AFTER TAX-TERM STRUCTURES OF REAL INTEREST RATES AND INFLATION COMPENSATION : INFERENCES FROM THE UK LINKED AND NON LINKED GILT MARKETS

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Abstract[1] : The behaviour of real interest rates is a core issue for financial economics in terms of both the dynamics of the short term rate and the characteristics of the overall term structure. Empirical investigations of the relationship between nominal interest rates, real interest rates and expected inflation have traditionally been hindered by the fact that, of the three variables, only nominal interest rates are able to be directly extracted from market prices. With the recent introduction of "real return" or "index linked" gilts (ILG's) in the UK, however, there now exists a means of directly extracting real interest rates from prevailing market prices. Furthermore, the existence of *both* linked and non linked gilts provides a means of examining the relationship between nominal interest rates, real interest rates and inflation. The purpose of this paper may be summarized as follows,

- To generate *after tax* real term structures from a cross section of ILG's using an alternative estimation procedure which allows for observation error and assumes that a marginal tax bracket exists for which all securities are priced correctly ;
- To estimate the parameters of the CIR one factor model for the after tax interest rate process with an approach similar to Brown & Schaefer (1994) but by using the "true" prices of ILG's which are implied by the term structure estimation. The "true" prices do not allow for buy and hold arbitrage opportunities ;
- To determine the spot rates of "inflation compensation" as measured by the difference between *after tax* nominal and real interest rates and to generate an inflation compensation term structure.

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I. Introduction

The behaviour of real interest rates is a core issue for financial economics in terms of both the dynamics of the short term rate and the characteristics of the overall term structure. Empirical investigations of the relationship between nominal interest rates, real interest rates and expected inflation have traditionally been hindered by the fact that, of the three variables, only nominal interest rates are able to be directly extracted from market prices. Without an observable measure of real interest rates, it has often been convenient to characterize this fundamental variable as both constant and exhibiting a flat term structure. This assumption, however, has been disputed in several studies including Nelson & Schwert (1977), Walsh (1987), and Rose (1988) who conclude, by various indirect techniques, that real rates do vary over time. With the recent introduction of "real return" or "index linked" gilts (ILG's) in the UK, however, there now exists a means of directly extracting real interest rates from prevailing market prices. Furthermore, the existence of *both* linked and non linked gilts provides a means of examining the relationship between nominal interest rates, real interest rates and inflation.

Although there have been a number of studies which have investigated the properties of real interest rates by indirect methods [1], none have attempted to characterize a zero coupon term structure of *after tax* real interest rates and observe its behaviour over time. Brown & Schaefer (1994) were the first to estimate the *before tax* real term structure of interest rates from a cross section of ILG's. Using the approach of Schaefer (1981), they estimate the real term structure as an optimization problem with the condition that the present value of each ILG cannot exceed its observed price. This approach, however, does not allow for observation error as it treats observed market prices as "true" prices.

Moreover, the method of Schaefer (1981) implicitly assumes a segmented market even

[1] Studies of this nature include Walsh (1987), Rose (1988) and Evans, Keef & Okunev (1993). Studies which have directly incorporated the yields of ILG's include Arak & Kreicher (1985), Pittman (1992), and Woodward (1990, 1993)

though Brown & Schaefer (1994) choose to ignore tax implications. They assume that all gilts are priced correctly for all tax brackets which may not be a reasonable assumption for the gilt market. As pointed out by Woodward (1990), the absence of capital gains taxes on gilts in the UK will very likely lead to segmented market equilibria with distinct tax clientele effects. By treating observed prices as "true" prices in a segmented market, the study of Brown & Schaefer (1994) is subject to systematic bias.

Brown & Schaefer (1994) also use observed prices to estimate the parameters of the CIR model. In a segmented market, however, this would also produce a bias as observed prices allow for buy and hold arbitrage opportunities. This is evident from the fact that no single set of discount factors is able to equate present values and observed prices for all gilts simultaneously.

The purpose of this paper may be summarized as follows,

- To generate *after tax* real term structures from a cross section of ILG's using an alternative estimation procedure to Brown & Schaefer (1994) which allows for observation error and assumes that a marginal tax bracket exists for which all securities are priced correctly;
- To estimate the parameters of the CIR one factor model for the after tax interest rate process with an approach similar to Brown & Schaefer (1994) but by using the "true" prices of ILG's which which are implied by the above term structure estimation. The "true" prices do not allow for buy and hold arbitrage opportunities;
- To determine the spot rates of "inflation compensation" as measured by the difference between *after tax* nominal and real interest rates and to generate an inflation compensation term structure.

Despite the existence of both linked and non-linked gilts, the relationship between nominal interest rates, real interest rates, and inflation cannot be determined in a straightforward manner. There are two significant secondary factors that must be considered. The first factor is the existence of an inflation risk premium which represents an additional

wedge between nominal and real interest rates over and above expected inflation. Although expected inflation is often assumed to equal the difference between expected nominal and real rates, this is only true when investors are risk neutral. Risk averse investors require compensation not only for expected future inflation but also for bearing the risk associated with the uncertainty of future inflation. For the purposes of this study, the inflation risk premium is simply aggregated with expected inflation as "inflation compensation".

The second factor is the non-contemporaneous indexation of inflation which, in the case of the ILG, involves a lag of eight months. The existence of this lag implies that an ILG is effectively non-linked over the eight month period just prior to its maturity. Suitable real interest rates may still be inferred, however, by adjusting the empirically estimated real term structure by a proxy for the inflation expected over the eight month lag period just prior to ILG maturity. The approach used by Brown & Schaefer (1994) to correct for the lag is to estimate expected inflation with an ARIMA model. In contrast, this study uses a procedure whereby inflation expectations are extracted directly from the relationship between the nominal and real after tax term structures. This method provides greater flexibility as it is able to incorporate non-seasonal price shocks into the estimate of inflation expectations.

