

Dynamic Fund Protection

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Presentation based on two papers:

Hans U. Gerber and Gerard Pafumi, “Pricing Dynamic Investment Fund Protection,” *North American Actuarial Journal*, Vol 4 (2), 2000

Hans U. Gerber and Elias S.W. Shiu, “Pricing Perpetual Fund Protection with Withdrawal Protection,” *North American Actuarial Journal*, Vol 7 (2), 2003

Presentation based on two papers by **Hans U. Gerber**:

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- Stock Index value at time t:

$$I(t) = I(0)e^{Y(t)}$$

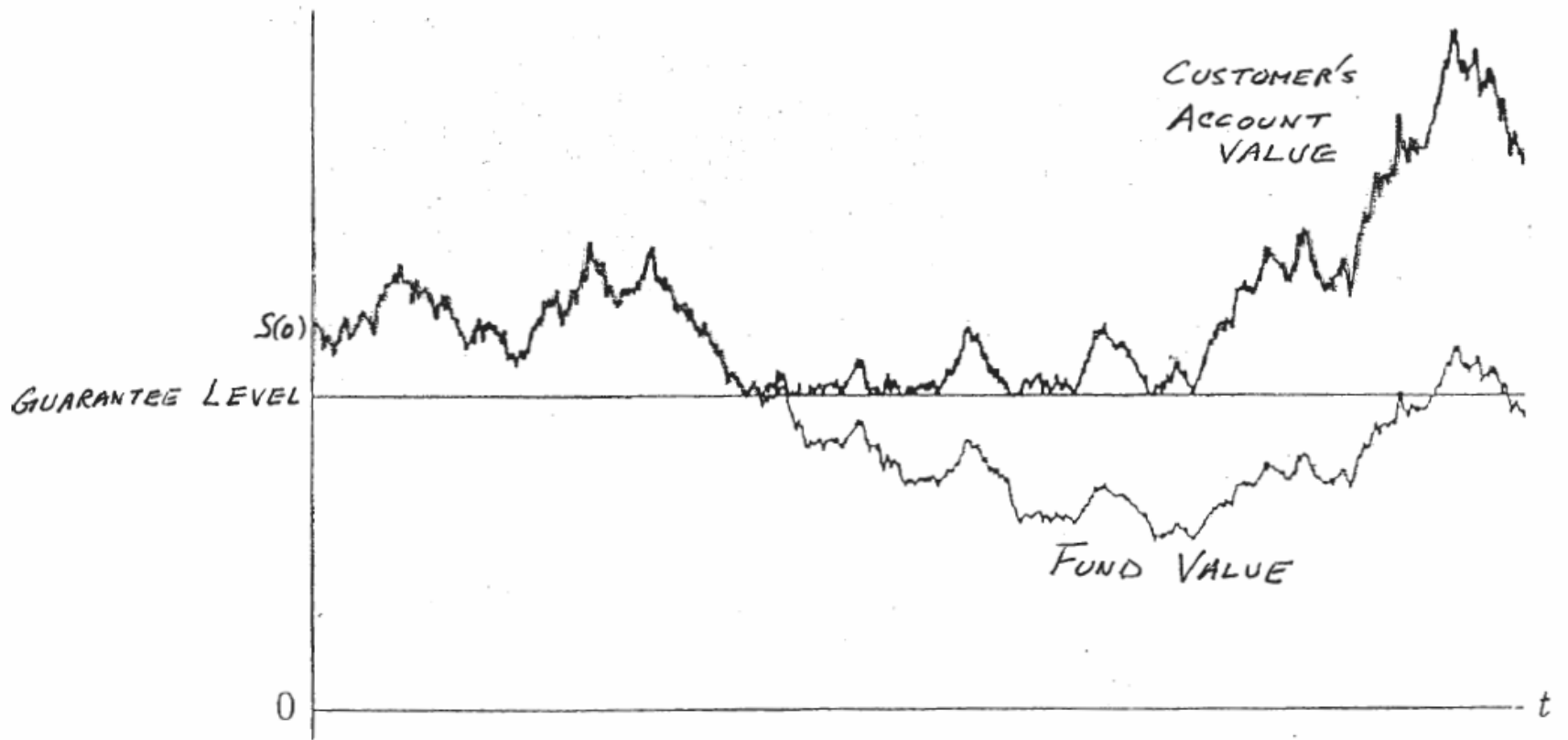
e.g., S&P 500

Assume $\{Y(t)\}$ is a *Brownian motion (Wiener process)*.

- Value of **one unit** of fund at time t:

$$S(t) = S(0)e^{\alpha Y(t)}$$

where α , called the *participation rate*, is usually < 1 .



$n(t)$ = number of fund units in the
customer's account at time t

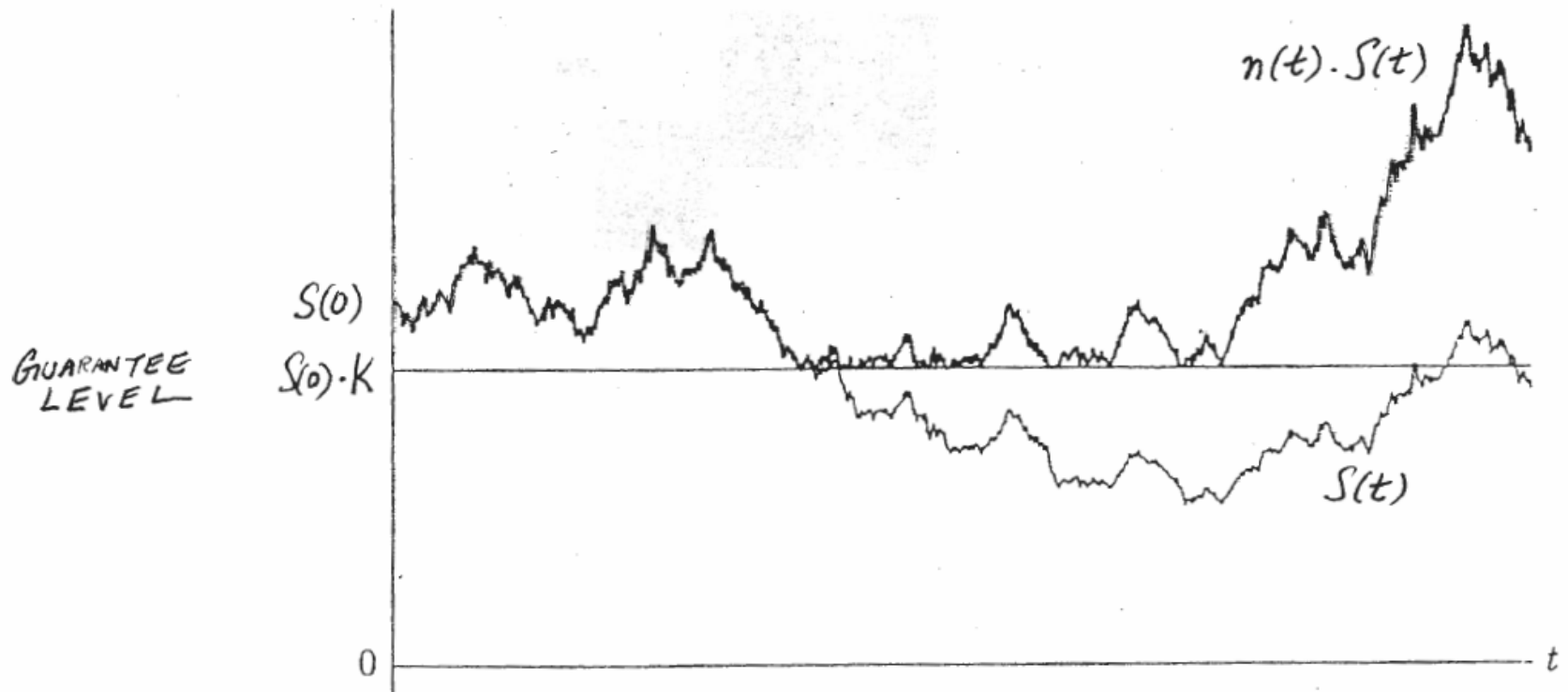
$$n(0) = 1$$

$$n(t)S(t) \geq S(0)K(t) \quad \text{guarantee boundary}$$

e.g., $K(t) = 0.9 (1.03)^t$ for satisfying U.S.

nonforfeiture laws

A Typical Sample Path of the Fund Values



- What is $n(t)$?

$$n(0) = 1$$

$$n(t) \geq \frac{S(0)K(t)}{S(t)}$$

$n(t)$ being *non-decreasing* means:

$$\text{For all } t \geq 0, \quad n(t) \geq \max_{0 \leq \tau \leq t} n(\tau)$$

Hence,

$$n(t) \geq \max \left\{ 1, \max_{0 \leq \tau \leq t} \frac{S(0)K(\tau)}{S(\tau)} \right\}$$

Therefore, the number of fund units in the customer's account at time t is

$$n(t) = \max \left\{ 1, \max_{0 \leq \tau \leq t} \frac{S(0)K(\tau)}{S(\tau)} \right\}$$

The customer's account value at time t is $n(t)S(t)$.

By the *Fundamental Theorem of Asset Pricing*, the time-0 “value” of this Dynamic Fund Protection contract is

$$E^* [e^{-rT} n(T)S(T)]$$

where * signifies that the expectation is taken with respect to the *risk-neutral* probability measure, r is the risk-free force of interest, and T is the maturity date of the contract.

This “value” should be compared with $S(0)$, the premium for one unit of the fund at time 0.

Because

$$S(T) = S(0)e^{\alpha Y(T)}$$

we have

$$\begin{aligned} E^* [e^{-rT} n(T)S(T)] \\ = S(0)e^{-rT} E^* [n(T)e^{\alpha Y(T)}] \end{aligned}$$

What is $E^* [n(T)e^{\alpha Y(T)}]$?

$$\begin{aligned}
& E^* [n(T)e^{\alpha Y(T)}] \\
&= \frac{E^* [n(T)e^{\alpha Y(T)}]}{E^* [e^{\alpha Y(T)}]} E^* [e^{\alpha Y(T)}] \\
&= E^{**} [n(T)] E^* [e^{\alpha Y(T)}]
\end{aligned}$$

where ** signifies a changed probability measure (an Esscher transform).

Now, $E^*[e^{\alpha Y(T)}]$ is the *moment-generating function* of the random variable $Y(T)$ (with respect to the risk-neutral probability measure) at the value α .

It is assumed that $\{Y(t)\}$ is a Brownian motion with *diffusion coefficient* σ . Under the risk-neutral probability measure, $Y(T)$ is a *normal* random variable, and

$$E^*[e^{\alpha Y(T)}] \\ = \exp(\alpha E^*[Y(T)] + \frac{1}{2} \alpha^2 \text{Var}^*[Y(T)]),$$

where

$$E^*[Y(T)] = \left(r - \frac{\sigma^2}{2} - \zeta \right) T,$$

and

$$\text{Var}^*[Y(T)] = \text{Var}[Y(T)] = \sigma^2 T.$$

Here, r is the risk-free force of interest, and ζ is the (constant) dividend-yield rate.

So it remains to determine $E^{**}[n(T)]$.

To determine $E^{**}[n(T)]$, we consider the case

$$K(t) = \beta e^{gt} \quad [\text{e.g., } K(t) = 0.9(1.03)^t]$$

Then,

$$n(t) = \max \left\{ 1, \max_{0 \leq \tau \leq T} \frac{S(0)K(\tau)}{S(\tau)} \right\}$$

$$= \max \left\{ 1, \max_{0 \leq \tau \leq T} \frac{\beta e^{g\tau}}{e^{\alpha Y(\tau)}} \right\}$$

$$= \max \left\{ 1, \beta \exp \left[\max_{0 \leq \tau \leq T} (g\tau - \alpha Y(\tau)) \right] \right\}$$

Define $X(\tau) = g\tau - \alpha Y(\tau)$.

Let

$$M(t) = \max_{0 \leq \tau \leq T} X(\tau)$$

be the *running maximum* of the process $\{X(\tau)\}$.

