APPLICATION OF INTERVALS OF POSSIBILITIES TO SOLVENCY

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ABSTRACT

We often meet with situations in which uncertainty, imprecision, vagueness, and lack of knowledge are present to such a degree, that probability and fuzzy models become improper tools for a description and analysis of the situation. In such cases the wealth of information assumed by these models, contradicts the lack of information that exists in reality.

Such situations may successfully be analyzed with intervals of possibilities, which provide inference about the uncertainty of the interval resulting from the applications of arithmetical operations to other input intervals. The interval's uncertainty is expressed by the width of the interval, ith its wealth of possible values, and is unrelated to probabilistic uncertainty which expresses randomness. We apply our results to equalization reserves, which are an important required element of the solvency rules in several countries.

1. INTRODUCTION

Real-life situations are often characterized by lack of knowledge and information. If we try to predict a situation in the future, vagueness and fuzziness often add to the uncertainty and imprecision that are due to randomness, lack of knowledge and lack of information.

This is the situation, for example, in the calculation of premiums for catastrophic insurance coverage. The calculation should take into consideration the risks and potential losses in the coverage period ahead as well as the risk category, including accommodation for projected future inflation of losses. Reserves must take in account incurred and fully reported losses; incurred but not yet reported losses (IBNR); incurred but only partially, or insufficiently, reported losses (IBNeR); and potential, not yet incurred, losses. Many of these reserves are, naturally, fully or partially unknown at the time the calculations are made.

Certain lines of insurance have considerable amount of experience to rely upon. These include, for example, life insurance, casualty insurance, motor insurance, and home insurance. These are actuarially well handled by risk theory, which essentially is an application of probability theory to insurance. However, insurance results are also greatly affected by vague, unpredictable factors such as recession, inflation, stagflation, a crash in the stock exchange market, introduction of trade restrictions, opening of the market between two countries, new laws with impact on insurance, newly discovered diseases like AIDS, and new discoveries like the impact of exposure to asbestos on health. These factors do not fit well within the theoretical framework of risk analysis, and often lead to the determination of insurance-specific quantities such as risk premiums, reserves and solvency margins by "feelings" and/or competitive market considerations. But is there really no way to improve actuarial calculations that may be "disturbed" or deviated by exogenous factors and lack of information?

We enhance the existing actuarial tools with intervals of possibilities that may be successfully applied even in cases where the information and/or knowledge about uncertain situations is very scant. The theory of intervals of possibilities that we introduce is closed with regard to interval arithmetic operations [1,10,12]. It can be coordinated with risk theory by including hardly predictable elements of risk.

2. THE CONTRADICTORY USE OF PROBABILITY THEORY AND FUZZY SETS

The theory of probability requires considerable amount of information. When a density function is selected to match a certain set of values or other experimental or environmental information, we resort to a vast amount of inherent knowledge. Often mathematically convenient functional forms are chosen, like Normal, Exponential, and Poisson distributions with specific parameters, to conform (or nearly conform) to the specific properties of the random values that potentially may be realized. In this way we reduce the informational load and simplify its representation. In many cases, however, the modeling accuracy of well known distribution functions, with their convenient analytical properties, and the choice of the parameters of the selected distributions may be questioned. This is certainly correct in real situations that are characterized by vagueness and lack of knowledge and information, as described in the introduction.

The experience that affects many insurable risks, like the health of a policy holder, usually changes risks continuously with time without abrupt jumps. However, when the risk premiums for a portfolio are determined, the usually non-homogeneous population of the policy holders in the portfolio is often divided into homogeneous categories; the premium for each category is then calculated under the assumption of homogeneity within the category. This process leads to artificial "jumps" when an insured risk passes the boundary between two categories, e.g., when a policy holder advances from one age group to another. This may lead to an abrupt, substantial change in premium that may not be justifiable. Lemaire [9] suggested to smooth the transition from one category to another, and make it gradual, through the introduction into insurance of <u>fuzzy set theory</u> and, more specifically, <u>fuzzy numbers</u>.

To describe and analyze gradual transitions, fuzzy set theory introduces <u>membership functions</u>. The fuzzy numbers which are used to determine fuzzy insurance premiums have membership functions which are continuous, strictly increasing below a certain value, and strictly decreasing above another, higher value [4]. Fuzzy numbers were further simplified into <u>triangular fuzzy numbers</u> [3, 5] in which the increasing and decreasing sections of the membership function are linear, and meet at the same point. Triangular fuzzy numbers, as well as (general) fuzzy numbers, are not closed with regard to arithmetical operations [1,3,4]; intervals of possibilities, in contrast, are closed under such operations [1]. As was the case of determining a density function for the description of an uncertain situation or event, one again has to resort to a vast amount of inherent knowledge to describe the membership functions of fuzzy sets and numbers.