The outline of the paper is as follows. Section two describes the market and the tax treatment for U.K. index linked gilts. Sections three and four generate after tax term structures and estimate the parameters of the CIR one factor model, respectively. Section five determines the spot rates of inflation compensation and estimates inflation compensation term structures. Section six provides conclusions.

II. The UK Index Linked Gilt Market

Index linked gilts (ILG's) were first issued in the UK in 1981 as part of the government's anti-inflation program. Three primary reasons were given by the treasury for the move (Rutterford 1983);

1. The issuance of gilts with payments linked to the inflation rate would reinforce the belief by the market in the government's anti-inflation policy and help to reduce both inflation and nominal interest rates.
2. The inflation risk premium for both the government and investors would be eliminated. Index linked gilts would reduce the goverment's cost of funding, since investors would not require an inflation risk premium on index linked bonds.
3. Index linked gilts would allow improved monetary control by offering a more complete set of debt instruments to meet the demands of different groups of investors.

Both the coupon and the principal of the ILG are linked to the UK retail price index (RPI). The basis for indexation is the RPI for the period eight months prior to gilt issue; two months lag to enable measurement of the RPI and six months lag to enable exact calculation of accrued interest based on a known forthcoming coupon payment.[2] The existence of the lag results in imperfect inflation protection except in the case where the inflation experienced eight months prior to the ILG issue is identical to the inflation experienced during the last eight months of the ILG's life. The magnitude of this imperfection diminishes as the holding period for the security increases.

There have been a total of seventeen unrestricted ILG issues in the UK (excluding an initial 1981 issue limited to pension funds), beginning in 1982 with maturity dates ranging from 1988 to 2030. ILG's were issued on a consistent basis throughout the 1980's with

[2] This differs from the accrual treatment for the Canadian index linked bond where the calculation of accrued interest is based on the forthcoming real coupon payment which is independent of price level. Thus, the additional six month lag is unnecessary.

coupon rates varying from 2% to 4.625%. Volumes have ranged in size from 300 million pounds to 1 billion pounds.

Table 1.

UK LINKED AND NON LINKED MARKETABLE DEBT

	Volume (£ Bn)		Turnover (£ Bn)	
Linked Debt				
1985	9.5	8.4 %	N/A	N/A
1990	17.5	16.2 %	2.2	2.9 %
Non Linked Debt				
1985	103.7	91.1 %	N/A	N/A
1990	90.5	81.1 %	56.4	89.1 %

The table provides volumes and turnovers for linked and non linked gilts in terms of total sterling marketable debt. Source: Bank of England

The total volume of ILG's has risen from 8.4% (9.5 billion pounds) of the overall volume of outstanding government issued gilts at March 31, 1985 to 16.2% (17.5 billion pounds) at March 31, 1990. The liquidity of the market is relatively thin, however, with a turnover of only 2.9% of the total gilt market turnover in 1990 (de Kock 1991). Holdings of ILG's are concentrated within a small number of investors including, predominantly, pensions funds and insurance companies. Individual investors are largely confined to the short end of the market.

Since 1986, both linked and and unlinked gilts have been exempt from capital gains taxes with the effect that lower coupon securities sold at a discount are taxed preferentially in comparison to those with higher coupons. In general, ILG's, whose inflation compensation accrues to both coupon and principal, possess a tax advantage over non-linked gilts, whose inflation compensation accrues only to the coupon which is taxed as ordinary income. Therefore, the effects of differential taxation must be considered in any comparison

of respective returns and subsequent measurement of expected inflation (Woodward 1990).

III. After Tax Term Structures

The impact of taxes on the estimation of the term structure of interest rates has been well documented. Pioneered by McCulloch (1975), this line of research has been extended by Schaefer (1982), Jordan (1984) and Litzenberger & Rolfo (1984) amongst others.[3] The existence of environments where ordinary income and capital gains are taxed differentially (even in the case where capital gains are non taxed) may make it impossible for investors in different tax brackets to agree on the relative pricing of bonds. In such a case, the difference between the observed price of a bond and its appropriate present value may include a tax effect over and above simple noise. The implication of this for the bond market is that a non-segmented equilibrium may be unobtainable due to the existence of buy and hold tax arbitrage opportunities. A *segmented* equilibrium characterised by tax clientele effects would be possible, however, if frictions exist to limit arbitrage profit. An example of such a friction would be a restriction on the ability to sell short. In such an environment, the resulting equilibrium is segmented in terms of investors only holding bonds which are priced correctly for their particular tax bracket.

Brown & Schaefer (1994) were the first to estimate the *before tax* real term structure of interest rates using cross sections of weekly ILG prices. Using the approach of Schaefer (1981), they estimate the real term structure as an optimization problem with the condition that the present value of each ILG cannot exceed its observed price. In a market characterized by a ban on short sales and a segmented equilibrium caused by differential taxation, this criterion would be satisfied by an infinite number of possible real term structures.

The specific term structure selected by the Schaefer (1981) approach is the one that

[3] Katz & Prisman (1991) provide a recent summary of the literature in this field.

maximizes the present value of an arbitrarily chosen stream of cashflows c_i subject to the constraint that the present value of each of the j outstanding ILG's cannot excede their market price. This approach can be formulated as follows,

$$\max_{d_i} \sum_i d_i c_i \tag{1}$$

Subject to:

$$\mathbf{Ad} \leq \mathbf{P}$$

where, $\mathbf{d}$ and d_i represent the vector of discount factors and its corresponding period i element, and where, $\mathbf{A}$ and $\mathbf{P}$ represent the payoff matrix of ILG cashflows and the vector of market prices, respectively.