Then,

$$n(t) = \max\{1, \beta e^{M(t)}\}.$$

Thus, to determine $E^{**}[n(T)]$, we need to know the probability distribution of running maximum $M(T)$ under the $**$ probability measure.

For a Wiener process $\{X(\tau)\}$ with *drift* μ and *diffusion coefficient* σ , it is known that

$$\Pr[M(t) \leq m] = \Phi\left(\frac{m - \mu t}{\sigma\sqrt{t}}\right) - e^{2m\mu / \sigma^2} \Phi\left(\frac{-m - \mu t}{\sigma\sqrt{t}}\right)$$

where $\Phi(\cdot)$ is the c.d.f. of the Normal (0, 1) random variable. For $X(\tau) = g\tau - \alpha Y(\tau)$, what are the drift and diffusion coefficient of $\{X(\tau)\}$ under the ** probability measure?

Recall:

$$\frac{E^* [n(T) e^{\alpha Y(T)}]}{E^* [e^{\alpha Y(T)}]} = E^{**} [n(T)]$$

Under the $**$ probability measure, the Brownian motion $\{Y(\tau)\}$ has drift

$$\begin{aligned} E^*[Y(1)] + \alpha\sigma^2 \\ = \left(r - \frac{\sigma^2}{2} - \zeta\right) + \alpha\sigma^2 \end{aligned}$$

and (unchanged) diffusion coefficient σ .

Now, $X(\tau) = g\tau - \alpha Y(\tau)$. Thus, under the $**$ probability measure, the process $\{X(\tau)\}$ has drift

$$g - \alpha \left[\left(r - \frac{\sigma^2}{2} - \zeta \right) + \alpha \sigma^2 \right]$$

and diffusion coefficient $\alpha\sigma$.

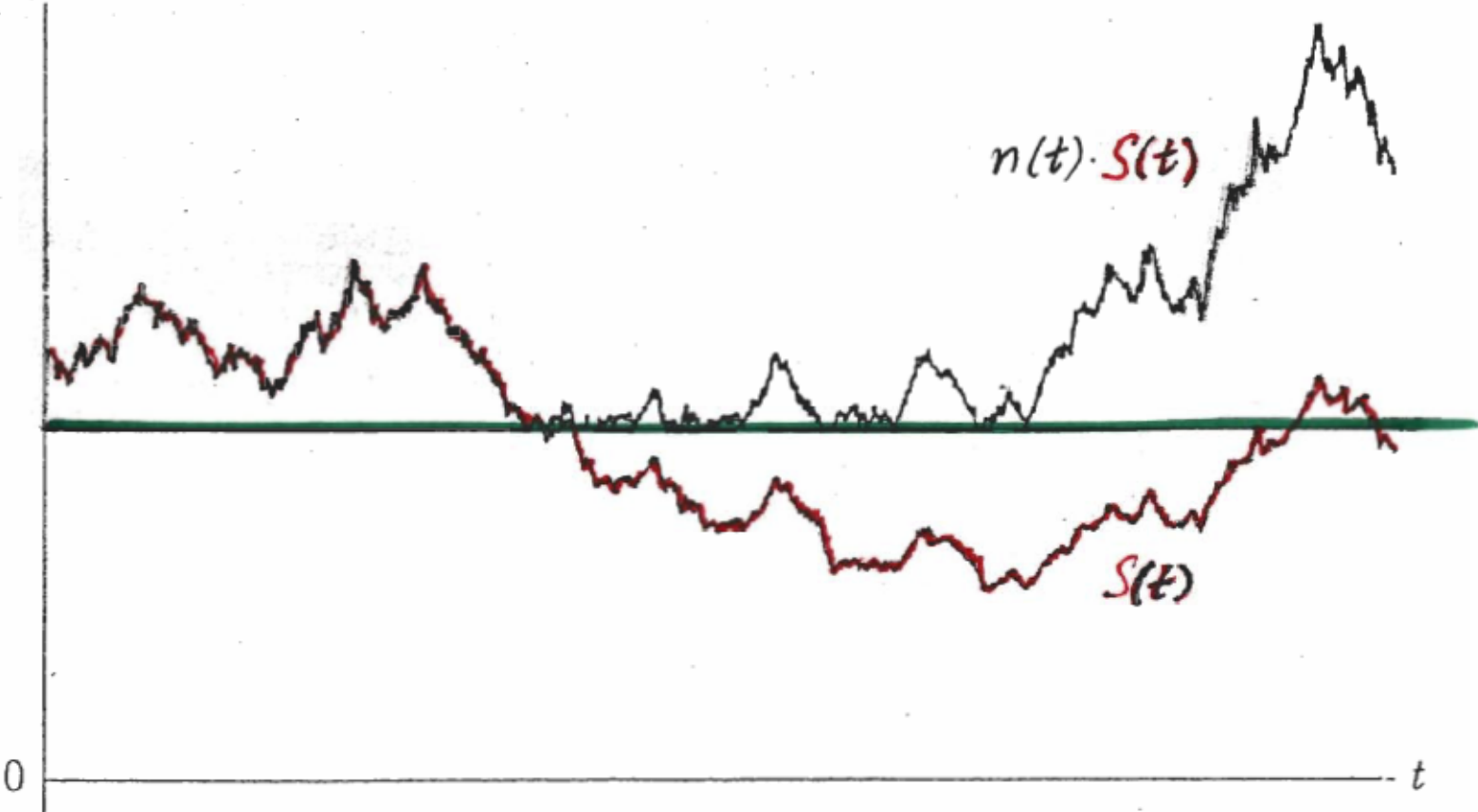
Hence, $E^{**}[n(T)]$ can be evaluated, and one can then write down a closed-form formula for $E^*[e^{-rT}n(T)S(T)]$.

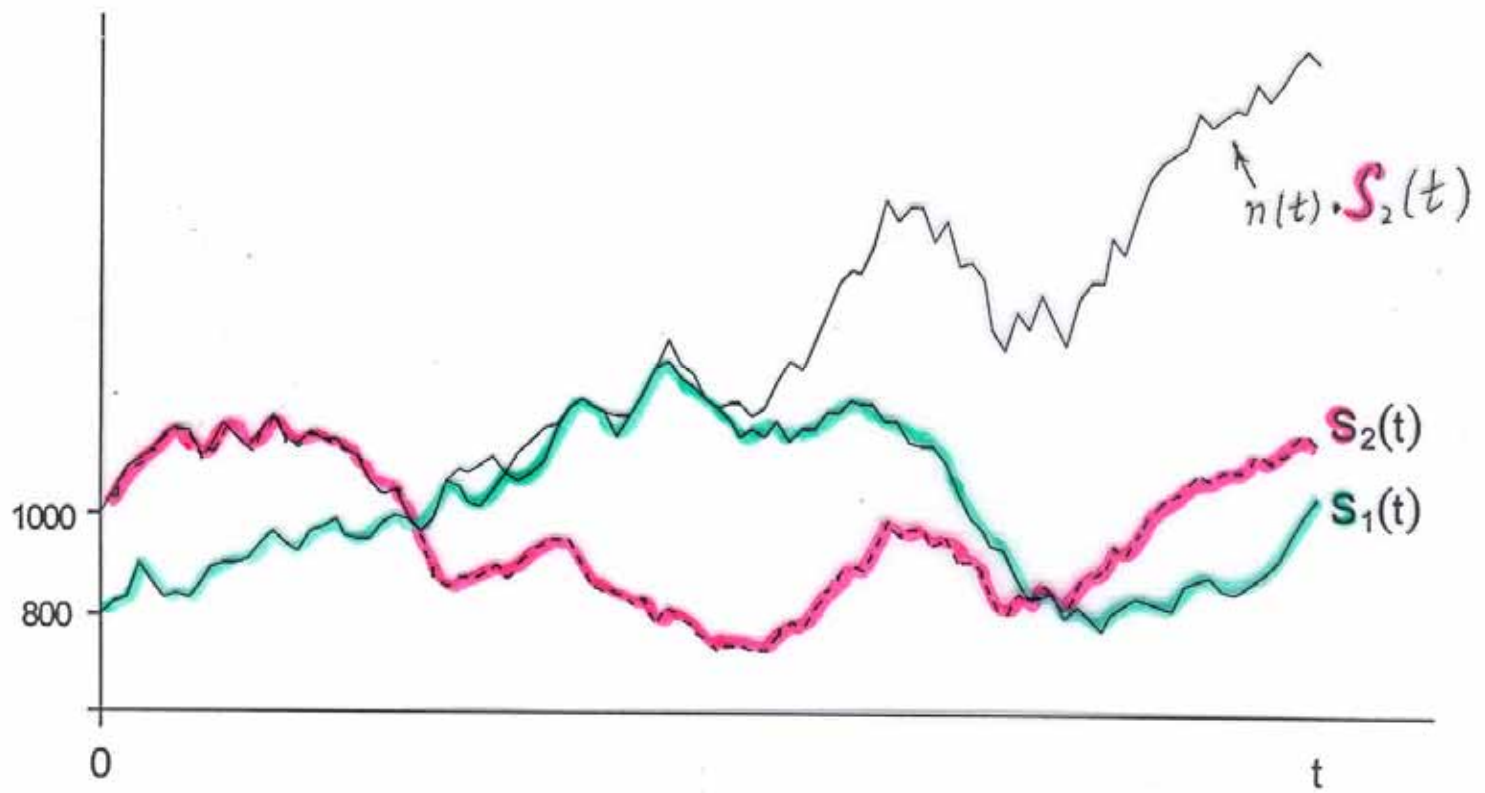
$$E^* \left[e^{-rT} \max \left\{ 1, \max_{0 \leq \tau \leq T} \frac{S(0)K(\tau)}{S(\tau)} \right\} S(T) \right]$$