<u>The Resulting Contradiction</u>

In both probability theory and fuzzy set theory, we are confronted with the following contradiction. Due to lack of suitable theories, situations that are characterized by vagueness and lack of information and knowledge, are analyzed by tools that require considerable exact knowledge and assume a level of precision that is not available in the existing data or in reality.

3. INTERVALS OF POSSIBILITIES (IPs)

We introduce intervals of possibilities (IP) to overcome the fundamental contradiction of analyzing a vague situation by tools that inherently require and assume amounts of knowledge and precision that are not available in the situation to be analyzed. The extreme values of an interval, its infimum and its supremum, are defined as the smallest and largest values that can be assumed or realized in an uncertain situation. These values can also be interpreted as the beliefs or estimates of a pessimist and an optimist, or as the "worst" and "best" realizations of an uncertain situation that an "average" person can imagine. Being beliefs, the chosen "extreme values" may or may not be actually extreme, and we cannot always assure that the defined IP should not be extended or reduced.

Example:

In industrial property insurance, the calculation of the risk premium is often based on the "maximum probable loss" or on the "estimated maximum loss". We can use an IP to analyze the insurance payment for a loss event and determine the absolute value of the loss payment in currency units. The infimum of the IP is, with certainty, not smaller than zero; it will equal zero, for example, if the loss event is excluded from the coverage, or if the loss amount does not exceed the insured's retention. The supremum of the IP, on the other hand, may be exceeded with some unknown but low probability if it is chosen to be the maximum probable loss, and it will be exceeded with higher probability if it is chosen as the estimated maximum loss. A supremum value for the loss, that with certainty will not be exceeded by the insurance payment, is the actual insured amount.

All values within an IP are possible values, while any value outside the IP is considered to be impossible or unachievable. In particular, in addition to associating the infimum and supremum of an IP with the perceptions of an "average" person, we introduce a plausible value that this person believes to be a reasonable possible realization. It is, naturally, somewhere within the IP; it should not, however, be identified as the "true value" that may result from a random or vague situation. While the plausible value may sometimes be ignored, it is a useful guide for decision making.

Analytically, an IP is an ordered triplet of real values $B = (\underline{b}, b, \bar{b})$, where $\underline{b}$ is the infimum value, b the plausible value and $\bar{b}$ the supremum value. We also denote the IP as $B = [\underline{b}, \bar{b}]$; this is used when the plausible value is immaterial. IPs are, of course, generalizations of real numbers. In a deterministic situation the IPs are "collapsed" into the limiting case of real numbers; thus we call a real number "a redundant IP" and denote it as $\bar{B}$

It should be recognized that the theory of intervals of possibilities is the theory that requires the least knowledge possible, among all analysis and decision theories that deal with ambiguity and uncertainty, if we wish to get better results than just "feelings".

4. IP ARITHMETICS

The IP arithmetics that we define below follows interval arithmetics techniques as developed within the computer science domain [6,10,12]. They define the IP that results from the application of arithmetical operations to input IPs.

A partial order can be applied to IPs:

$A \leq B$ if $\underline{a} \leq \underline{b}$, $a \leq b$ and $\bar{a} \leq \bar{b}$, while $A < B$ if $A \leq B$ and at least one of the three defining inequalities is strict. In particular, $\bar{A} \leq B$ if $a \leq \underline{b}$ and $\bar{A} \geq B$ if $\bar{b} \leq a$.

Sometimes we are interested in IPs which contain only positive values, e.g., when we consider a loss IP (as in the former example) from which non-payment of insurance is excluded. We call these <u>positive intervals of possibilities (PIP)</u>. An IP B is a PIP when $\underline{\bar{0}} < B$, i.e., when $0 < \underline{b}$.

These definitions also imply that the IP B is negative if $\underline{\bar{0}} > B$, and that $A = B$ if $\underline{a} = \underline{b}$, $a=b$ and $\bar{a} = \bar{b}$, and $A \neq B$ otherwise.