The use of this approach for term structure estimation may be problematic, however. No allowance is made for observation error as it treats observed prices as "true prices". This seems unrealistic given the non-synchronous trading of securities in the bond market.[4] In addition, given that the choice of the cashflow stream in the maximization is arbitrarily chosen then, as pointed out by Katz & Prisman (1991), so also may be the term structure determined by the optimization. Consequently, the estimation will be biased in terms of the specific cashflow stream chosen and cannot be used to identify mispriced bonds or a representative marginal tax rate.

In their study, Brown & Schaefer (1994) assume that no significant tax clientele effect exists in the ILG market and, thus, choose to ignore taxes. If this is indeed the case, then there is no a priori reason to expect that the error between market price and the present value contains anything beyond simple noise. Their method, in fact, estimates the term structure based on the assumption that the prices of bonds only contain positive error terms. It is, therefore, not clear why the Schaefer (1981) approach is preferable to estimating a term structure based on the minimization of the sum of squared error terms,

[4] This issue is referred to in Dybvig (1989)

particularly given that Brown & Schaefer (1994) later appeal to price "observation error" in their estimation of the CIR model parameters.

As pointed out by Woodward (1990), the absence of capital gains taxes on gilts since 1986 in the UK will very likely lead to segmented market equilibria with distinct tax clientele effects. Thus, rather than assuming the absence of tax clientele effects, this study utilizes the approach of McCulloch (1975) to generate an *after tax* term structure for ILG's based on the assumption that a given marginal tax bracket exists for which all bonds are priced correctly.

The marginal tax rate of the representative investor is determined as the specific rate which produces the lowest sum of squared errors from amongst each of the after tax term structures estimated. This approach is based on the Rolfo & Litzenberger (1984) study and assumes that, on a particular date, the representative investor's tax rate is the same for all maturities of gilts. In the UK gilt market, the regression is simplified by the absence of capital gains taxes which eliminates the need for the instrumental variable adjustment utilized in McCulloch (1975).

The data employed in this study is that of Woodward (1990, 1993) and includes non linked and ILG prices as well as the monthly RPI collected over ninety-four periods from January 25, 1986 until October 26, 1993. Eighty-five non linked gilts and seventeen ILG's were used throughout the period with callable and convertible gilts excluded from the sample.

The specific methodology used is that of Litzenberger and Rolfo (1984) whereby the time interval to the longest gilt maturity date is partitioned into k subintervals, each containing approximately an equal number of payment dates, by placing $k+1$ knot points at $t = 0, 1, ..., T$. The spot real after tax (marginal tax rate τ) discount factor, $z_\tau^{0,t}$, for term of length t, thus, takes the form,

$$z_\tau^{0,t} = 1 + \alpha t + \beta t^2 + \sum_{i=1}^{k} \gamma_i t_i^3 D_i(t) \tag{2}$$

where if $t < t_{i-1}$, then,

$$D_i(t) = 0$$

and if $t > t_{i-1}$, then,

$$D_i(t) = 1$$

The price of an ILG can be expressed as,

$$P_{n,0} = \sum_{j}^{n} c_{\tau,j} z_{\tau}^{0,j} + \epsilon \tag{3}$$

where, $c_{\tau,j}$ represent the period j real after tax cash flows and where, ϵ captures noise and tax effects.

The substitution of equation (2) into equation (3) produces the following regression equation,

$$P_{n,0} - \sum_{j}^{n} c_{\tau,j} = \alpha_0 + \alpha \sum_{j}^{n} c_{\tau,j} t_j + \beta \sum_{j}^{n} c_{\tau,j} t_j^2 + \sum_{i=1}^{k} \gamma_i \sum_{j}^{n} c_{\tau,j}(t_j - t_i)^3 + \epsilon \tag{4}$$

The discount function is, thus, determined by substituting the resultant regression coefficients into equation (2).

Regressions are run over a range of marginal tax rates with the rate which minimizes the sum of squared error terms presumed to be that of the representative investor. For the representative investor the ϵ term is assumed to be free of tax effects. An identical methodology is utilized to determine nominal term structures of interest rates. Unlike Woodward (1990), however, it is not assumed that the same representative investor is appropriate for both the ILG and the non linked gilt markets.

The marginal tax rates of the representative investor calculated for both nominal gilts and ILG's are provided in Appendix A. The representative investor clearly differs across the linked and non-linked markets which further underscores the importance of considering after tax effects when determining the relationship between interest rates and inflation. The marginal tax rate appropriate for the non-linked market exhibits a declining trend from

approximately 20% in 1986 to approximately 10% in 1993 which is consistent (although slightly lower in magnitude) with Woodward (1990). In contrast, the tax rate appropriate for the ILG market increases from essentially 0% in 1986 to approximately 10% in 1993.

The approach of real term structure estimation described above ignores the eight month lag of inflation indexation on ILG's. Without further refinement, this approach will provide a true estimation of the real term structure only if the expectation of inflation over the last eight months of the ILG's life is equal to actual inflation eight months prior to ILG issue and that no corresponding inflation risk premium exists. If this is not the case, an inflation bias will be inherent in the estimate of real spot rates which diminishes proportionately with the length of the term. To correct for this bias, a method of estimating inflation expectations for the non-indexed eight month period is required.