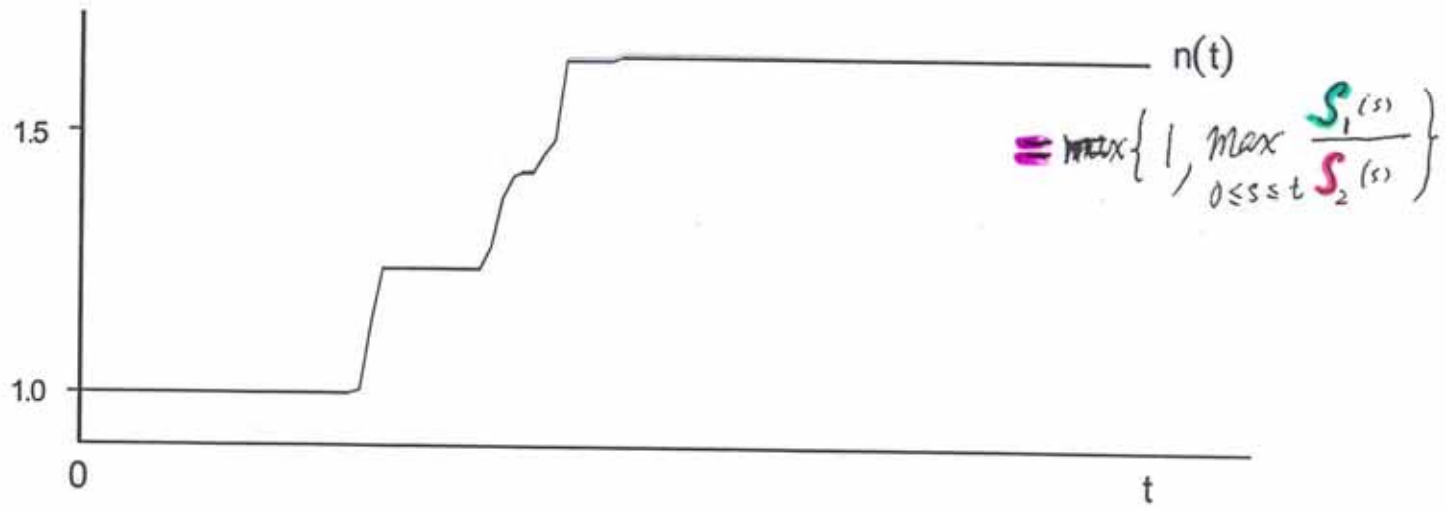
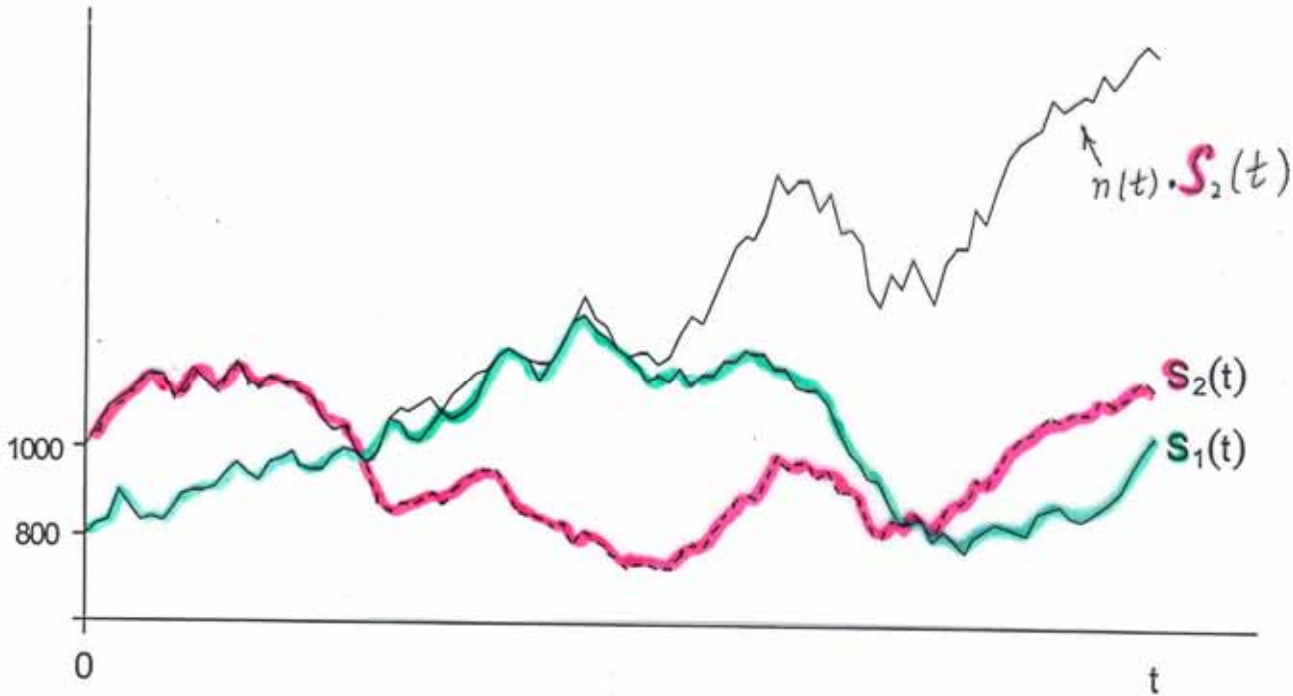
Generalize to stochastic guarantee boundary:

$$E^* \left[e^{-rT} \max \left\{ 1, \max_{0 \leq \tau \leq T} \frac{S_1(\tau)}{S_2(\tau)} \right\} S_2(\tau) \right]$$

A Typical Sample Path of the Fund Values







Earlier, T was a fixed maturity date, i.e., we were pricing “European” options.

How about “American” options?

$$\sup_{\text{All stopping times } T} E^* \left[e^{-rT} \max \left\{ 1, \max_{0 \leq t \leq T} \frac{S_1(t)}{S_2(t)} \right\} S_2(T) \right]$$

Note: The payoff is *path-dependent*.

Two special cases:

$$S_2(t) = S(t), \quad S_1(t) = S(0)(0.9)(1.03)^t$$

$$S_1(t) = S(t), \quad S_2(t) = \text{constant}$$

L. Shepp & A. N. Shiryaev, “The **Russian Option**: Reduced Regret,” *Annals of Probability*, Vol 3, 1993.

The Russian option is an American *lookback* or *high watermark* option without maturity date. Its time-0 price is:

$$\sup_T E^* \left[e^{-rT} \max \left\{ k, \max_{0 \leq t \leq T} S(t) \right\} \right]$$

The constant k can be viewed as the historical maximum of the stock prices (the maximum before time 0).

- Let $S_j(t) = S_j(0) e^{X_j(t)}$, $j = 1, 2$, and we assume that $\{X_1(t), X_2(t)\}$ is a bivariate Brownian motion.
- Assume that each stock (or stock index) pays dividends continuously at a rate proportional to its price. That is, for $j=1, 2$, there is a constant $\zeta_j > 0$, such that stock j pays dividends of amount $\zeta_j S_j(t) dt$ between time t and time $t+dt$.

- Then, under the risk-neutral probability measure, $\left\{ e^{-(r-\zeta_j)t} S_j(t) \right\}$, $j = 1, 2$, are *martingales*.

- Again, r is the risk-free force of interest.

- Then, under the risk-neutral probability measure, $\left\{ e^{-(r-\zeta_j)t} S_j(t) \right\}$, $j = 1, 2$, are *martingales*.

Address of the Society of Actuaries is:
475 North **Martingale** Road
Schaumburg, Illinois, U.S.A.

- Address of the Institute of Actuaries of Australia is:

4 **Martin** Place, Sydney

But

- Address of the Institute of Actuaries of Australia is:

4 **Martin** Place, Sydney

But its current president is

Andrew **Gale**

- How about the Swiss Association of Actuaries?

- How about the Swiss Association of Actuaries?

Hans U. Gerber, “**Martingales** in Risk Theory,” *Bulletin of the Swiss Association of Actuaries* (1973), 205-216.

Under the risk-neutral probability measure,

$$\left\{ e^{-(r - \zeta_j)t} S_j(t) \right\}, j = 1, 2,$$

are martingales. Also, there are **two** martingales of the form

$$\left\{ e^{-rt} [S_1(t)]^\theta [S_2(t)]^{1-\theta} \right\}.$$

The martingale condition is

$$e^{-rt} E^* [e^{\theta X_1(t) + (1-\theta)X_2(t)}] = 1.$$

This leads to the quadratic equation

$$-r + E^*[\theta X_1(1) + (1 - \theta)X_2(1)] \\ + \frac{1}{2} \text{Var}[\theta X_1(1) + (1 - \theta)X_2(1)] = 0.$$

Its solutions are $\theta_1 < 0$ and $\theta_2 > 1$. Thus, the two processes,

$$\left\{ e^{-rt} [S_1(t)]^{\theta_j} [S_2(t)]^{1-\theta_j} \right\}, \quad j = 1, 2,$$

are martingales under the risk-neutral probability measure.

$$\text{Let } n(T) = \max \left\{ 1, \max_{0 \leq \tau \leq T} \frac{S_1(\tau)}{S_2(\tau)} \right\}$$

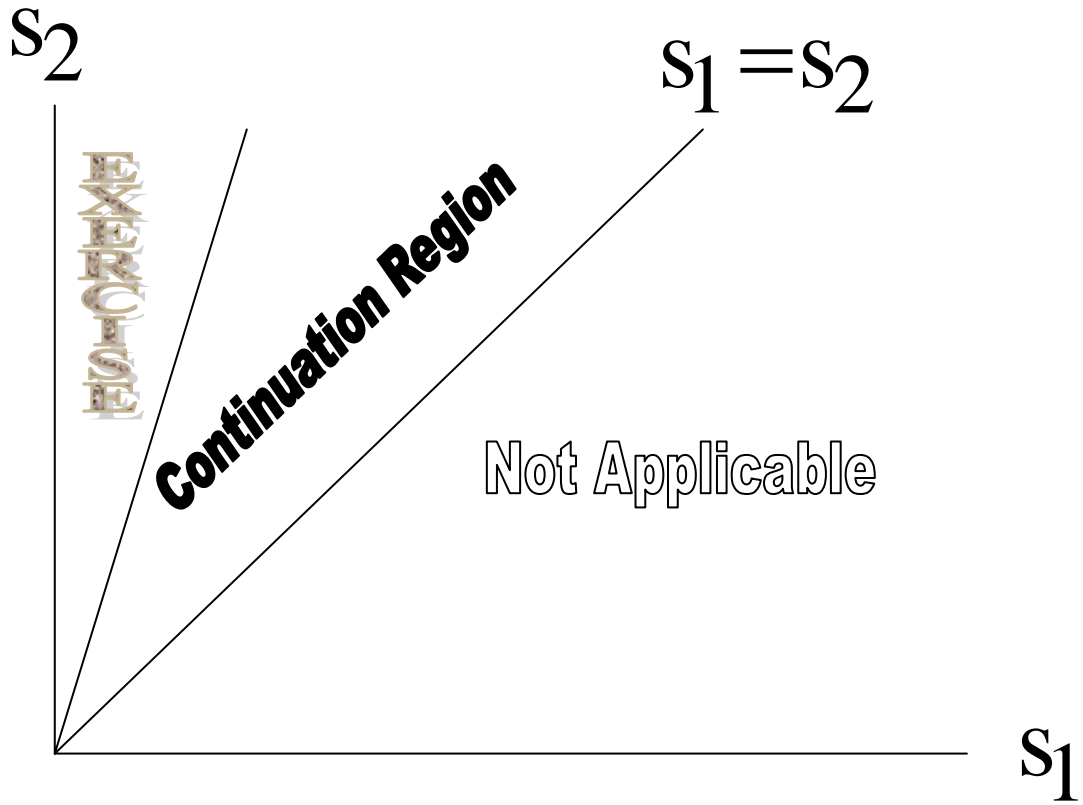
For $s_1 > 0$, $s_2 > 0$, define

$$V(s_1, s_2)$$

$$= \sup_T E^* [e^{-rT} n(T) S_2(T) | S_1(0) = s_1, S_2(0) = s_2]$$

The supremum is taken over all stopping times T . There is no fixed expiry date.

This is the price of the perpetual *dynamic protection option*.

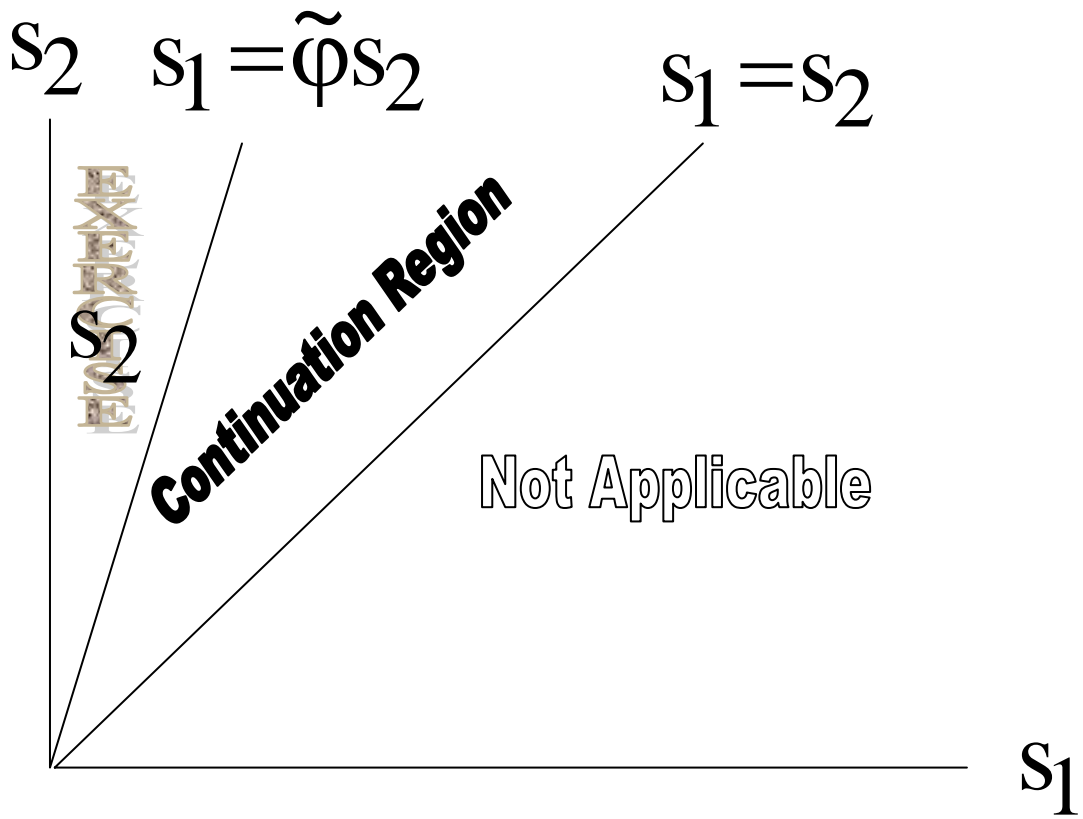


$$h(s) = (\theta_2 - 1)s^{\theta_1} + (1 - \theta_1)s^{\theta_2}, \quad s > 0$$

$$\tilde{\varphi} = \left(\frac{-\theta_1(\theta_2 - 1)}{(1 - \theta_1)\theta_2} \right)^{\frac{1}{\theta_2 - \theta_1}}$$

$$0 < \tilde{\varphi} < 1$$

$$V(s_1, s_2) = \begin{cases} \frac{h(s_1 / s_2)}{h(\tilde{\varphi})} s_2 & \text{if } \tilde{\varphi} < \frac{s_1}{s_2} \leq 1 \\ s_2 & \text{if } 0 < \frac{s_1}{s_2} \leq \tilde{\varphi} \end{cases}$$



Instead of $n(T) = \max \left\{ 1, \max_{0 \leq t \leq T} \frac{S_1(t)}{S_2(t)} \right\}$,

we now consider $n(T) = \max \left\{ 1, \frac{S_1(T)}{S_2(T)} \right\}$.