IP are intervals, and as such are subjected to set operations:

An IP A is included in an IP B, or $A \subseteq B$, if $\underline{b} \leq \underline{a} \leq \bar{a} \leq \bar{b}$. Intersection is defined by $A \cap B = [\max(\underline{a}, \underline{b}), \min(\bar{a}, \bar{b})]$ if $\max(\underline{a}, \underline{b}) \leq \min(\bar{a}, \bar{b})$. The intersection is empty, in which case we define $A \cap B = \underline{\bar{0}}$ if $\bar{b} < \underline{a}$ or $\bar{a} < \underline{b}$. The union always is non-empty, as it is defined by $A \cup B = [\min(\underline{a}, \underline{b}), \max(\bar{a}, \bar{b})]$.

Note that the plausible value is not defined for the set operations. Rather, the resulting plausible value may be anywhere within the resulting IP.

The <u>width</u> of an IP A, which is defined as $w[A] = w[\underline{a}, \bar{a}] = \bar{a} - \underline{a}$, can be used as a measure for the interval's dispersion. The smaller the width, the narrower the interval, and the less uncertain is the knowledge of the plausible value. We then have:

Theorem 1:

a) $w[A \cap B] \leq w[A]$, $w[A \cap B] \leq w[B]$

b) $w[A \cup B] \geq w[A]$, $w[A \cup B] \geq w[B]$

Proof:

a) If the intersection is not empty,

$w[A \cap B] = \min(\overline{a},\overline{b}) - \max(\underline{a}, \underline{b}) \leq \min(\overline{a},\overline{b}) - \underline{a} \leq \overline{a} - \underline{a} = w[A]$

and similarly for $w[A \cap B] \leq w[B]$.

For $A \cap B = \underline{\overline{0}}$ we have

$w[A] = \overline{a} - \underline{a} \geq 0 = w[A \cap B]$, and similarly for $w[B] \geq w[A \cap B]$.

b) $w[A \cup B] = \max(\overline{a},\overline{b}) - \min(\underline{a}, \underline{b}) \geq \max(\overline{a},\overline{b}) - \underline{a} \geq \overline{a} - \underline{a} = w[A]$

and similarly for $w[A \cup B] \geq w[B]$. q.e.d.

To assure that the IPs are closed under arithmetical operations (in contrast to fuzzy numbers and triangular fuzzy numbers), we demand that the resulting IP should contain all the points generated from the application of an arithmetical operator to pair of points in the input IPs. Thus, if A and B are IPs and $\odot$ is an arithmetical operator, we require that

$$A \odot B = (\min_{a'\varepsilon A, b'\varepsilon B} (a' \odot b'), a \odot b, ,\max_{a'\varepsilon A, b'\varepsilon B}(a' \odot b')) \tag{1}$$

Theorem 2:

The resulting interval on the right hand side of (1) is the smallest of all possible resulting intervals.

Proof:

All pair combinations $a' \odot b'$, with $a'\varepsilon A$ and $b'\varepsilon B$, are included in the resulting interval. Thus any resulting IP must have an infimum $\leq \min_{a'\varepsilon A, b\varepsilon B'} (a' \odot b')$ and a supremum $\geq \max_{a'\varepsilon A, b'\varepsilon B} (a' \odot b')$.

q.e.d.

When equation (1) is applied to addition and substraction, the result can be simplified:

$\underline{A + B} = (\underline{a} + \underline{b}, a+b, \overline{a} + \overline{b})$ (2a)

$\overline{0} - B = -B = (-\overline{b}, -b, -\underline{b})$ (2b)

$A - B = A + (-B) = (\underline{a} - \overline{b}, a - b, \overline{a} - \underline{b})$ (2c)

Addition is easily seen to be associative and commutative. Furthermore, $\overline{0} = (0,0,0)$ is the unit element for addition and substraction. Note, though, that

$B - B = (\underline{b} - \overline{b}, 0, \overline{b} - \underline{b}) \neq \overline{0}$ for $\underline{b} \neq \overline{b}$ (2d)

Theorem 3:

If A and B are IPs, then $w[A + B] = w[A - B] = w[A] + w[B]$

Proof:

$w[A + B] = w[\underline{a} + \underline{b}, \overline{a} + \overline{b}] = (\overline{a} + \overline{b}) - (\underline{a} + \underline{b}) =$

$= (\overline{a} - \underline{a}) + (\overline{b} - \underline{b}) = w[A] + w[B]$

Similarly, $w[A - B] = (\overline{a} - \underline{b}) - (\underline{a} - \overline{b}) =$

$= (\overline{a} - \underline{a}) + (\overline{b} - \underline{b}) = w[A] + w[B]$ q.e.d.