The approach used by Brown & Schaefer (1994) to correct for this bias is to estimate expected inflation through the use of an ARIMA model which accomodates the seasonality of inflation expectations. In contrast, this study uses an alternative procedure to correct for the lag by extracting inflation expectations directly from the relationship between the nominal and real after tax term structures. This method provides greater flexibility as it is able to incorporate non seasonal price shocks into the estimate of inflation expectations.

The continuously compounded *real*, n month spot rate, k_0^n, extracted from the estimated ILG term structure when $n \geq 8$ can be expressed as,

$$k_0^n = r_0^n + \left(\frac{8}{n}\right)(\pi_{n-8}^n - \hat{\pi}_{-8}^0) \tag{5}$$

and when $n < 8$,

$$k_0^n = r_0^n + (\pi_0^n - \hat{\pi}_{-8}^{n-8}) \tag{6}$$

where, r_0^n is the true n period real spot rate and where, $\hat{\pi}$ and π represent actual inflation experienced prior to ILG issue and expected inflation over the last months of the ILG's life, respectively. Without loss of generality it can be assumed that π also contains the inflation risk premium appropriate for the period in question. For $n \geq 8$, the lag bias in the real

spot rate estimate is reflected by the difference between π^n_{n-8} and $\hat{\pi}^0_{-8}$. The magnitude of this bias is clearly proportional to the term of the spot rate and at long terms to maturity, becomes negligible. Note that for terms of eight months or less, the real rate essentially possesses the characteristics of a nominal rate.

The continuously compounded *nominal*, n month spot rate, K^n_0, extracted from the estimated non linked gilt term structure is free of any lag bias and can be simply expressed as,

$$K^n_0 = r^n_0 + \pi^n_0 \tag{7}$$

Under the assumption that an "expectations hypothesis" for the term structure of inflation compensation is valid, the following relationship holds,

$$\pi^{n+8}_0 = \left(\frac{n}{n+8}\right)\pi^n_0 + \left(\frac{8}{n+8}\right)\pi^{n+8}_n \tag{8}$$

Given the following $n+8$ and n month nominal spot rates,

$$\begin{aligned} K^{n+8}_0 &= r^{n+8}_0 + \pi^{n+8}_0 \qquad (9)\\ &= r^{n+8}_0 + \left(\frac{n}{n+8}\right)\pi^n_0 + \left(\frac{8}{n+8}\right)\pi^{n+8}_n \\ K^n_0 &= r^n_0 + \pi^n_0 \qquad (10)\end{aligned}$$

and the following $n+8$ month real spot rate,

$$k^{n+8}_0 = r^{n+8}_0 + \left(\frac{8}{n+8}\right)(\pi^{n+8}_n - \hat{\pi}^0_{-8}) \tag{11}$$

then, the true n spot real rate derived from the estimated nominal and real term structures can be expressed as,

$$r^n_0 = K^n_0 - \left(\frac{n+8}{n}\right)(K^{n+8}_0 - k^{n+8}_0) + \left(\frac{8}{n}\right)\hat{\pi}^0_{-8} \tag{12}$$

This approach estimates expected future inflation by using the information directly contained in the market prices of linked and non-linked gilts. The previously noted difference between the representative investor's marginal tax rate appropriate for the ILG and

non-linked gilt markets is critical to the estimation and underscores the need to consider nominal and real rates on an after tax basis.

After tax real and nominal term structures are estimated over monthly periods from January 25, 1986 until October 25, 1993 and are illustrated in Figures 1 and 2. Table 2 contains before and after tax mean spot rates and standard deviations over the sample period.

Several observations can be made from the estimates. First, as noted by Brown & Schaefer (1994), real term structures are neither flat nor constant over time. The shapes of the real term structures, while predominantly "humped" shaped, can be either upwardly or downwardly sloping. Most of the volatility, however, is at the short end as spot rates for longer terms become remarkably constant.

The impact of the eight month lag for inflation indexation is apparent as the lag adjusted real spot rates are much less variable than the unadjusted rates at the short end of the term structure. This is not an unexpected result given that, unadjusted spot rates are highly influenced by the inflation expected over the eight month period just prior to maturity.

A comparison of before tax and after tax rates reveals only a small reduction in the means and a negligible change in the volatilities for ILG's due to the fact that the marginal tax bracket for these securities over much of the period was 0%. Means of after tax spot rates were much lower for non-linked gilts with a non-negligible decline in volatility. This result is important for the determination of inflation compensation term structures.

Although the January 1986 until December 1989 sample period of the Brown & Schaefer (1994) study is not identical to that of this study, some comparisons can still be made. Their results indicate significantly greater volatility at the short end of the term structure which is likely due to the different approaches used to correct for the lag in indexation. It is also noteworthy that the Schaefer (1981) approach to term structure estimation is likely to produce an upward bias in the estimation of spot rates. The Brown & Schaefer (1994)

After Tax Term Structure of Nominal Interest Rates

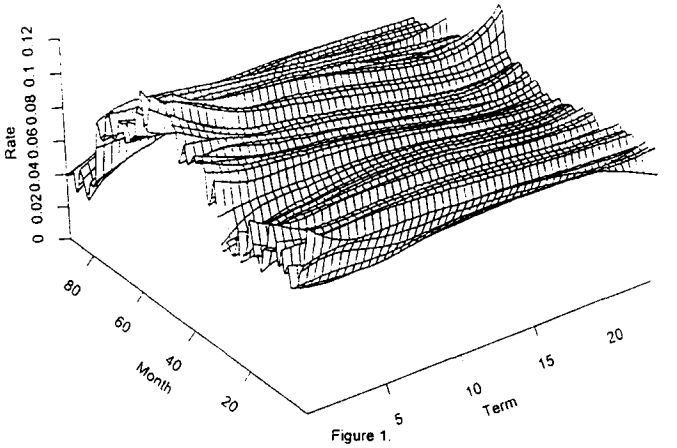

Figure 1.