Then, $n(T)S_2(T) = \max\{S_1(T), S_2(T)\}$,

which is the payoff of the *maximum option* (also called *alternative option* or *greater-of option*). This is a simpler option since the payoff is not path-dependent.

The price of the American maximum option without a fixed expiry date is:

$$W(s_1, s_2)$$

$$= \sup_T E^*[e^{-rT} \max\{S_1(T), S_2(T)\} \mid S_1(0)=s_1, S_2(0)=s_2] ,$$

$$s_1 > 0 \text{ and } s_2 > 0.$$

This option has been evaluated in the paper Gerber and Shiu, “Martingale Approach to Pricing Perpetual American Options on Two Stocks,” *Mathematical Finance*,” Vol 6 (1996).

$$\sup_T E^* [e^{-rT} n(T) S_2(T) | S_1(0) = s_1, S_2(0) = s_2]$$

$$\text{For } V(s_1, s_2), \quad n(T) = \max \left\{ 1, \max_{0 \leq t \leq T} \frac{S_1(t)}{S_2(t)} \right\}.$$

$$\text{For } W(s_1, s_2), \quad n(T) = \max \left\{ 1, \frac{S_1(T)}{S_2(T)} \right\}.$$

Obviously, $V(s_1, s_2) \geq W(s_1, s_2)$.

$$\sup_T E^* [e^{-rT} n(T) S_2(T) | S_1(0) = s_1, S_2(0) = s_2]$$

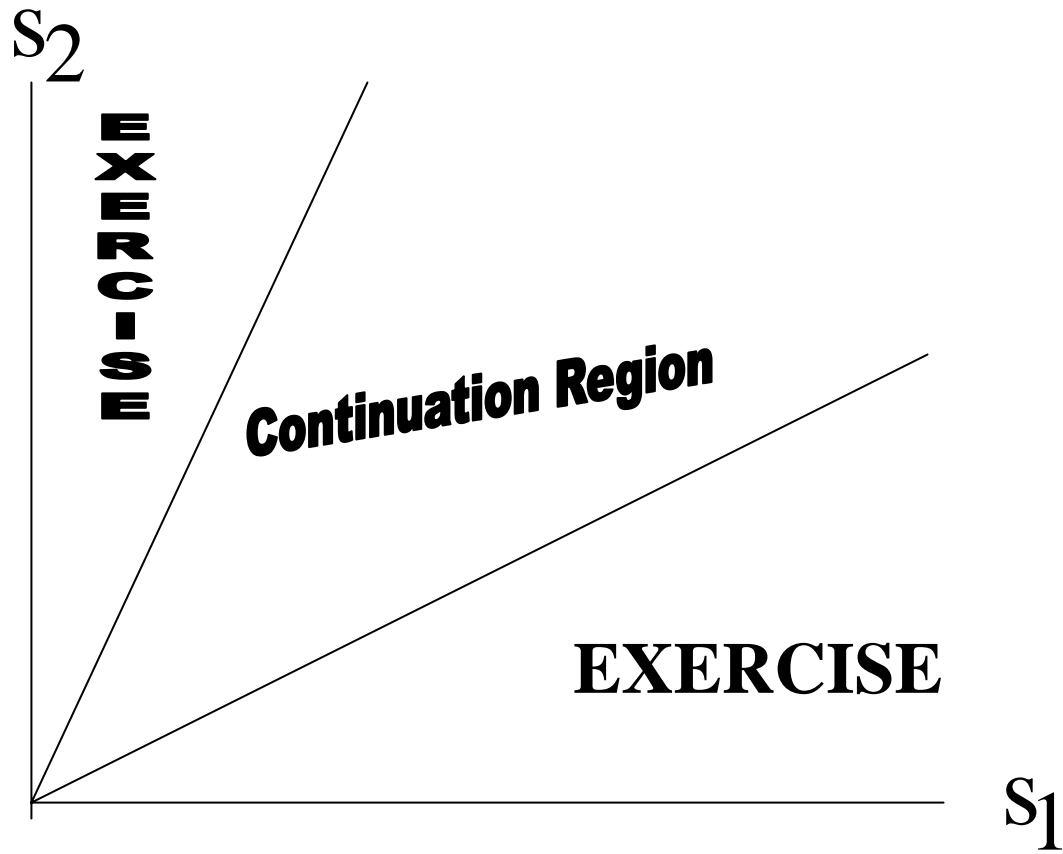
$$\text{For } V(s_1, s_2), \quad n(T) = \max \left\{ 1, \max_{0 \leq t \leq T} \frac{S_1(t)}{S_2(t)} \right\}.$$

$$\text{For } W(s_1, s_2), \quad n(T) = \max \left\{ 1, \frac{S_1(T)}{S_2(T)} \right\}.$$

Obviously, $V(s_1, s_2) \geq W(s_1, s_2)$.

Surprisingly, there is a constant $\tilde{c} > 1$, such that

$$V(s_1, s_2) = W(\tilde{c}s_1, s_2).$$

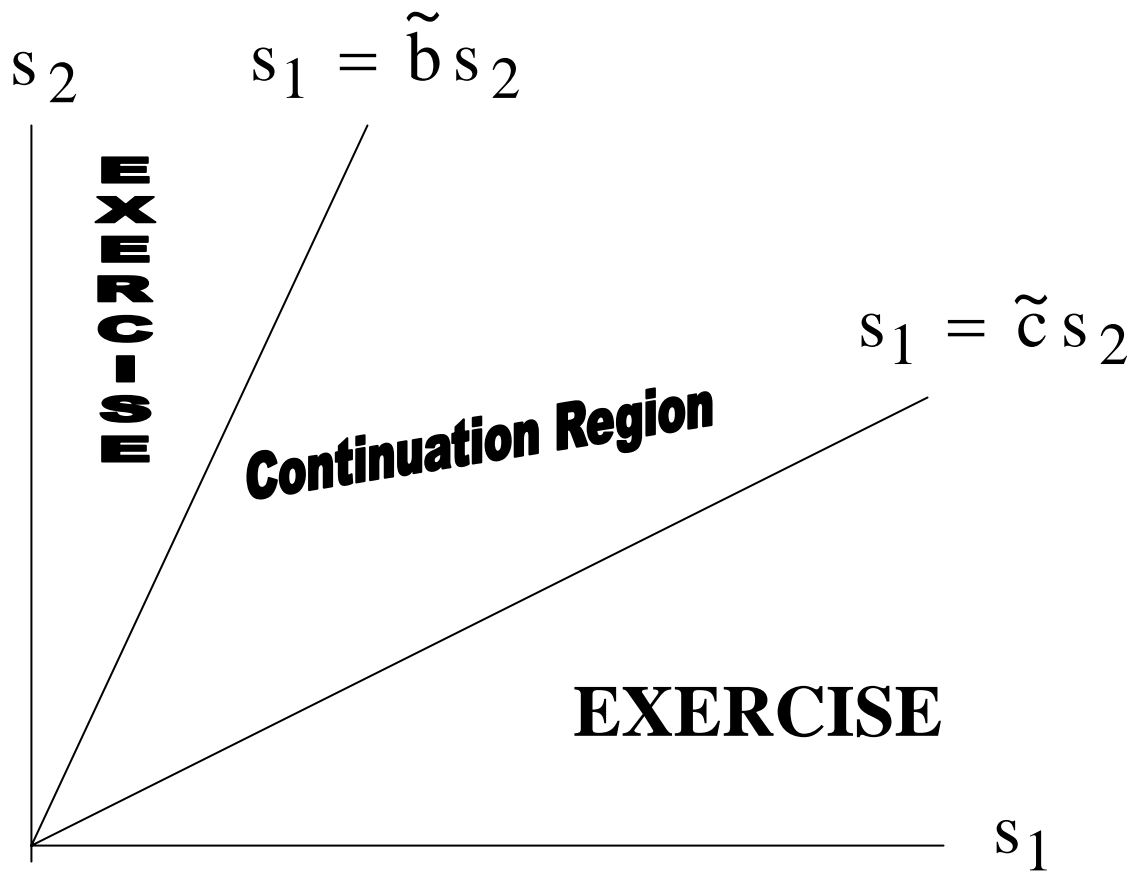


$$\tilde{\mathbf{b}} = \left(\frac{-\theta_1}{1-\theta_1} \right)^{(1-\theta_1)/(\theta_2-\theta_1)} \left(\frac{\theta_2}{\theta_2-1} \right)^{(\theta_2-1)/(\theta_2-\theta_1)}$$

$$\tilde{\mathbf{c}} = \left(\frac{-\theta_1}{1-\theta_1} \right)^{-\theta_1/(\theta_2-\theta_1)} \left(\frac{\theta_2}{\theta_2-1} \right)^{\theta_2/(\theta_2-\theta_1)}$$

$$\mathbf{k}(\mathbf{x}) = \frac{\theta_2 (\mathbf{x}/\tilde{\mathbf{b}})^{\theta_1} - \theta_1 (\mathbf{x}/\tilde{\mathbf{b}})^{\theta_2}}{\theta_2 - \theta_1} \quad \mathbf{x} > 0$$

$$W(s_1, s_2) = \begin{cases} s_2 & \text{if } \frac{s_1}{s_2} \leq \tilde{b} \\ s_2 k\left(\frac{s_1}{s_2}\right) & \text{if } \tilde{b} < \frac{s_1}{s_2} < \tilde{c} \\ s_1 & \text{if } \frac{s_1}{s_2} \geq \tilde{c} \end{cases}$$