Corollary 1:

If A and B are non-redundant IPs, their addition and substraction intervals have a larger width (and thus also uncertainty) than the input intervals.

Proof:

Since A and B are non-redundant IPs, $w[A] = \overline{a} - \underline{a} > 0$ and $w[B] = \overline{b} - \underline{b} > 0$. Therefore $w[A + B] = w[A - B] = w[A] + w[B] >$

$> w[A]$, and similarly for $w[B]$. q.e.d.

Corollary 2:

Addition and substraction of a redundant IP $\overline{B}$ does not change the width of the other input interval.

Proof:

Since B is a redundant IP, $w[B] = \overline{b} - \underline{b} = 0$. Therefore

$w[A + B] = w[A - B] = w[A] + w[B] = w[A]$. q.e.d.

Theorem 3 and its corollaries show that addition and substraction are non-decreasing operations with regard to the width, and thus also the uncertainty, that is described by the IPs. Furthermore, for non-redundant IPs, these operations result in a strictly increasing uncertainty.

With regard to multiplication, we distinguish between 3 categories of IPs: positive IPs (PIP for which $\underline{a} > 0$), negative IPs (with $\overline{a} < 0$), and IPs that include 0 (for which $\underline{a} \leq 0 \leq \overline{a}$). Consequently, nine distinguished results may occur, according to the categories of the two multiplicands (see, e.g., Petkovic [12]). For example:

$$A \cdot B = (\underline{a} \cdot \underline{b},\ a \cdot b,\ \overline{a} \cdot \overline{b}) \quad \text{if} \quad 0 \leq \underline{a} \quad \text{and } 0 \leq \underline{b} \tag{3a}$$

$$A \cdot B = (\overline{a} \cdot \overline{b},\ a \cdot b,\ \underline{a} \cdot \underline{b}) \quad \text{if} \quad \overline{a} \leq 0 \quad \text{and } \overline{b} \leq 0 \tag{3b}$$

$$A \cdot B = (\underline{a} \cdot \overline{b},\ a \cdot b,\ \overline{a} \cdot \underline{b}) \quad \text{if} \quad \overline{a} \leq 0 \quad \text{and } 0 \leq \underline{b} \tag{3c}$$

$$A \cdot B = (\overline{a} \cdot \underline{b},\ a \cdot b,\ \underline{a} \cdot \overline{b}) \quad \text{if} \quad 0 \leq \underline{a} \quad \text{and } \overline{b} \leq 0 \tag{3d}$$

Note that in these four cases, as opposed to the other five cases, 0 is neither included in A nor in B. Further, in contrast to addition, where (2a) is valid irrespective of what categories A and B belong to, we have to distinguish for A·B between nine different cases that depend on the signs of the end-points of A and of B. In practice, (3a) is the most useful case; it also includes, as a special case, the multiplication of two PIPs.

<u>Theorem 4</u>:

Suppose A and B are PIPs, then

$w[A \cdot B] = \bar{b} \cdot w[A] + \underline{a} \cdot w[B] = \underline{b} \cdot w[A] + \bar{a} \cdot w[B]$

<u>Proof</u>

$w[A \cdot B] = \bar{a} \cdot \bar{b} - \underline{a} \cdot \underline{b} = \bar{a} \cdot \bar{b} - \underline{a} \cdot \bar{b} + \underline{a} \cdot \bar{b} - \underline{a} \cdot \underline{b} = \bar{b} \cdot w[A] + \underline{a} \cdot w[B]$

$= \bar{a} \cdot \bar{b} - \bar{a} \cdot \underline{b} + \bar{a} \cdot \underline{b} - \underline{a} \cdot \underline{b} = \bar{a} \cdot w[B] + \underline{b} \cdot w[A]$ q.e.d.

<u>Corollary 3</u>:

Suppose A is a PIP and $b \geq 1$. Then a multiplication of A by B increases uncertainty.

<u>Proof</u>

$w[A \cdot B] = \underline{b} \cdot w[A] + \bar{a} \cdot w[B] \geq w[A]$ q.e.d.

<u>Corollary 4</u>:

Suppose $\underline{a} > 0$ and $\bar{b} = b = \underline{b} > 0$ i.e. $B = \underline{\bar{B}}$ Then $w[A \cdot B] = b \cdot W[A]$, and thus

a) for $b < 1$: $w[A \cdot B] < w[A]$

b) for $b = 1$: $w[A \cdot B] = w[A]$

c) for $b > 1$: $w[A \cdot B] > w[A]$

To see the usefulness of corollary 4, consider the calculation of the future value of a portfolio that is represented by a PIP A. If the interest rate is $i > 0$, the value of the portfolio after n years will be $(1+i)^n \cdot A$, which according to corollary 4 is more uncertain than the present IP A.