After Tax Term Structure of Real Interest Rates

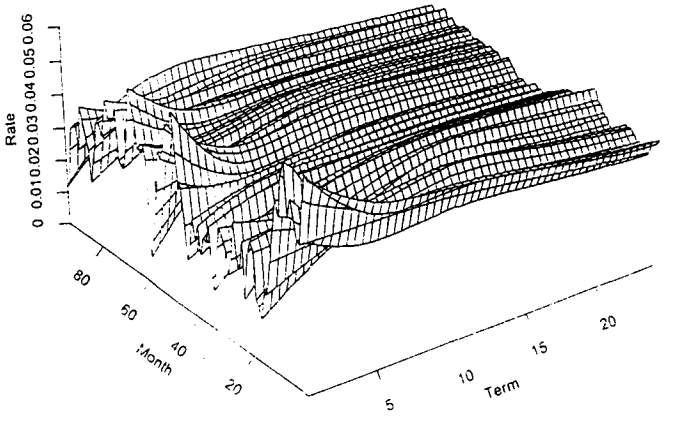

Figure 2.

spot rate estimates show some indication of this.

Table 2.

ZERO COUPON SPOT RATES

Panel 1. Before Tax

Term	Mean (Standard deviation) Nominal	Real (Unadj.)	Real (Adj.)	B/S
1	9.80 (2.14)	3.32 (1.33)	3.09 (1.08)	3.56 (2.48)
2	9.43 (1.87)	3.35 (0.93)	3.15 (0.81)	3.63 (1.86)
3	9.35 (1.67)	3.38 (0.73)	3.20 (0.63)	3.69 (1.48)
5	9.56 (1.39)	3.49 (0.51)	3.41 (0.49)	3.78 (1.09)
10	9.66 (1.03)	3.86 (0.40)	3.83 (0.42)	3.87 (0.74)
20	8.89 (0.71)	3.91 (0.33)	3.89 (0.37)	3.69 (0.52)

Panel 2. After Tax

Term	Mean (Standard deviation) Nominal	Real (Unadj.)	Real (Adj.)	B/S
1	8.15 (2.01)	3.21 (1.33)	3.05 (1.01)	N/A
2	8.12 (1.75)	3.23 (0.93)	3.12 (0.74)	N/A
3	8.14 (1.60)	3.25 (0.73)	3.15 (0.61)	N/A
5	8.21 (1.34)	3.40 (0.51)	3.36 (0.49)	N/A
10	8.41 (1.00)	3.75 (0.39)	3.76 (0.40)	N/A
20	7.94 (0.64)	3.77 (0.32)	3.76 (0.33)	N/A

The table provides means and standard deviations of before and after tax spot rates for both nominal and real interest rates over monthly periods from January 25, 1986 until October 25, 1993. Although the sample periods are not identical, the results of Brown & Schaefer (1994) are provided for comparison purposes.

IV. CIR Model for the Real Term Structure

Brown & Schaefer (1994) fit the Cox, Ingersoll & Ross (1985) term structure model to market prices of ILG's in order to assess the extent to which the single factor model captures the main features highlighted by their empirical approach. They find results that are both consistent and inconsistent with the theoretical model. On the one hand,

the term structures estimated by Brown & Schaefer (1994) consistently demonstrate the general shapes, the high correlation of crossectional spot yields, and the low volatility of long term yields predicted by the CIR model. On the other hand, intertemporal stability of the CIR parameters is firmly rejected.

This study endeavors to estimate CIR model parameters in a manner similar to Brown & Schaefer (1994) but to determine the theoretical term structure utilizing the "clean" prices implied by the term structure estimation procedure in contrast to their approach which utilizes directly observed market prices. Given that observed prices may not be free from buy and hold arbitrage opportunities for all investors, then their use cannot be consistent with an equilibrium model for the term structure of interest rates. As pointed out by Hull (1993), a model of the term structure which is comfortably within a 1% error margin for pricing a bond could produce an error as large as 50% for pricing a derivitive security on that bond. In order to adjust for this inconsistancy, Brown & Schaefer (1994) appeal to observation error and fit the CIR model by minimizing the sum of squared errors between the market and "model" prices. This approach produces a bias, however, if a tax clientele effect, such as that demonstrated in the previous section, were to exist over and above simple noise. In this study, the theoretical after tax prices are determined from the term structure estimation approach of this study which, given the assumption of a representative investor, are free from clientele effects.

As a first step, able 3. illustrates the correlation of crossectional spot rates for both before and after tax real spot rates using the term structure estimation approach of this study.

Consistent with Brown & Schaefer (1994), a fairly high degree of correlation is found for changes in real spot rates which suggests some promise for the one factor approach to modelling real interest rates. Note that the correlation for after tax real rates is higher than that for before tax rates [5] which is consistent with Rumsey (1993) who finds that

[5] The after tax results demonstrate higher correlation than both the before tax results of this study and the

Table 3.

CORRELATION OF SPOT REAL INTEREST RATES

Panel 1. After Tax

	Spot Rate Term					
	1	2	3	5	10	20
1	1.00					
2	0.99	1.00				
3	0.95	0.98	1.00			
5	0.77	0.85	0.93	1.00		
10	0.67	0.64	0.72	0.86	1.00	
20	0.51	0.57	0.66	0.73	0.86	1.00

Panel 2. Before Tax

	Spot Rate Term					
	1	2	3	5	10	20
1	1.00					
2	0.97	1.00				
3	0.93	0.97	1.00			
5	0.71	0.79	0.93	1.00		
10	0.59	0.61	0.72	0.85	1.00	
20	0.55	0.59	0.58	0.74	0.87	1.00

The table provides correlations across monthly before and after tax interest rate changes for spot rates of varying terms over the period from January 25, 1986 until October, 25 1993.

interest rate models, in general, provide a better fit to after tax interest rates than to before tax interest rates.