It can be readily checked that

$$\tilde{\varphi} = \frac{\tilde{\mathbf{b}}}{\tilde{\mathbf{c}}}.$$

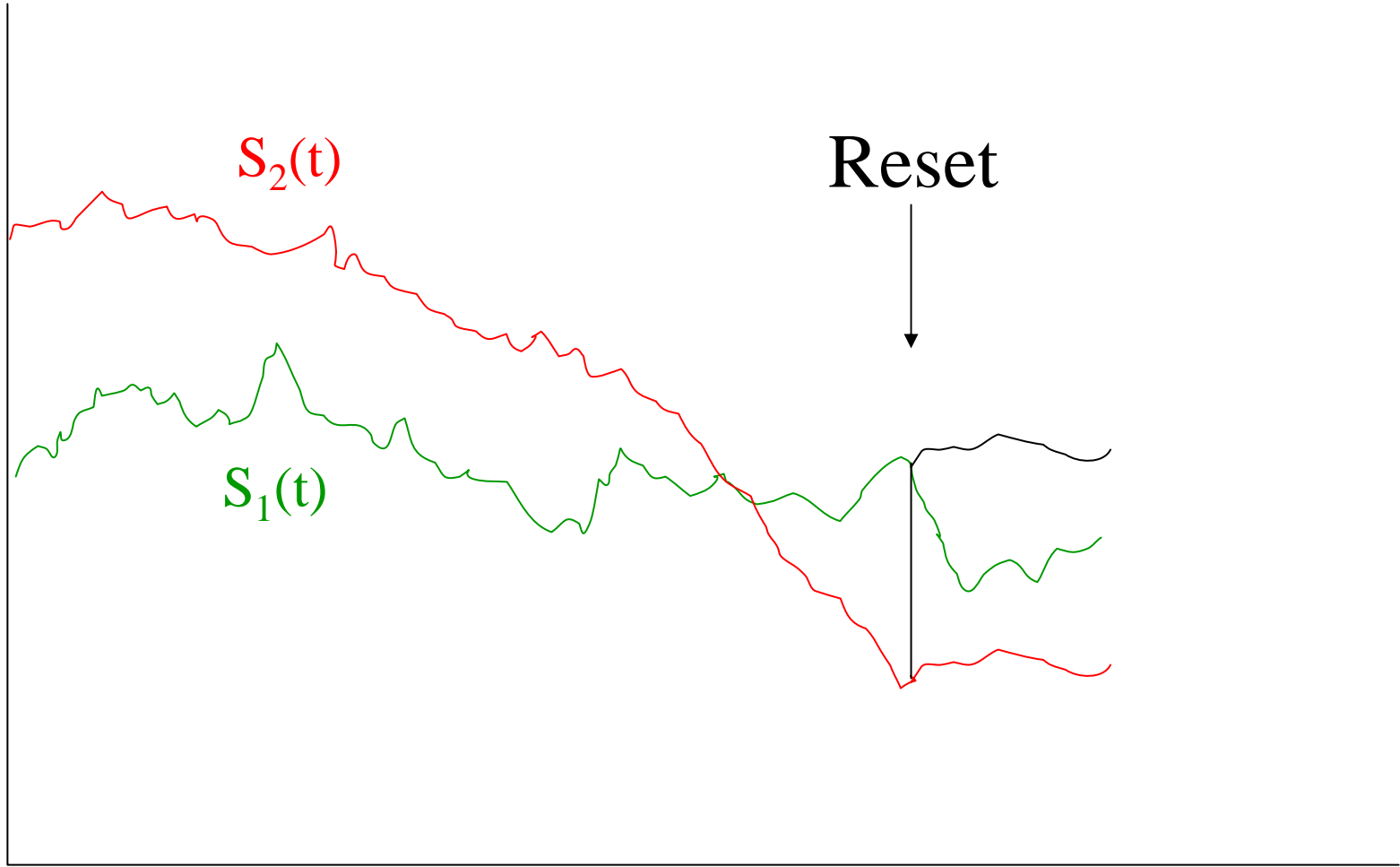
From this, we realized that

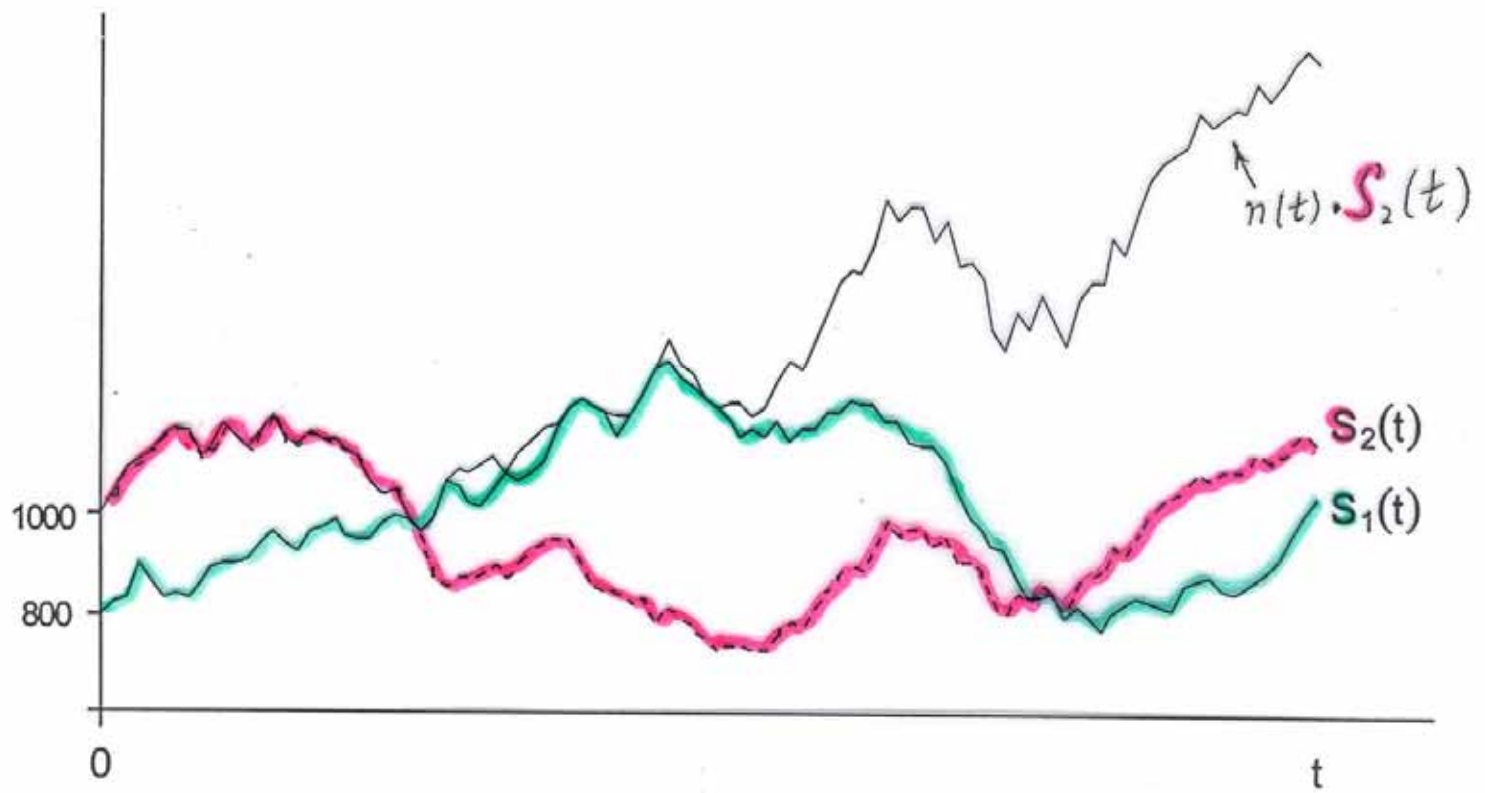
$$V(s_1, s_2) = W(\tilde{\mathbf{c}}s_1, s_2)$$

But why is this formula true?

Y K Kwok and C C Chu wrote a discussion on “Pricing Perpetual Fund Protection with Withdrawal Protection,” *North American Actuarial Journal*, Vol 7 (2), 2003. They introduced the concept of a perpetual option with “up to n resets”. When the number of possible resets n becomes ∞ , we have

$$V(s_1, s_2) = W(\tilde{c}s_1, s_2)$$





For $n = 1, 2, 3, \dots$, and let $V_n(s_1, s_2)$ denote the price of the option with *up to* n resets, where $s_1 = S_1(0) > 0$ and $s_2 = S_2(0) > 0$. The option has no fixed expiry date. Thus,

$$V_{n+1}(s_1, s_2) = \sup_T E^* [e^{-rT} \max\{V_n(S_1(T), S_1(T)), S_2(T)\} \mid S_1(0) = s_1, S_2(0) = s_2].$$

Because $V_n(s_1, s_2)$ is a homogeneous function of degree 1,

$$V_n(S_1(T), S_1(T)) = V_n(1, 1)S_1(T)$$

Define

$$\kappa_n = V_n(1, 1).$$

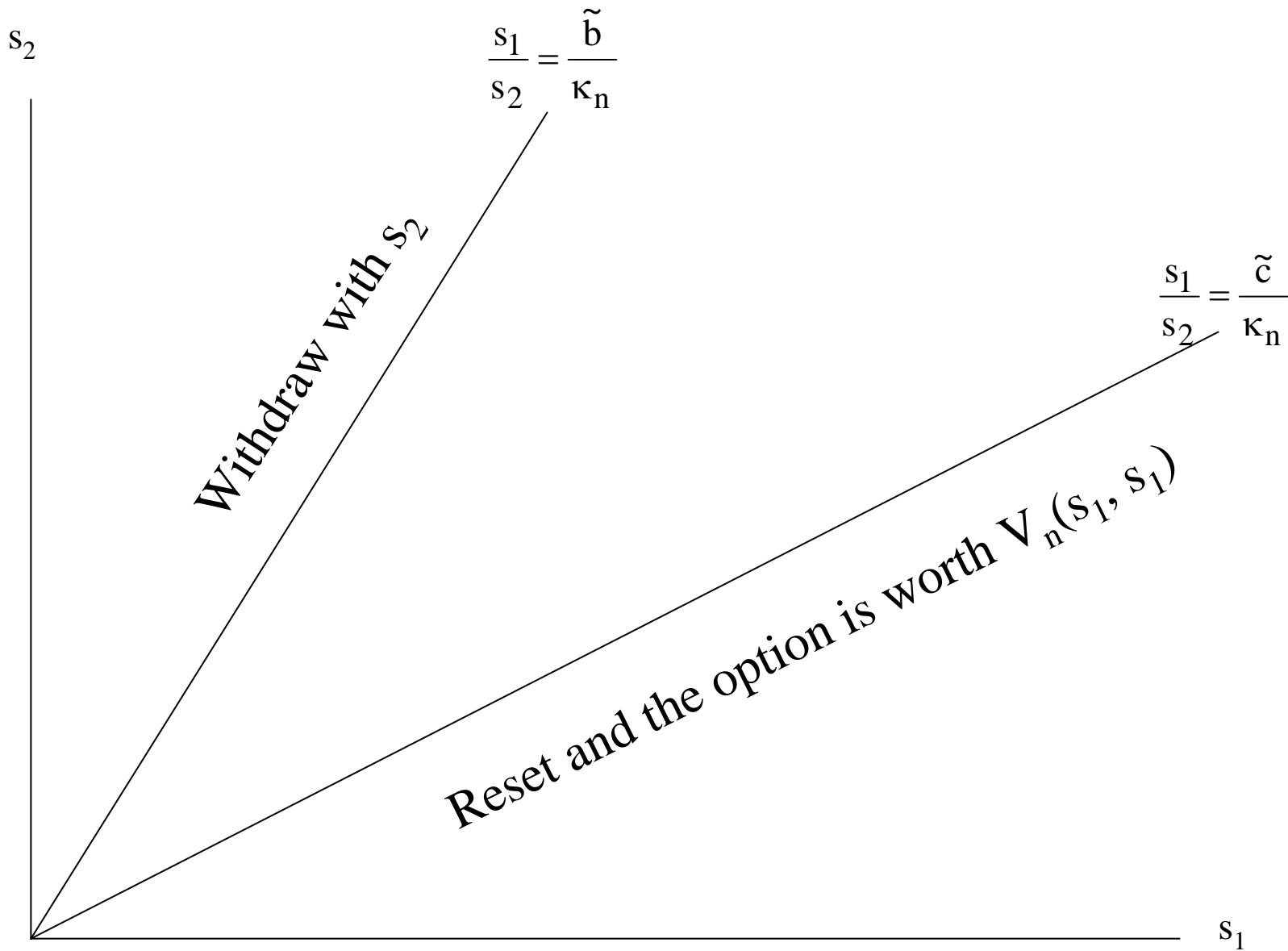
Then,

$$V_{n+1}(s_1, s_2)$$

$$= \sup_T E^* [e^{-rT} \max\{\kappa_n S_1(T), S_2(T)\} \mid S_1(0) = s_1, S_2(0) = s_2]$$

$$= \sup_T E^* [e^{-rT} \max\{S_1(T), S_2(T)\} \mid S_1(0) = \kappa_n s_1, S_2(0) = s_2]$$