Multiplication, like addition, is associative and commutative. The unit element is $\underline{\bar{E}}(1,1,1)$. It also obeys $A \cdot \bar{0} = \underline{\bar{0}} A = \underline{\bar{0}}$ Multiplication is not distributive, as $A \cdot (B-B) \neq A \cdot B - A \cdot B$ whenever B is non-redundant0 IPs are, however, sub-distributive (see [1]), i.e., obey $A \cdot (B+C) \subseteq A \cdot B + A \cdot C$. Furthermore, when A,B and C are PIPs we have $A \cdot (B+C) = A \cdot B + A \cdot C$, justifying our statement that PIPs can often be applied more comfortably and lead to more specific results than the more general IPs.

Interval division is well defined when division by zero is precluded, i.e., when 0 is not included in the denominator interval. In this case, (1) leads to

$$1/B = \{1/b' : b' \varepsilon B\} = (1/\bar{b}\ ,\ 1/b,\ 1/\underline{b}) \tag{4a}$$

with the bounded width $1/\underline{b} - 1/\bar{b}$. In particular, when A and B are PIPs, we get from (3a) and (4a)

$$A/B = A\cdot(1/B) = (\underline{a}/\bar{b},\ a/b,\ \bar{a}/\underline{b}) \tag{4b}$$

Similarly, when both A and B are negative IPs, we have from (3b) and (4a)

$$A/B = (\bar{a}/\underline{b},\ a/b,\ \underline{a}/\bar{b}) \tag{4c}$$

The other cases, corresponding to (3c) and (3d), follow easily

$$A/B = (\underline{a}/\underline{b},\ a/b,\ \bar{a}/\bar{b}) \ \text{ if } \ \bar{a} < 0 \ \text{ and } \ 0 < \underline{b} \tag{4d}$$

$$A/B = (\bar{a}/\bar{b},\ a/b,\ \underline{a}/\underline{b}) \ \text{ if } \ 0 < \underline{a} \ \text{ and } \ \bar{b} < 0 \tag{4e}$$

If 0 is included in the divisor IP, i.e., $\underline{b} < 0 < \bar{b}$, we have that $1/b$ approaches $+\infty$ or $-\infty$ as b approaches zero from above or below. 1/B then becomes the whole real line, excluding the interval $(1/\underline{b}\ ,\ 1/\bar{b})$. As an IP it does not exclude any other possibility, and therefore is too "rich" and has little useful information. This is an extreme case, when an operation (division, in this case) moves us from a finite, information containing interval, to an infinite interval. It describes the transition from limited uncertainty to almost total uncertainty, in contrast to the other borderline case in which an IP B collapses into a real number $\bar{B}$ describing the transition from uncertainty to certainty.

Therefore, to be of value, infinite intervals should be excluded from considerations; only finite intervals are taken in account. Consequently, we exclude division by an IP which includes zero. Thus, while (4b)-(4e) are in a sense the inverse operations to (3a)-(3d), there are no inverse operations to

3 of the other 5 multiplication cases in which the second argument includes zero. It is of central importance to emphasize that the IPs we deal with include only the minimal requirements that are needed for getting meaningful results when dealing with uncertain situations.

When power operations are considered, namely when the IP A^B is defined, the usefulness of PIPs becomes eminently clear. The controlling equation is again (1), which in many cases may be simplified. For practical purposes, we will assume that A is a PIP; $\underline{a} < 0$ may lead to meaningful or invalid results for certain exponents B. In financial mathematics, A will usually be some functional of an interest or discount factor, and thus indeed a PIP.

As was the case in multiplication, we must distinguish several cases. For $A > 0$, we should distinguish between A greater than 1 (i.e., $\underline{a} > 1$), A less than 1 (where $\bar{a} < 1$), and A that contains 1 (in which $\underline{a} < 1 < \bar{a}$). For B we have to consider positive B, negative B, and an IP B which includes the zero. These cases are particularly meaningful when cash flows are evaluated. For example, $\underline{a} > 1$ is an inflation factor, while $\bar{a} < 1$ implies deflation or discounting. When A contains 1, we must take in account both inflation and deflation, and make our decisions accordingly. Similarly, when A represents the interest factor (1+i), $\underline{b} > 0$ can be used for future values while $\bar{b} < 0$ is useful for present values. If $\underline{b} = \bar{b}$, i.e., $B = \underline{\bar{B}}$ we are sure about the timing of payments in the future, while if $\underline{b} < \bar{b}$, the number of payments and the timing of, for example, the last payment are uncertain. These notions are illustrated in [1] where IPs for callable bonds are demonstrated.

