek (1977) and Dothan (1978), while the other two, Merton (1973) and CIR (1985), are GET.

The most commonly applied models are those by Vasicek (1977) and CIR (1985). However, the interesting result reached by CKLS is that these models perform poorly relative to less well-known models such as Dothan (1978).

First we summarize the approach used by CKLS, then we draw some conclusions from their main results. Their starting point is the following stochastic differential equation for the short-term interest rate:

$$(4.2.1) \quad dr = (\alpha + \beta r)dt + \sigma r^\gamma dz$$

This is an unrestricted process within which, by imposing appropriate restrictions on the four parameters  $\alpha$ ,  $\beta$ ,  $s$  and  $g$ , it is possible to describe the dynamics of the short rate riskless rate  $r$  implied by most models (both single-factor and multifactor). CKLS estimate the parameters of the process (4.2.1) by using the generalized method of moments (GMM) technique, test the restrictions imposed on the parameters by the eight alternative models and

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(16) Given their different nature, we have not included in the comparison the models described in section 2.4. For some comments on their empirical properties see Boero and Torricelli (1994).

analyse the capacity of the models to describe the volatility of the term structure. Their empirical analysis is conducted by using one-month Treasury bill yields over the period 1964-1989.

The GMM procedure (see Hansen, 1982) has been commonly applied in tests of interest rate and term structure models (see Harvey, 1988, Longstaff, 1989, and Longstaff and Schwartz, 1992). CKLS estimate the parameters of the continuous process (4.2.1) by applying the GMM technique to the discrete-time approximation:

$$(4.2.2) \quad r_{t+1} - r_t = \alpha + \beta r_t + e_{t+1}$$

$$(4.2.3) \quad E(e_{t+1})=0, E(e_{t+1}^2) = \sigma^2 r_t^{2\gamma}$$

where the variance of interest rate changes depends on the level of the interest rate.

The advantages of the GMM procedure are the following. First, it does not require normality for the distribution of interest rate changes. This is important as it allows a comparison of models of the term structure with different assumptions about the distribution of changes in  $r$ . For example, among the models listed in table 1, Vasicek and Merton assume normality for interest rate changes, while CIR (1985) assume proportionality to a non-central chi-squared. The asymptotic justification of the GMM relies only on the following assumptions: a) stationarity; b) ergodicity; c) existence of the relevant expectations. Second, both the GMM estimators and their standard errors are consistent, even in the presence of conditional heteroscedastic errors, serially correlated errors, or errors which are correlated across maturities. This is also important as heteroscedasticity is likely to occur in the presence of a temporal aggregation problem arising from estimation of a continuous time process with discrete time data.

The empirical analysis of CKLS is conducted in three steps. They compare the goodness of fit of the eight models considered. This is provided by a  $\chi^2$  measure, a model being misspecified if this measure has a high value. They evaluate the restrictions of each model against the general specification and the nested models with each other, by conducting an hypothesis testing procedure based on a  $\chi^2$  test analogous to the likelihood ratio test (Newey and West, 1987). Finally, they evaluate the ability of the models to capture the volatility of changes in the riskless rate. The latter plays a fundamental role in two important applications of models of the term structure: valuing contingent

claims; valuing hedging interest rate risk. An analysis of the latter applications goes beyond the purpose of the present paper.

The empirical results obtained by CKLS can be summarized as follows.

(i) In terms of  $\chi^2$  goodness of fit, the results indicate that the models by Merton (GET), Vasicek (APT) and CIR (1985) (GET) are misspecified, while the Dothan (APT) model could not be rejected, as they all have low values of  $\chi^2$ .

(ii) The ranking of the models based on the goodness of fit measure is strictly related to the value obtained for the parameter  $\gamma$ , which indicates the relation between interest rate volatility and the level of the interest rate. Those models for which  $\gamma$  is negative have been rejected, while those which assume a positive  $\gamma$  are not rejected. These results therefore suggest that the relation implied by the value of the parameter  $\gamma$  is very important, perhaps more relevant, as CKLS indicate, than other issues, such as negative interest rates, on which the debate over the relative merits of the different models has focused. Models as Vasicek and Merton allow for negative interest rates, and in fact they are often criticized for this feature. However, according to CKLS' results, the implication that interest rate changes are homoscedastic is a far more serious limit of these models. More on this will be said below, when we discuss the use of the GARCH approach to capture the dynamics of the variance of financial processes.