$$= W(\kappa_n s_1, s_2).$$



$$V_{n+1}(s_1, s_2) = W(\kappa_n s_1, s_2)$$

$$V_{n+1}(s_1, s_2) = W(\kappa_n s_1, s_2)$$

Put $s_1 = s_2 = 1$. Then

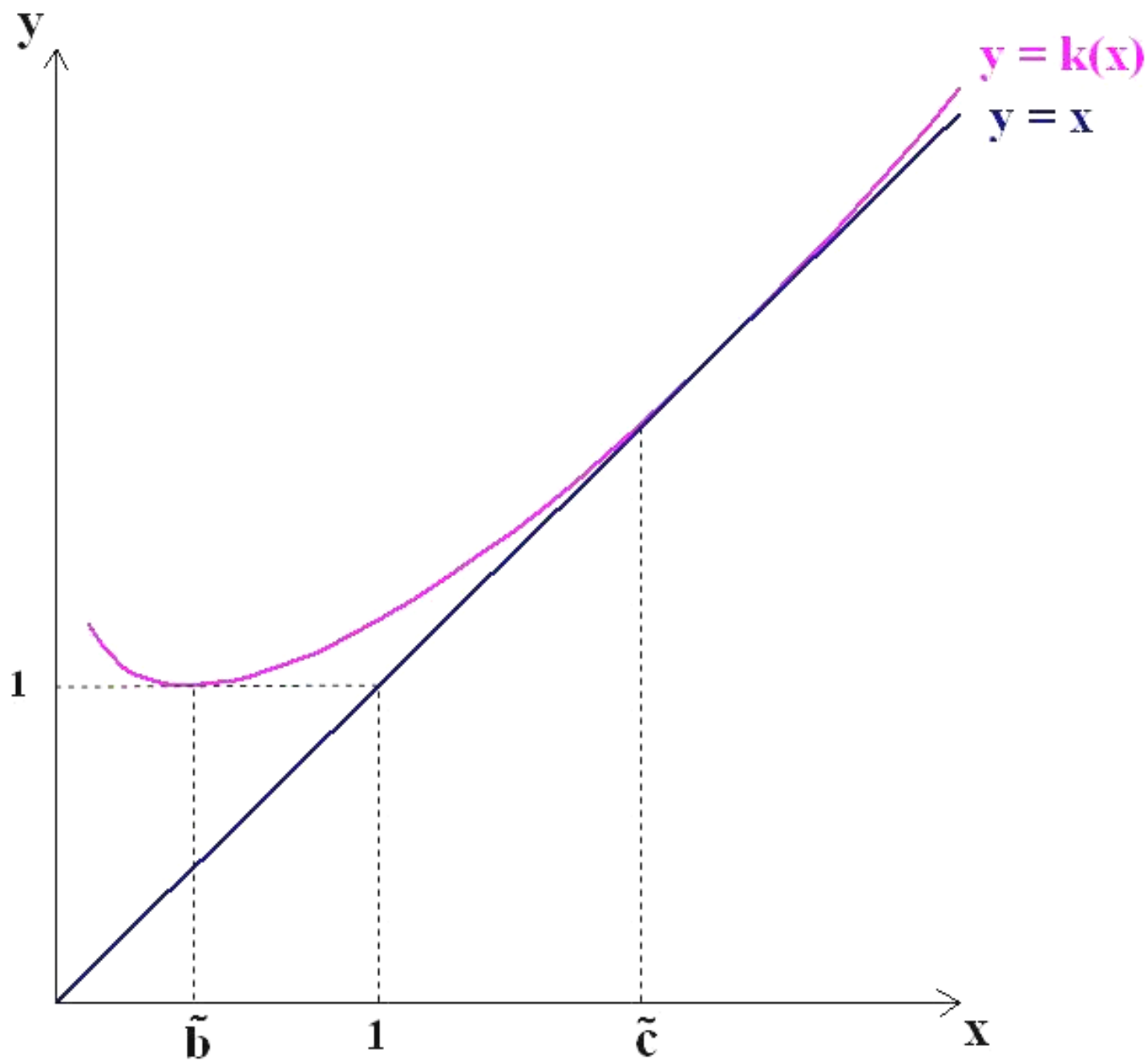
$$V_{n+1}(1, 1) = W(\kappa_n, 1),$$

or

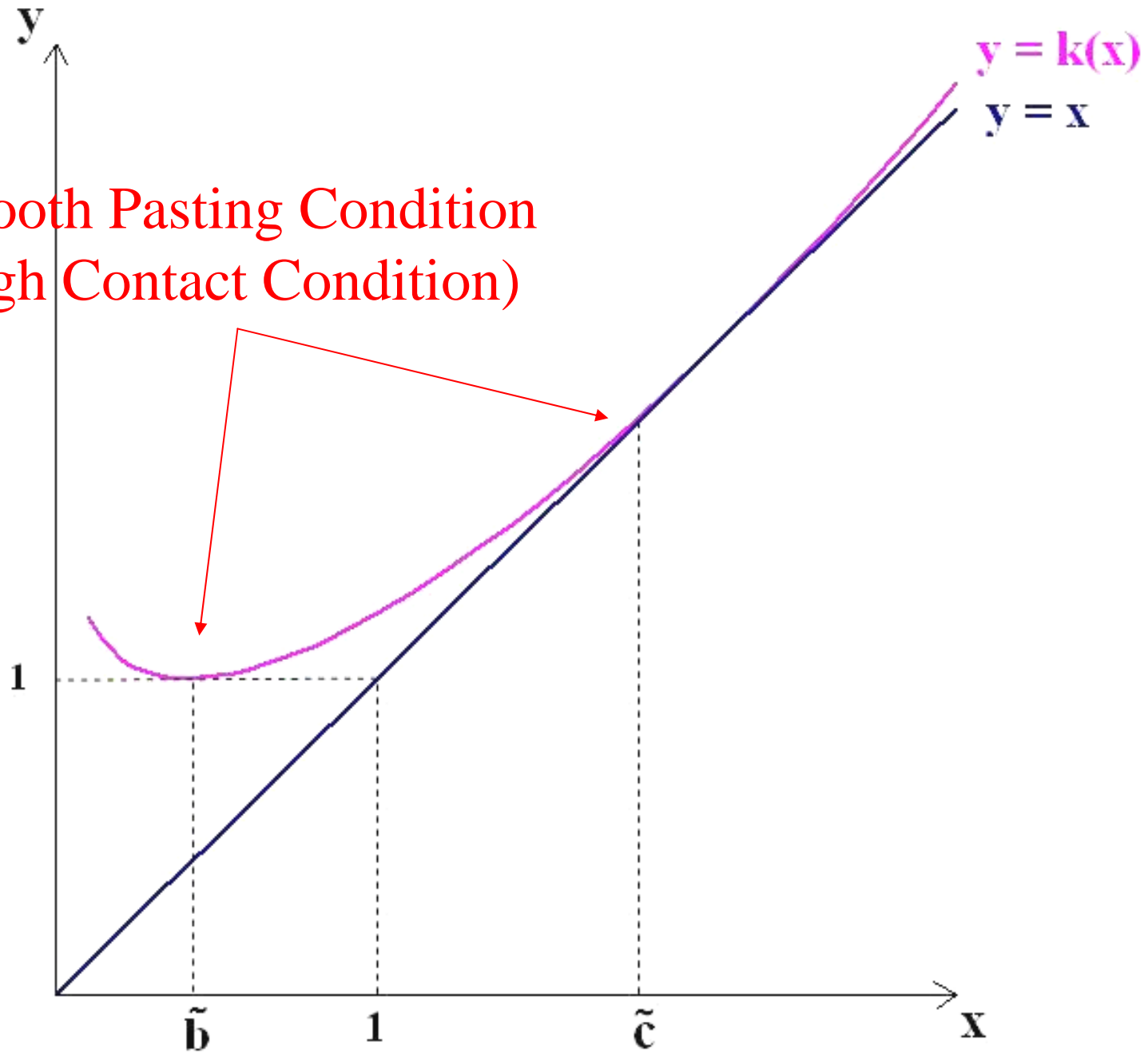
$$\begin{aligned}\kappa_{n+1} &= W(\kappa_n, 1) \\ &= k(\kappa_n),\end{aligned}$$

where

$$k(x) = \frac{\theta_2 (x/\tilde{b})^{\theta_1} - \theta_1 (x/\tilde{b})^{\theta_2}}{\theta_2 - \theta_1} \quad x > 0.$$



Smooth Pasting Condition
(High Contact Condition)



$$\kappa_1 = W(1, 1) = k(1)$$

$$\kappa_2 = k(\kappa_1)$$

$$\kappa_3 = k(\kappa_2)$$

.