Consequently, we must distinguish between the following nine cases:

$$A^B = (\underline{a}^{\underline{b}}, a^b, \bar{a}^{\bar{b}}) \quad \text{if } \underline{a} \geq 1 \text{ and } \underline{b} \geq 0 \tag{6a}$$

$$A^B = (\underline{a}^{\bar{b}}, a^b, \bar{a}^{\underline{b}}) \quad \text{if } 0 < \underline{a} < \bar{a} \leq 1 \text{ and } \underline{b} \geq 0 \tag{6b}$$

$$A^B = (\underline{a}^{\bar{b}}, a^b, \bar{a}^{\bar{b}}) \quad \text{if } 0 < \underline{a} < 1 < \bar{a} \text{ and } \underline{b} \geq 0 \tag{6c}$$

$$A^B = (\bar{a}^{\underline{b}}, a^b, \underline{a}^{\bar{b}}) \quad \text{if } \underline{a} \geq 1 \text{ and } \bar{b} \leq 0 \tag{6d}$$

$$A^B = (\bar{a}^{\bar{b}}, a^b, \underline{a}^{\underline{b}}) \quad \text{if } 0 < \underline{a} < \bar{a} \leq 1 \text{ and } \bar{b} \leq 0 \tag{6e}$$

$$A^B = (\bar{a}^{\underline{b}}, a^b, \underline{a}^{\underline{b}}) \quad \text{if } 0 < \underline{a} < 1 < \bar{a} \text{ and } \bar{b} \leq 0 \tag{6f}$$

$$A^B = (\bar{a}^{\underline{b}}, a^b, \bar{a}^{\bar{b}}) \quad \text{if } \underline{a} \geq 1 \text{ and } \underline{b} < 0 < \bar{b} \tag{6g}$$

$$A^B = (\underline{a}^{\bar{b}}, a^b, \underline{a}^{\underline{b}}) \quad \text{if } 0 < \underline{a} < \bar{a} \leq 1 \text{ and } \underline{b} < 0 < \bar{b} \tag{6h}$$

$$A^B = (\min[\underline{a}^{\bar{b}}, \bar{a}^{\underline{b}}], a^b, \max[\underline{a}^{\underline{b}}, \bar{a}^{\bar{b}}]) \quad \text{if } 0 < \underline{a} < 1 < \bar{a} \text{ and } \underline{b} < 0 < \bar{b} \tag{6i}$$

Note that when $B = \bar{0}$ we get, as expected, that $A^B = \bar{1}$ for all cases. Further, the nine cases are demarcated by the unit elements for addition and multiplication, $\bar{0}$ and $\bar{1}$. Of particular interest is case (6i), where both A and B include the unit elements (1 and 0) as inner elements (rather than as infimum or supremum). It is the most complex case, and the only one where a closed form expression for the extreme values cannot be directly provided.

The arithmetic operators discussed above can be extended to any real function. Petkovic [12] shows that if f is a rational function of n real variables, and F is its interval extension in which the real variables of f are extended to intervals, then the interval value of F contains the range of values of the corresponding real function f. This theorem is valid whenever the real arguments of f lie in the corresponding intervals used in the evaluation of F.

5. APPLYING INTERVALS OF POSSIBILITIES TO SOLVENCY OF INSURERS

Investigations of minimum solvency conditions for insurance companies are made for the protection of insurance consumers by the state. To assure that insurance consumers get the protection they deserve for the premium they pay, even in economically difficult times, the state has to enforce minimum solvency conditions on insurance companies. Such conditions constrain the ability of the insurance companies to provide coverage. Therefore, the companies must carefully plan for and monitor the fulfillment of their solvency constraints.

Recent developments in risk theory and data processing were used in different countries to determine and improve long-term insurance companies' strategies [11,13]. However, it is unclear as to what extent risk theory can be used to describe complex future portfolio situations for insurance companies. Major components of this limitation are the vagueness, lack of knowledge and imprecision that exist for insurance companies throughout the determination of strategy to comply with irregularly changing solvency rules with changing conditions.