This study estimates the parameters of the CIR square root process in a manner similar to Brown & Dybvig (1986) and Brown & Schaefer (1994) by assuming the following process

before tax results of Brown & Schaefer (1994) although it must be noted that the sample periods of the two studies are not identical.

for the instantaneous short term rate,

$$dr = \vartheta(\theta - r)dt + \sigma\sqrt{r}dz \tag{13}$$

where, θ is the mean, ϑ is the mean reversion coeficient and σ is the volatility. Given the above process and the arbitrage pricing condition, it can be shown that the price, $P(r,t,T)$, of a default free, zero coupon bond must satisfy the following partial differential equation,

$$\vartheta(\theta - r)P_r + P_t + \frac{1}{2}\sigma^2 r P_{rr} - rP = \lambda r P_r \tag{14}$$

where λ represents the market price of risk. The terminal condition,

$$P(r,T,T) = 1 \tag{15}$$

implies that, for any τ,

$$P(r,t,t+\tau) = A(\tau)e^{-B(\tau)r} \tag{16}$$

where,

$$A(\tau) = \left(\frac{2\gamma e^{\frac{1}{2}(\vartheta+\lambda+\gamma)\tau}}{(\vartheta+\lambda+\gamma)(e^{\gamma\tau}-1)+2\gamma}\right)^{\frac{2\vartheta\theta}{\sigma^2}} \tag{17}$$

$$B(\tau) = \left(\frac{2(e^{\gamma\tau}-1)}{(\vartheta+\lambda+\gamma)(e^{\gamma\tau}-1)+2\gamma}\right)$$

and,

$$\gamma = \sqrt{(\vartheta+\lambda)^2 + 2\sigma^2} \tag{18}$$

The τ period zero coupon spot rate defined by this process is

$$r_0^\tau = \frac{B(\tau)r - \ln(A(\tau))}{\tau} \tag{19}$$

where,

$$\lim_{\tau\to\infty} r_0^\tau = \frac{2\vartheta\theta}{\vartheta+\lambda+\gamma} \tag{20}$$

Given that prices are dependent only on the risk adjusted interest rate process, implies that the parameters to be estimated are, r, σ^2, $\vartheta\theta$, and $\vartheta + \lambda$.

The parameter values are estimated in an approach similar to Brown & Dybvig (1986) and Brown & Schaefer (1994) by solving the following optimization problem across j ILG's with respect to the risk adjusted CIR parameters.

$$\min \sum_j (\hat{P}_j - P_j)^2 \tag{21}$$

This approach differs from the above studies in that P_j represents the theoretical price for the CIR model and $\hat{P}_j$ represents the "clean" price determined from the empirical estimation procedure in the previous section rather than the observed market price. The "clean" price represents the observed price less the observation error.

The CIR after tax term structure estimates over the sample period are illustrated in Figure 3. with mean spot rates and standard deviations provided in table 4.

CIR After Tax Term Structure of Real Interest Rates

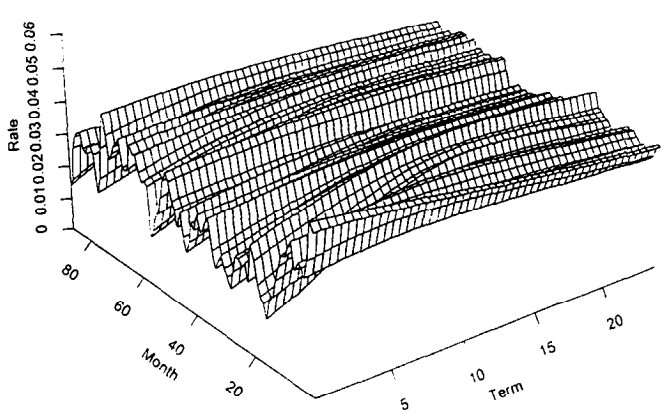

Figure 3.

Table 4.

CIR ZERO COUPON REAL SPOT RATES

	Mean (Standard deviation)		
Term	CIR	Estimate	Estimate (Adj)
1	3.09 (0.89)	3.21 (1.33)	3.05 (1.01)
2	3.14 (0.78)	3.23 (0.93)	3.12 (0.74)
3	3.20 (0.68)	3.25 (0.73)	3.15 (0.61)
5	3.40 (0.50)	3.37 (0.51)	3.36 (0.49)
10	3.71 (0.40)	3.75 (0.39)	3.76 (0.40)
20	3.78 (0.32)	3.77 (0.32)	3.76 (0.33)

The table provides mean and standard deviations of after tax spot real interest rates based on the CIR term structure model for the period from January 25, 1986 until October 25, 1993. The results are compared to the adjusted and unadjusted estimates from the previous section.

Similar to the conclusions reached by Brown & Schaefer (1994), the CIR model is quite able to accomodate the general shapes of the observed term structures. The agreement between the mean and standard deviations of the CIR model and the observed term structures for each term is good and is particularly strong for longer maturities. The understatement of the real rate standard deviation that the CIR model suggests for the shorter end is significantly less than that found by Brown & Schaefer (1994) which is again likely due to the different approaches used for measuring inflation expectations.