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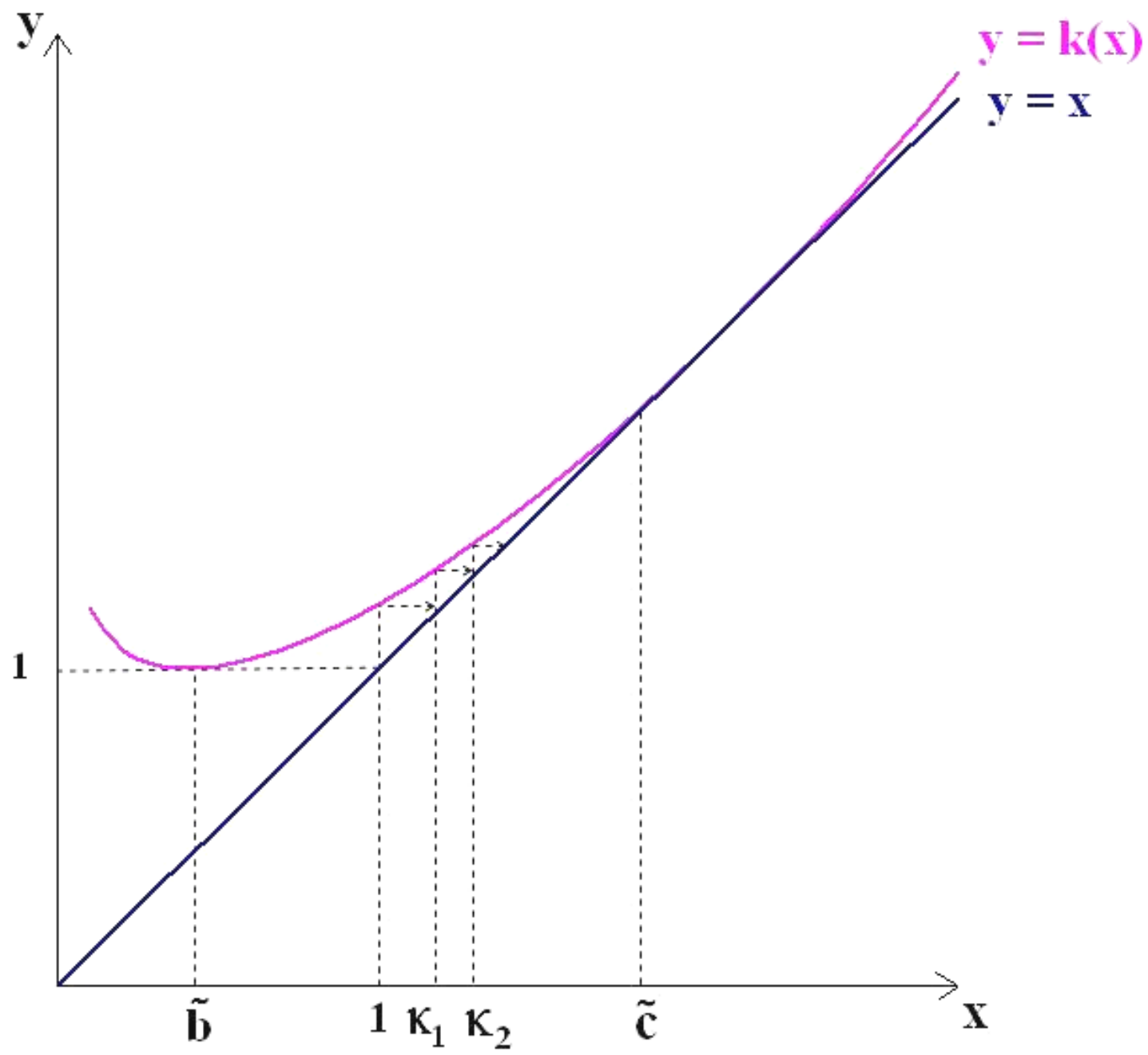
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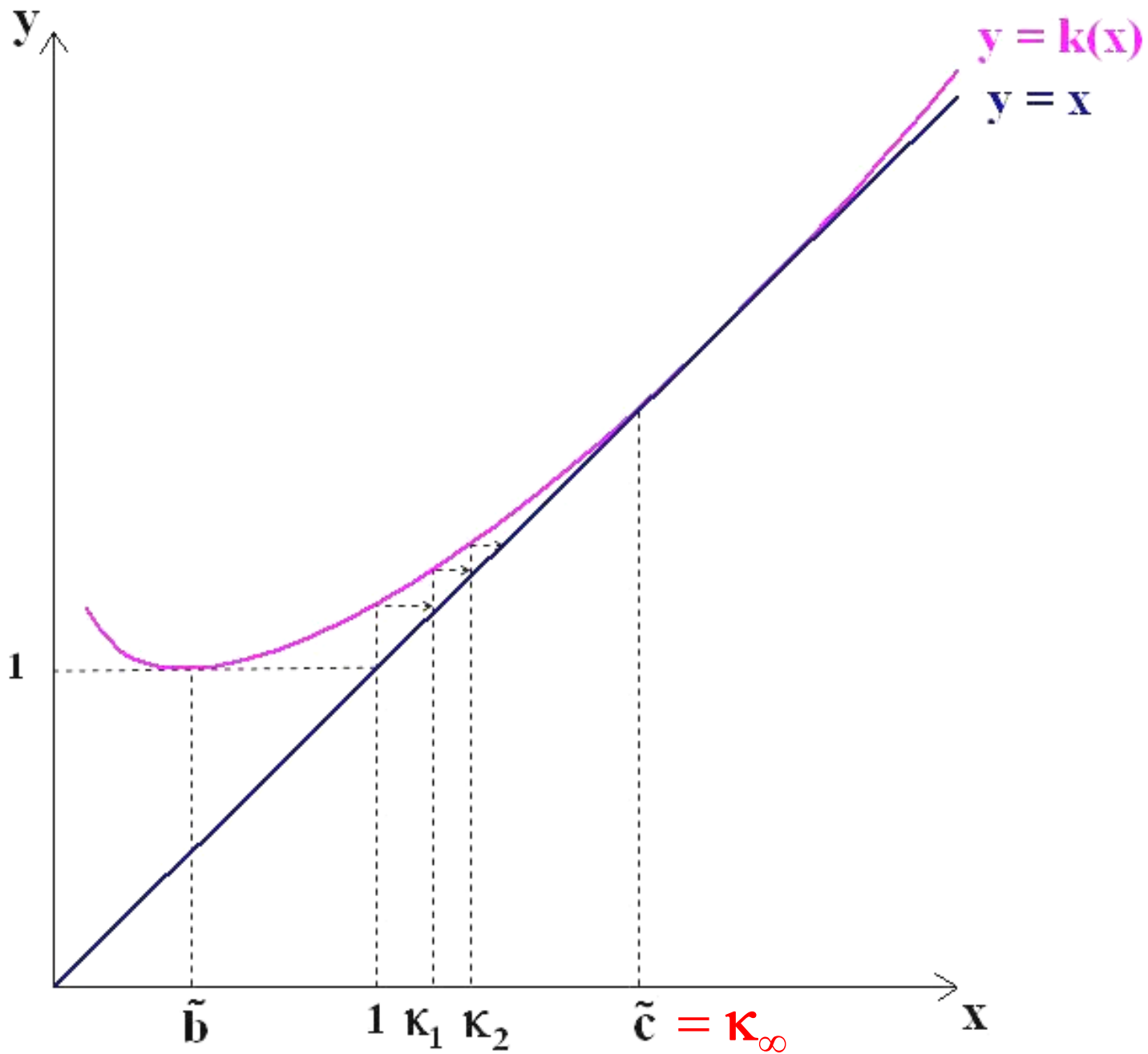
$$\kappa_{n+1} = k(\kappa_n),$$

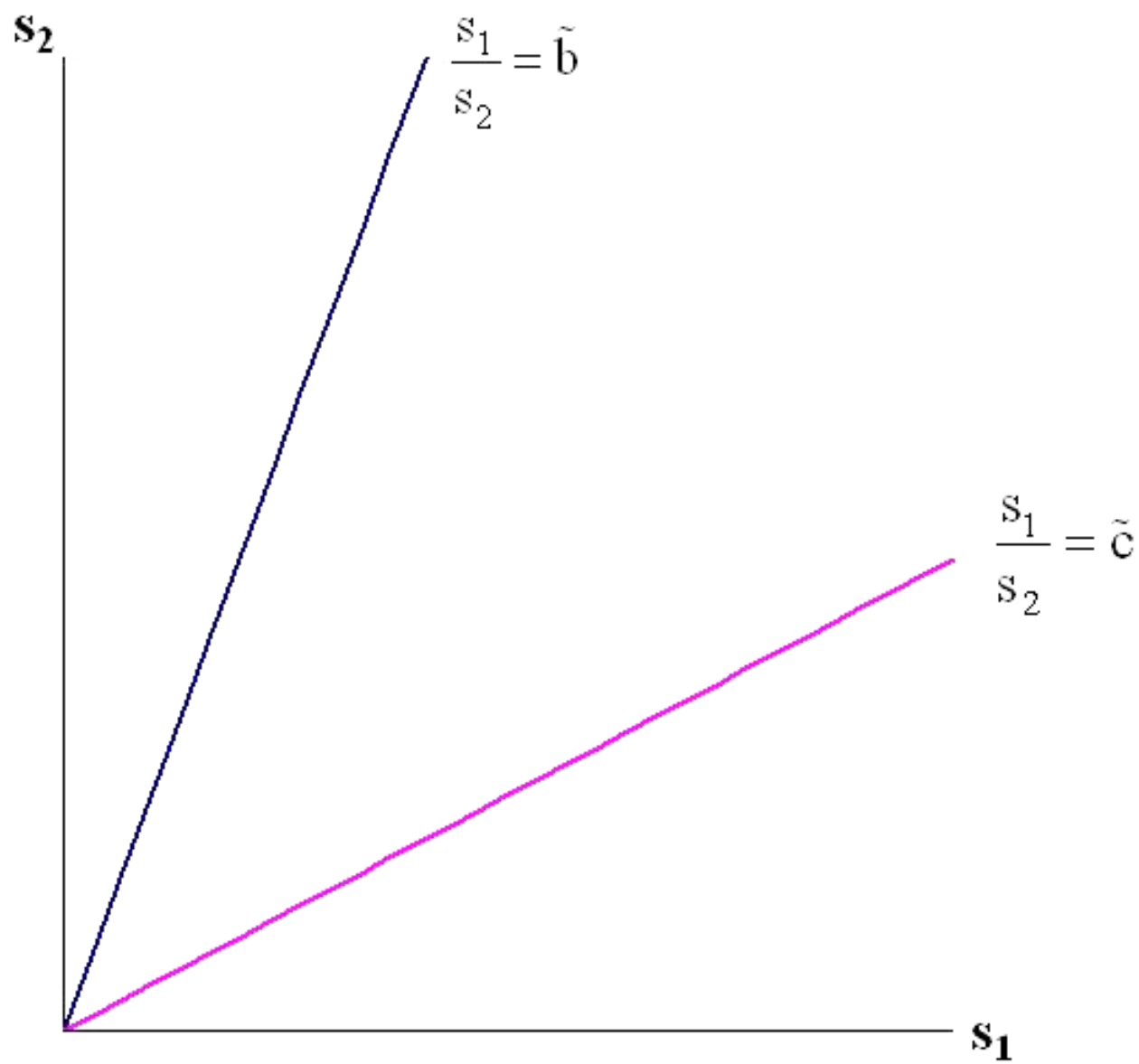
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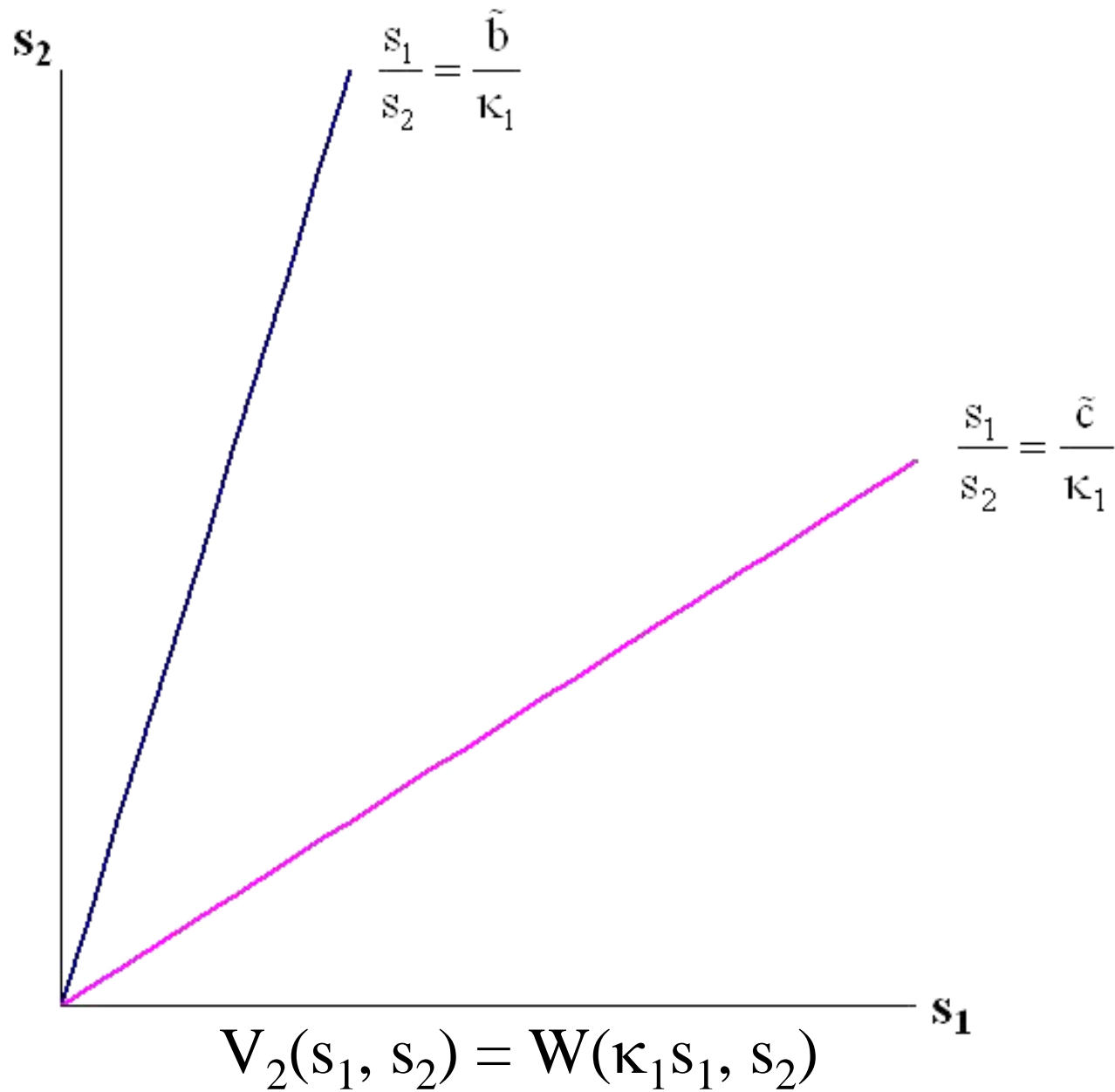
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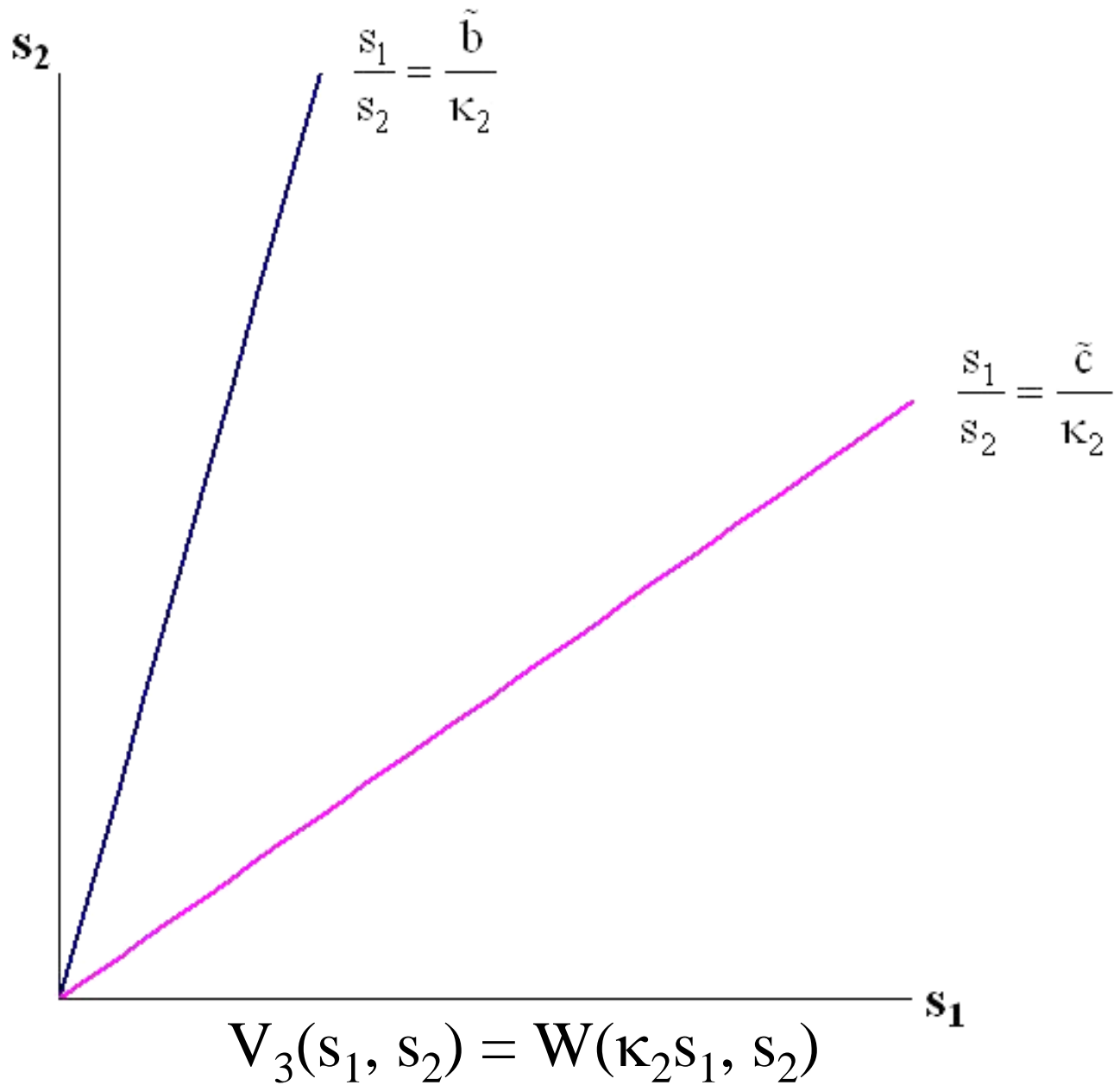