(iii) In the unrestricted model there seems to be weak evidence of mean reversion in the short term interest rate, as the parameter  $\beta$  is in general insignificant. The implication of this is that the additional complexity of models of the term structure which allow for mean reversion may not be justified empirically.

(iv) The results indicate a high sensitivity of the conditional volatility of the process to the level of interest rate, with an unconstrained estimate of the parameter  $\gamma$  of 1.499.

(v) Finally, in pairwise comparison no model which assumes the dependence of the conditional volatility on the interest rate level can be rejected in pairwise comparisons with models which do not make that assumption. For example, the Merton model is preferred to the Vasicek model. Therefore, a main result of the CKLS' empirical evaluation is that the most important feature of models of the interest rate is their ability to explain its time-varying volatility, rather than its identification with one (APT) or the other (GET) class of models.

**Table 1**  
**Description of models and implications for the dynamics of the short term interest rate**

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|                         |  |                         |
|-------------------------|--|-------------------------|
| Unrestricted process    | $dr = (\alpha + \beta r)dt + \sigma r^\gamma dz$ |                         |
| 1) Merton (1973) (GET)  | $dr = \alpha dt + \sigma r dz$                   | Brownian Motion + drift |
| 2) Vasicek (1977) (APT) | $dr = (\alpha + \beta r)dt + \sigma dz$          | Ornstein-Uhlenbeck      |
| 3) CIR (1985) (GET)     | $dr = (\alpha + \beta r)dt + \sigma r^{1/2} dz$  | Square Root process     |
| 4) Dothan (1978) (APT)  | $dr = \sigma r dz$                               | Geometric BM            |

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Models 1 and 2 imply constant conditional volatility of changes in riskless rate.

Model 3 implies conditional volatility of changes in  $r$  proportional to  $r$

Models 4 implies conditional volatility of changes in  $r$  proportional to  $r^2$

CKLS also report the coefficients of determination, the  $R^2$ , as a further measure of the relative performance of alternative models. This measure provides information about the goodness of fit (or forecast power) of each model for interest rate changes and interest rate volatility (squared interest rate changes). The  $R^2$  values are reported in Table 2 with the parameter estimates of the alternative models. The explanatory power for interest rate changes ( $R^2_1$ ) is very low for Vasicek and CIR (about 1% of the total variation in interest rate changes), while Merton and Dothan have zero explanatory power.  $R^2_2$  is zero for those models which assume constant volatility (Merton and Vasicek), while Dothan explains 13% of interest rate volatility and CIR only 5%.

It is interesting to note that the ranking indicated by the  $R^2$  measures is in line with that based on the GMM criterion, the important conclusion from CKLS being that the models which best describe the dynamics of interest rate over time are those which allow its volatility to be dependent on the level of interest rate. In particular both Vasicek which implies that the volatility of interest rate is constant, and the square root process which appear in the CIR single-factor general equilibrium term structure model (CIR) perform poorly relative to other less well known models (Dothan) with a better description of interest rate dynamics.

**Table 2 Empirical results**  
 source: Chang, Karolyi, Longstaff and Sanders (1992)

|                         | $\gamma$ | $\chi^2$<br>(p-value) | <i>df.</i> | $R^2_1$ | $R^2_2$ |
|-------------------------|----------|-----------------------|------------|---------|---------|
| Unrestricted process    | 1.499    |                       |            | 0.026   | 0.20    |
| 1) Merton (1973) (GET)  | 0.0      | 6.7579<br>(0.034)     | 2          | 0.0     | 0.0     |
| 2) Vasicek (1977) (APT) | 0.0      | 8.8467<br>(0.003)     | 1          | 0.0132  | 0.0     |
| 3) CIR-SR (1985) (GET)  | 0.5      | 6.1512<br>(0.013)     | 1          | 0.0164  | 0.05    |
| 4) Dothan (1978) (APT)  | 1.0      | 5.6018<br>(0.133)     | 3          | 0.0     | 0.134   |