An important element of solvency strategies is the equalization reserve. The EC solvency margin rules were set up in the 1970's for all direct insurance companies, as opposed to reinsurance companies, that work in the EC. Further regulations were introduced in Germany for minimal equalization reserves, to protect insurance companies against cyclical variations in risk exposure. The regulation of equalization reserves as special solvency rules in Germany is based on works by Becker [2], Helten [7], Karten [8] and others.

An equalization effect is applied in the following way. When the underwriting results are positive, the profit is fully or partially carried over into the equalization reserve (ER). On the other hand, if the results are negative, the loss can be reimbursed from the ER. The equalization effect thus leads to the regulation of the transfer into or out of the ER; analytically,

$$\Delta u = f(u\ , p\ , l\ , \bar{l}\ , r_t\ , a) \tag{7}$$

(see Pentikainen [11], § 7.3, 7.5 and 7.8], where

u is the initial amount of ER, for example at the beginning of the year,

p are the net premiums earned,

l is the loss ratio, i.e. claims (paid and outstanding) divided by premiums p,

$\bar{l}$ is a moving average loss ratio, often for the last 10 years

$r_t = 1 + i_t$ is an interest factor for time t

a is a control parameter, which each insurer may select individually within certain limits (for example, $0 \leq a \leq 0.15$).

The control parameter "a" can well be represented by an interval of possibilities. Pentikainen demonstrates with a sensitivity test [11, § 7.12.2] that the control parameter in the transfer equation (7.5.1.) of [11] plays a central role, when a target zone for ER with an upper limit u_2 and a lower limit u_1 is introduced. Replacing the control parameter with an IP, thus better matching reality, can therefore lead to more extensive and non-deterministic results for Δu in (7).

An extension of the control parameter from a real variable to an interval variable leads to an interval-dependent extension of the original function. In particular, using (7.8.1) of Pentikainen [11], we get the interval function

$$U(t) = r_t \cdot U(t\text{-}1) + (\bar{l}(t) - l(t) + A) \cdot p(t) \tag{8}$$

that includes the real value

$u(t) = r_t \cdot u(t\text{-}1) + (\bar{l}(t) - l(t) + a) \cdot p(t).$

In this extension, U and A are the interval extensions of the real values u and a. It is clear that U(.) coincides with u(.) if the interval A degenerates to the control parameter a. We also assume that the interest factor $r = r_k = (1 + i_k) > 1$, k=1,...n changes with time.

Now denote

$B(t) = (\bar{l}(t) - l(t) + A(t)) \cdot p(t),$

and assume that the control IP A(t) also changes with time. B(t) is a function of A(t), and thus is itself an IP. $\bar{l}(t)$, l(t) and p(t) may be considered as degenerate IPs. B(t) is, of course, the interval extension of $b(t) = (\bar{l}(t) - l(t) + a) \cdot p(t)$. (8) then simplifies into

$$U(t) = r_t \cdot U(t\text{-}1) + B(t) \quad \text{and} \quad u(t) = r_t \cdot u(t\text{-}1) + b(t). \qquad (8a)$$

We assume that $r_t > 1$; then, according to corollaries 1 and 4 we conclude from (8a) that

$$w[U(t)] > w[U(t-1)] \quad for \quad t=1,...,n, \quad so\ that$$

$$w[U(n)] > w[U(n-1)] > ... > w[U(1)] > w[U(0)]$$

In other words, the part of the equalization reserve's uncertainty that depends on lack of knowledge, and is described by intervals of possibilities, is increasing the farther we look into the future.

We would like now to understand the impact of the time span, from estimation to realization, on the estimate's uncertainty in the IP theory. What, for example, is the difference in the ER-uncertainty for year n, between an estimate that is based on year n-1 data (1 year apart) and an estimate that is based on the initial year 0 data (n year difference)? A-priori, we would expect that the former estimate will be more exact, due to the shorter time span and the increased level of knowledge (we know already the data for years 1,2,...,n-1).