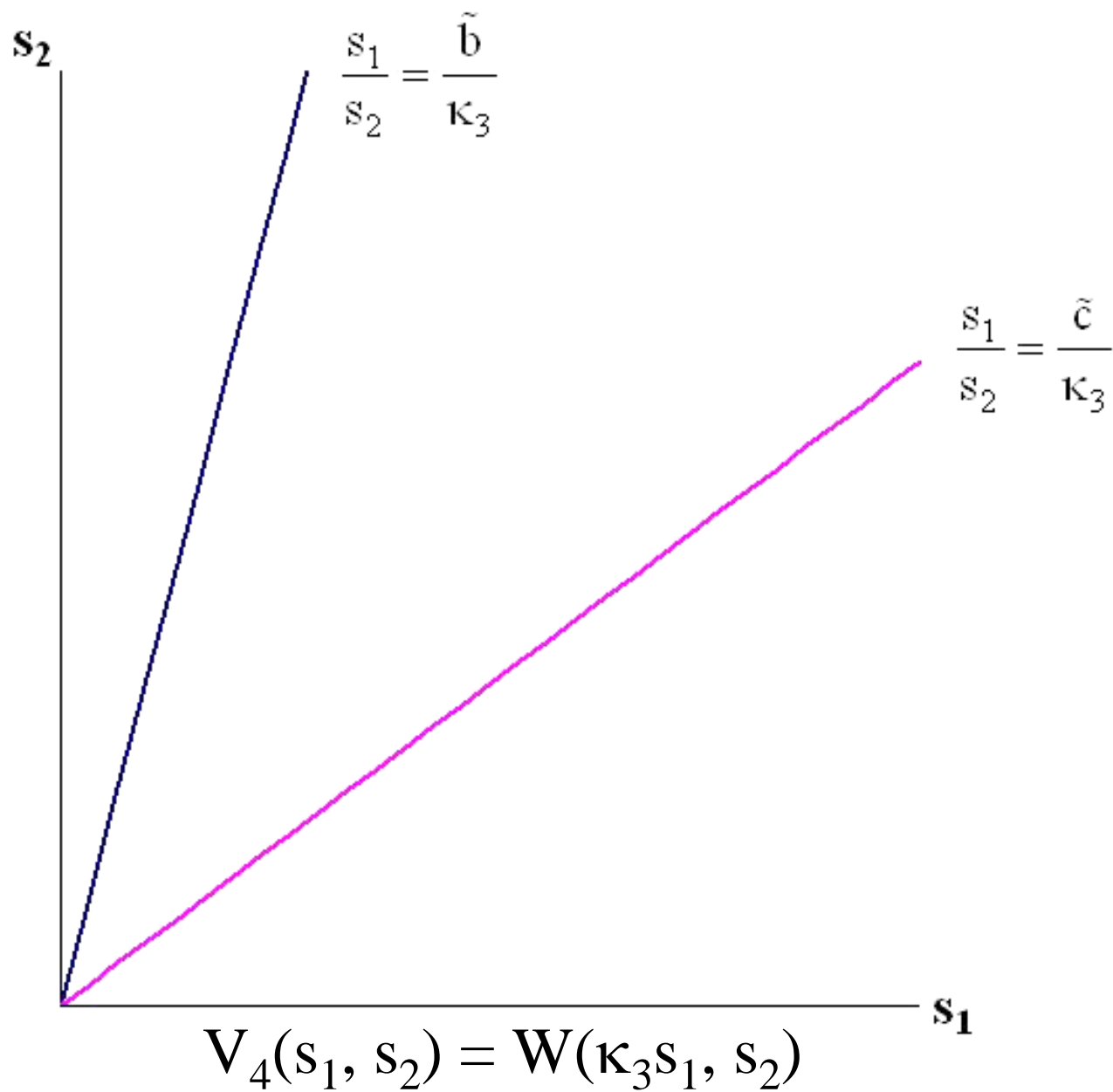


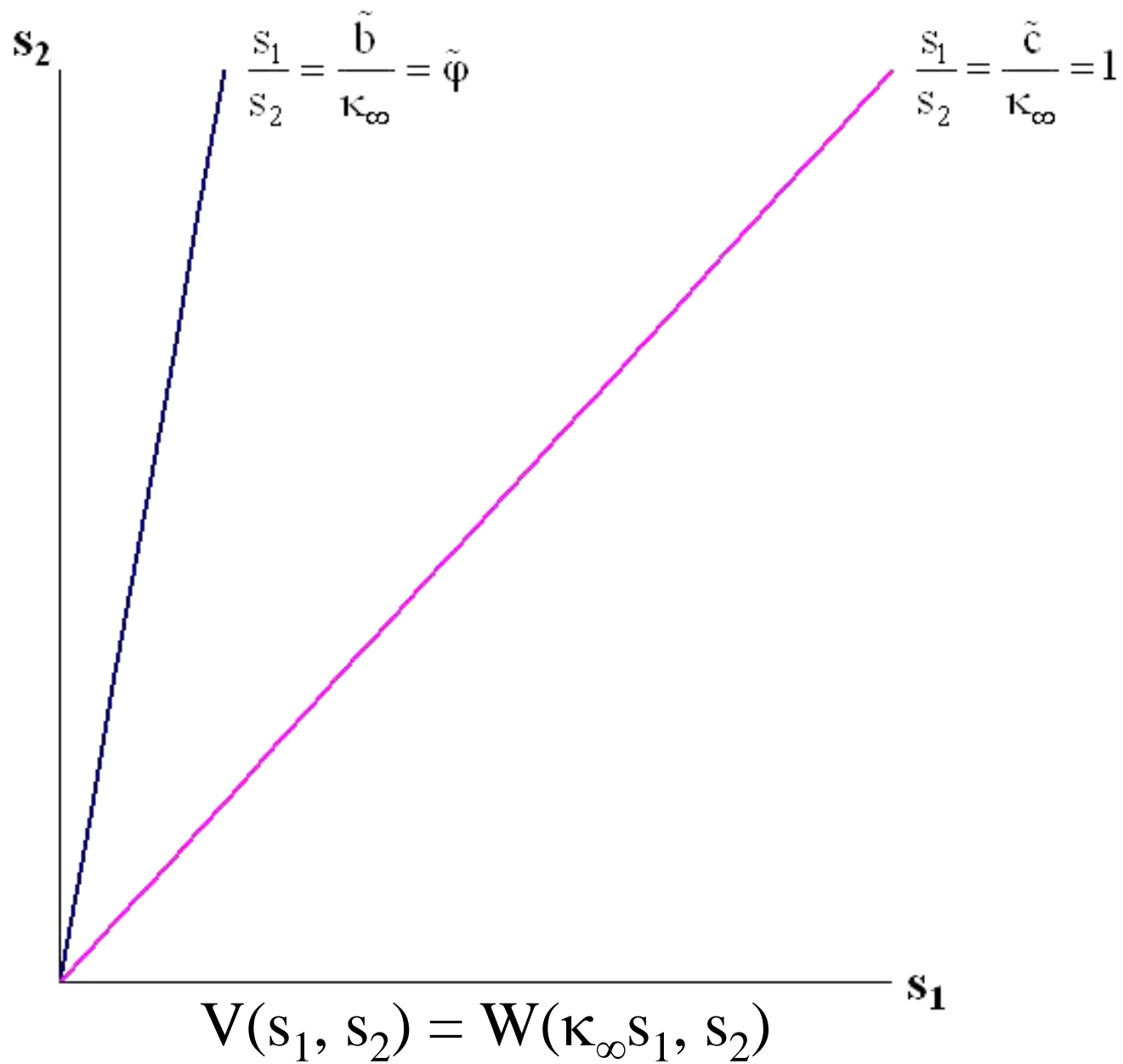


$$V_1(s_1, s_2) = W(s_1, s_2)$$









H. U. Gerber and G. Pafumi, “Pricing Dynamic Investment Fund Protection,” *North American Actuarial Journal*, Vol 4 (2), 2000.

J. Imai and P. P. Boyle, “Dynamic Fund Protection,” *North American Actuarial Journal*, Vol 5 (3), 2001.

H. U. Gerber and E. S.W. Shiu, “Pricing Perpetual Fund Protection with Withdrawal Protection,” *North American Actuarial Journal*, Vol 7 (2), 2003.

H.-K. Fung and L. K. Li, “Pricing Discrete Dynamic Fund Protection,” *North American Actuarial Journal*, Vol 7 (4), 2003.

C. C. Chu and Y. K. Kwok, “Reset and Withdrawal Rights in Dynamic Fund Protection,” *Insurance: Mathematics and Economics*, Vol 34, 2004.

Thank you for your patience

Time for lunch