**4.2.2 Different applications of the CIR model**

In a recent application of the CIR model to the italian term structure, Fornari and Mele (FM) (1994) have concentrated on the conditional heteroscedasticity of the residuals in the discrete time approximation

$$r_t = \alpha + (1-\beta)r_{t-1} + \sigma r_{(t-1)}^{0.5} e_t, \quad \text{with } e_t \text{ NID}(0,1) .$$

of the diffusion process implied by the CIR model:  $dr = (\alpha - \beta r)dt + \sigma r^{0.5} dz$  and estimate the CIR model by using the GARCH (Generalized Autoregressive Conditional Heteroscedasticity) technique. This procedure has been widely used to capture the complex dynamics of the volatility of most financial processes (see Engle, 1982, Bollerslev, 1986, and Pagan and Schwert, 1990) and therefore it seems to provide a better way to estimate the CIR model.

The GARCH representation is given by:

$$r_t = \alpha + \beta Z_t + e_t, \text{ where } e_t | I_{t-1} \sim N(0, h_t)$$

$$h_t = a_0 + a_1 e^2_{t-1} + \dots + a_p e^2_{t-p} + b_1 h_{(t-1)} + \dots + b_q h_{t-q}$$

where  $r(t) = \log(P_t/P_{t-1})$  ( $r_t$  is the return of the financial asset, and  $P_t$  is the price),  $Z_t$  is a vector of explanatory variables and  $I(t-1)$  is the information set

at time  $t-1$ . The model, defined as a GARCH (p,q) is estimated with ML methods.

(Under certain conditions (see Nelson, 1990) GARCH models adequately represents diffusion approximations). This representation explicitly accounts for the instability of the volatility over time and therefore represents a generalization of the one-factor CIR model. A further advantage of this approach to estimate the CIR model is that the conditional variance is endogenously estimated (joint estimation of the model), rather than being exogenously fixed. Finally, this procedure also gives an estimate of the unconditional variance which can then be used to price bonds according to the closed form solution (CIR, 1985).

Previous applications of the CIR model to the Italian term structure have followed two different approaches. One uses a market rate as an approximation to the instantaneous rate (for example, Barone and Cesari, 1986, employ the auction return of 3-months T-bills) and directly estimate the univariate discrete form of the CIR model. This specification, however, does not take into account the complex dynamics of the variance of  $r$  which can show significantly autocorrelated, while this can be conveniently captured by the GARCH models. The other approach uses a cross section of the prices of zero-coupon bonds (traded on both the over-the-counter and the screen-based markets) all observed at time  $t$  (see Barone, Cuoco and Zautzik, 1989) and estimate a nonlinear regression of the price vector on the matrix of the future income streams, following Brown and Dybvig (1986). Problems with nonlinear estimation of cross-section data identified by FM include small sample bias due to limited availability of data, and sensitivity of the algorithm to the initial values of the parameters and to the presence of outliers.

#### **4.2.3 Multifactor models**

Another direction which has been suggested to take heteroscedasticity into account is to generalise one factor models by describing volatility as a stochastic factor itself.

The theoretical two-factor models presented in the literature capture the time-variation of volatility, but their discrete approximations are either too hard to implement or they are not accurate. The first, by Fong and Vasicek (1991) has a discretised form which is very hard to implement (a stochastic variance

model with conditional heteroscedastic volatility). The second, by Longstaff and Schwartz, suggests estimation of a GARCH model in mean, which is not a good approximation of the continuous time model (see FM for a discussion of this point).

Here we concentrate on some empirical aspects of the LS model. Longstaff and Schwartz show that their model is able to capture the level and volatility of the term structure, and a very specific cross-sectional structure of yield changes, whereby changes in yields are known functions of changes in the short term interest rate and changes in its volatility, and this relationship depends on only four parameters.

To test their model they proceed in two steps. First they estimate the volatility of the short term interest rate by using the GARCH technique. Then they use these estimates to test the cross-sectional restrictions imposed by the model on the structure of yield changes. The model was tested for eight different maturities, ranging from three months to five years, and the results of the test, conducted by using the GMM procedure, support the model for both short-term and medium-term maturities. This is in contrast with previous empirical work which showed that equilibrium models of the term structure did not hold for maturities longer than one year (Longstaff, 1989). Longstaff and Schwartz provide further evidence against the CIR single-factor model. By applying the same procedure used to test their model, they find that the cross-sectional restrictions imposed by the CIR model on the evolution of the term structure are not consistent with the data. In particular, the CIR model does not imply a dependence of yield changes on volatility changes, while the evidence supports this relationship for all maturities.