To find an answer to this question, we must be able to explicitly express the increased knowledge that is accumulated each year. From (8a) we immediately get that

$$\boxed{U(k) = \Big(\prod_{i=1}^{k} r_i\Big)\, U(0) + \sum_{i=1}^{k} \Big(\prod_{j=i+1}^{k} r_j\Big)\, B(i) \quad \mathit{for} \quad k=1,\ldots,n \qquad (9)}$$

We will now assume that at year's end the IP B(k-1) has been realized and thus has been degenerated into a real value b_{k-1} for k=2,...,n. We then replace the IP B(i) in (9) with the real number b_i. Similar replacements can be made if other parameters, such as r_i, are actually IPs. Thus, immediately at the beginning of year k, with only the information gathered up to the end of year k-1, we have the ER for start of year k which is based on actual knowledge of data up to the end of year k-1 (i.e., just before the beginning of year k):

$$\boxed{U^{b}(k) = \Big(\prod_{i=1}^{k-1} r_i\Big)\, U(0) + \sum_{i=1}^{k-1} \Big(\prod_{j=i+1}^{k-1} r_j\Big)\, b_i \quad \mathit{for} \quad k=2,\ldots,n \quad (10)}$$

and thus at the end of year k:

$$\overline{U}(k) = r_k U^b(k) + B(k)$$
$$= (\prod_{i=1}^{k} r_i)\ U(0) + B(k) + C$$
$$\text{where} \quad C = \sum_{i=1}^{k-1} (\prod_{j=i+1}^{k} r_j)\ b_i \quad \text{for } k=2,\ldots,n \qquad (11)$$

Consequently, U(k) is the estimate of the ER, using IPs, as made at year 0 for a period k years ahead, while $\overline{U}(k)$ is an estimate made at the beginning of year k, just a year ahead. Comparing by (9), (10) and (11) the widths of these estimates, we find that theorems 3 and 4 and their corollaries then imply

$$w[U(k)] - w[\overline{U}(k)] = \sum_{i=1}^{k-1} (\prod_{j=i+1}^{k} r_j)\ w[B(i)] > 0 \quad \text{for} \quad k=2,\ldots,n$$

which demonstrates explicitly the increase in the width of U(k) if we move the estimation of U(k) from the beginning of year k backwards to year zero.

We now turn our discussion from the special case of equalization reserves to the general concept of the required solvency of insurance companies. The concept of solvency margin in the EC is comprised of the margins of:

a. equity capital such as shares capital, statutory and free reserves, and profits carried forward;
b. underestimation of assets; and
c. overestimation of liabilities, such as provisions for losses in investments and for extra amounts in insurance liabilities.

The EC solvency ratio standards concerning non-life insurance were set up in 1973. The solvency margin has to be at least equal to the larger of:

a. 18 % on the first 10 million ECU of the company's net premium income after reinsurance, and 16 % on the excess; and

b. 26 % of the claims net expenditure (on own account) up to 7 million ECU and 23 % on the excess.

Another constraint on the premium as well as the claims is the requirement that the net figures must be at least 50 % of the gross figures. Otherwise, the required solvency margin is increased by reinsured amounts.

The capital for the solvency margin is supposed to be freely available whenever needed. In many countries the solvency margin contains buffers against risk fluctuations, like the equalization reserve discussed above. Other countries do not include them in the determination of the solvency margin, since these buffers are not freely available for any purpose, in contrast to free reserves. These buffers can only be used according to strictly defined transfer rules, to equalize the fluctuation in the specific risk business for which they are set up.

In the U.S. the National Association of Insurance Commissioners (NAIC) has outlined recommendations for the effective safeguarding of solvency. These include an "early warning system", under which insurance companies are obliged to provide certain information on their financial standing. Based on a number of indicators warning limits are determined; when the indicators flash a warning signal, the authorities impose stricter supervision. In the U.S. the term "surplus" is used for the concept of solvency margin, and the inverse premium/surplus ratio (as compared to the EC solvency ratio) is used as the main indicator of solvency.

In solvency margins we see very useful possibilities for the application of the theory of intervals of possibilities. All the terms which are included in the solvency margin and solvency ratio requirements in the EC include considerable uncertainty, due to lack of knowledge and vagueness. The long and short terms aspects of solvency margin call for application of IPs. The early warning system of the NAIC in the U.S. provides flexibilities for different grades of severity for rules and supervision, to assure the safety of the insurance consumers. These flexibilities have upper and lower limits and can be dealt with, possibly better and more effectively, through the introduction of IPs into the calculations.

Inflation is an important factor for solvency margins; it can be effectively handled with the power operations discussed in section 4. Further, the premium as well as the minimally required solvency margin and surplus are positive. Thus, we can apply the results for PIPs, and considerably simplify the presentation and calculations. Moreover, we can apply the division operator to the solvency ratio.

Still, the applicability of the theory of IPs to the concept of solvency margins is just emerging. We hope that this paper will stimulate interesting, far reaching applicable research in the directions indicated here.

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