The estimates of the CIR parameters are shown in table 5. While stability of these parameters is rejected, there appears to be less variation in the estimates for after tax interest rates over time than for before tax rates in this study or in that of Brown & Schaefer (1994). This is again consistent with Rumsey (1993) and provides further support for fitting one factor interest rate models to after tax interest rates.

As a framework for comparison, the parameter estimates of Brown & Schaefer (1994) are compared with the results of this paper in order to value a fifteen year zero coupon

Table 5.

CIR PARAMETER ESTIMATES

Panel 1. Before Tax

Year	CIR Parameters					
	$\vartheta + \lambda$	$\vartheta\theta$	σ^2	r_{st}	r_1	r_{20}
1986	-.0059	.00142	.0082	.0340	.0277	.0345
1987	-.1274	.00110	.0137	.0327	.0333	.0406
1988	-.0930	.00098	.0095	.0338	.0343	.0409
1989	-.0477	.00092	.0068	.0282	.0284	.0387
1990	-.0745	.00098	.0060	.0271	.0296	.0404
1991	-.0171	.00214	.0042	.0331	.0351	.0426
1992	-.0212	.00311	.0159	.0321	.0381	.0430
1993	-.1153	.00310	.0342	.0268	.0362	.0355

Panel 2. After Tax

Year	CIR Parameters					
	$\vartheta + \lambda$	$\vartheta\theta$	σ^2	r_{st}	r_1	r_{20}
1986	-.0059	.00142	.0082	.0374	.0277	.0345
1987	-.1274	.00110	.0137	.0226	.0333	.0406
1988	-.0952	.00107	.0095	.0219	.0243	.0392
1989	-.0481	.00104	.0071	.0251	.0270	.0357
1990	-.0728	.00109	.0086	.0260	.0261	.0382
1991	-.0200	.00175	.0042	.0228	.0284	.0376
1992	-.0238	.00186	.0065	.0321	.0344	.0386
1993	-.0543	.00040	.0048	.0268	.0381	.0370

The table provides estimates for the CIR one factor interest rate model for both before and after tax interest rates over the period from January 25, 1986 until October, 25 1993.

bond and hypothetical five year call options on that bond. Three European call options are considered with strike prices of \$40, \$50 and \$60, respectively. The modified binomial approach of Tian (1993) is used to construct a lattice of the CIR interest rate process. Under this approach, a path independent binomial tree is created from a transformation of the assumed interest rate process to one which has constant volatility. In general, for

an interest rate process described as,

$$dr = \mu(r,t)dt + \sigma(r,t)dz \tag{22}$$

the transformed constant volatility process can be defined as,

$$d\Theta = q(r,t)dt + \Theta_r \sigma dz \tag{23}$$

where,

$$q(r,t) = \Theta_t + (\mu - \lambda^* \sigma)\Theta_r + \frac{1}{2}\sigma^2 \Theta_{rr} \tag{24}$$

For the CIR interest rate process an appropriate transformation is simply,

$$\Theta = \sqrt{r} \tag{25}$$

which implies a transformed stochastic process of

$$d\Theta = qdt + vdz \tag{26}$$

where,

$$\begin{aligned} q &= [\vartheta(\theta - r) - \lambda^* \sigma r]\,\Theta_r + \frac{1}{2}\sigma^2 r \Theta_{rr} \\ &= \frac{\alpha_1}{\Theta} - \alpha_2 \Theta \\ v &= \frac{\sigma}{2} \end{aligned} \tag{27}$$

and where,

$$\begin{aligned} \alpha_1 &= \frac{4\vartheta\theta - \sigma^2}{8} \\ \alpha_2 &= \frac{\vartheta + \lambda^* \sigma}{2} \end{aligned} \tag{28}$$

Tian (1993) finds the speed of convergence to the closed form solution to be within approximately .05% for a zero coupon bond with only a ten step lattice.[6]

[6] This is strictly true only when $\alpha_1 > 1$ (Tian 1993)

Table 6.

BOND AND CALL VALUES FOR CIR INTEREST RATE PROCESS

TIAN MODIFIED BINOMIAL LATTICE

Year		Estimated Values		Brown & Schaefer	
	Strike	Bond	Call	Bond	Call
1986	40	47.44	14.90	47.90	15.31
	50		6.77		7.16
	60		0.19		0.36
1987	40	31.22	2.05	30.93	0.93
	50		0.01		0.00
	60		0.00		0.00
1988	40	43.27	8.51	42.72	8.08
	50		0.38		0.21
	60		0.05		0.01
1989	40	52.28	17.71	52.95	18.95
	50		9.01		10.43
	60		0.41		1.52

The table provides values for a fifteen year zero coupon bond and five year European call options of varying strike price. The estimation is based on the Tian (1993) modified binomial lattice model used to construct the CIR real interest rate process comparing Brown & Schaefer (1994) parameter estimates to those of this paper.

The values of the bond and call options estimated by both the Brown & Schaefer (1994) approach and that of this paper are demonstrated in table 6.

The results demonstrate the magnitude of error inherent in a misspecification of parameter estimates for an interest rate process. Although there is a fair degree of consistency across both sets of CIR parameter estimates for the values of a bond, the results obtained for derivitives can vary widely. Hence, if observed market prices do not preclude buy and hold arbitrage opportunities, then to use these to estimate the parameters of an interset rate process can lead to large errors when valuing interest rate derivitive securities.