Thus, the Longstaff and Schwartz model is theoretically appealing and it seems to be consistent with the data. However, as already pointed out, a major drawback is that the GARCH model in mean suggested by LS for the estimation is not a good approximation of the continuous time model, as indicated by FM. Therefore, it remains to be seen whether this model can provide a practical useful tool, and maintain its advantage over alternative models in more complex applications (in the valuation of contingent claims, for example).

Fornari and Mele (1994) propose a two-factor model (with instantaneous returns and variance) which overcomes the analytical difficulties of the

previous models. They derive a closed-form solution for the price of zero-coupon bonds whose discrete approximation is an AR(1)-GARCH(1,1) model, which gives some satisfactory results in an application to the Italian overnight rates. However, although the empirical results are so far satisfactory, its adequacy needs to be further tested in a wider range of applications and in comparison with alternative models.

## 5. Conclusions

The theory of the TSIR has historically developed in three main stages. In the first stage, the assumption of an economy with no uncertainty coupled with the no arbitrage assumption gave rise to the simplest term structure theory whose main implication was that the forward rate must be equal to the future spot rate. In the second stage, uncertainty was simply introduced as a modification of the previous theory and the stochastic version of the above-mentioned equality naturally followed: the forward rate must be equal to the expected value of the future spot rate. This approach gave rise to the different version of the traditional expectation hypothesis theory (e.g. the unbiased expectation hypothesis, the local expectation hypothesis).

The problem with traditional theories is that they are not always consistent with a notion of equilibrium in that they sometimes allow for profitable riskless arbitrage. It is only in the third stage, identifiable with the period that goes from the seventies to nowadays, that uncertainty has received an appropriate treatment.

The literature in this period has been very prolific following two mainstreams: the no-arbitrage approach and the general equilibrium approach. Both of them can be usefully compared with the traditional theories.

Yet, we think that on a theoretical ground GET is superior mainly for two reasons. First, relevant variables such as the spot interest rate and the interest risk premium are endogenous. Secondly, the relationship between the real and the financial side of the economy becomes a clear and important element in the understanding of the term structure.<sup>(17)</sup>

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<sup>(17)</sup>We have decided to dichotomize the literature in the way proposed in order to stress such a relationship. Other authors have followed different criteria to classify models (e.g. De Munnik).

As regards the applications, however, the advantages of GET over APT models are not so clear. In fact, the most important feature of TSIR models is their ability to capture the volatility of changes in the interest rate, rather than their identification with one or the other class of models. The importance of this feature emerges clearly from the comparative study by CKLS which shows that the relative performance of alternative models is closely related to their ability to capture the volatility of changes in the riskless rate. Thus, an interesting and surprising result of their study is that the CIR model (GET) performs poorly relative to the less well known model of Dothan (APT), as the latter capture the dynamics of the volatility better.

Within the GET approach, a solution to the shortcoming of the CIR model is given by the two-factor model of Longstaff and Schwartz, which explicitly captures the time-variation of volatility. LS have shown that the single-factor CIR model is not consistent with the data, the most important reason for the rejection of the model being that yield changes are only allowed to depend on changes in the short-term interest rate and not on changes in its volatility. However the LS generalization of the model increases its analytical difficulties, and this may limit its practical use in more complex applications.

Finally, based on practical rather than theoretical considerations, Fornari and Mele have proposed a two-factor model with a convenient analytical representation which can be adequately estimated using the recent econometric technique of the GARCH models. The same technique has been applied to the single-factor CIR model yielding more satisfactory results than previous estimation procedures.

To conclude, volatility plays a major role in important applications of term structure models, such as valuing contingent claims and hedging interest rate risk, and it is therefore a desirable property of any TSIR model to provide an adequate representation of it.

At the current state of the literature, there is no model that we can say outperforms the others both theoretically and empirically. Particularly on the empirical side, much comparative work across alternative models needs to be done to evaluate their consistency with the data, their relative performance and applicability to a wider range of applications.

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