V. Premium for Inflation Compensation

The difference between the nominal interest rate and the real interest rate is a premium for inflation compensation which consists of both expected inflation and an inflation risk premium. The existence of an inflation risk premium has been demonstrated theoretically by Fischer (1975), Amihud & Barnea (1977) amongst others but empirically its magnitude is of some debate. Pittman (1991) using yields on ILG's estimates its magnitude at approximately 80-90 basis points while Flesaker & Ronn (1987) and Chu (1991), using the short lived US CPI futures data, estimate it in the order of 52-163 basis points. Without an exogeneous proxy for inflation expectations, it is not possible to extract the inflation risk premium from inflation compensation. This study is concerned solely with inflation compensation which, in general, will not be equal to expected inflation unless the inflation risk premium is assumed to be zero.

The results of section two and of Woodward (1990) clearly demonstrate that if inflation compensation is not measured using after tax nominal and real rates, then its magnitude will be contaminated by tax effects. If the marginal tax bracket of the representative investor is different across the linked and the non linked gilt markets then estimates based on before tax interest rates will be biased. Figures 4 and 5 illustrate term structures of inflation compensation both on a before tax basis and on an after tax basis from January 25, 1986 until October 25, 1993. Summary results are provided in table 7.

It is evident that the term structures of inflation compensation are more volatile than that of real interest rates, thus, providing the bulk of the volitility of nominal interest rates. The results also indicate that the general level of inflation compensation is overestimated by roughly 100 - 150 basis points if the estimation is based on before tax interest rates. This is consistent with the findings of Woodward (1990).

Table 8 demonstrates the correlation between after tax real spot rates and spot inflation compensation for varying terms on both a lag adjusted and unadjusted basis. The impact of the lag adjustment is very clear as noted by distinct negative correlation in

Before Tax Term Structure of Inflation Compensation

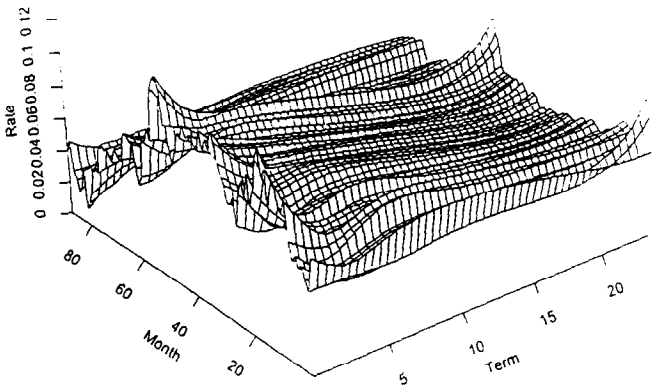

Figure 4.

After Tax Term Structure of Inflation Compensation

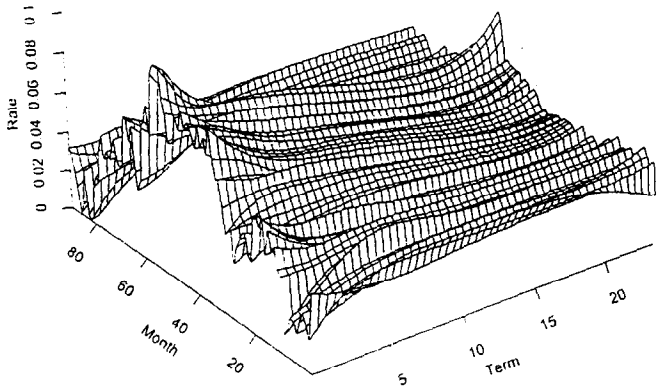

Figure 5.

the unadjusted case which tapers off significantly with increasing term to maturity. In contrast, a constant level of low correlation is present across all terms in the adjusted case.

Table 7.

INFLATION COMPENSATION SPOT RATES

Term	Mean (Standard deviation) Before Tax	After Tax
1	6.48 (2.17)	4.93 (1.99)
2	6.08 (1.88)	4.91 (1.70)
3	5.96 (1.67)	4.88 (1.56)
5	6.06 (1.38)	4.77 (1.35)
10	5.79 (1.01)	4.42 (0.94)
20	5.57 (0.64)	4.35 (0.58)

The table provides mean and standard deviations of inflation compensation as calculated by the difference between after tax nominal and real interest rates as estimated from the term structure estimation approach of this study over the period from January 25, 1986 until October 25, 1993.

Table 8.

CORRELATION OF AFTER TAX REAL SPOT INTEREST RATES AND INFLATION COMPENSATION

	1	2	3	5	10	20
Unadjusted Real Rates	-.302	-.255	-.211	-.172	-.160	-.033
Adjusted Real Rates	-.150	-.113	-.084	-.108	-.155	-.029

The table provides correlations of spot real interest rates and spot inflation compensation over the period from January 25, 1986 until October 25, 1993.

VI. Conclusions

Utilizing an alternative approach to Brown & Schaefer (1994), this study estimates the *after tax* term structure of real interest rates from information contained in both the UK linked and non-linked gilt markets over the period from January 25, 1986 until October 25, 1993. Given the estimation of both nominal and real term structures, the approach of

this study also enables the estimation of the term structure of "inflation compensation" over the same period.

Two major observations can be made regarding the estimates for spot real interest rates. Firstly, the volatility of the short term rate is much lower than that found by Brown & Schaefer (1994) which privides a better fit with the predictions of the CIR single factor model for interest rates. This is likely due to the different approaches used by the two studies to correct for the lag in indexation.

Secondly, consistent with Rumsey (1993), there appears to be some evidence to suggest that single factor interest rate models produce a better fit to interest rates on an after tax basis than on a before tax basis.

The estimation of the term structure of inflation compensation also produced some interesting observations. Given the differing marginal tax brackets across the linked and non linked gilt markets, estimates of spot rates of inflation compensation are overestimated by 100 - 150 basis points if term structures are not calculated on an after tax basis. This is consistent with the results of Woodward (1990).

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