Dynamic Fund Protection

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Presentation based on two papers:


Presentation based on two papers by **Hans U. Gerber**:


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• Stock Index value at time t: 
  \[ I(t) = I(0)e^{Y(t)} \]
  e.g., S&P 500

Assume \{Y(t)\} is a Brownian motion (Wiener process).

• Value of one unit of fund at time t:
  \[ S(t) = S(0)e^{\alpha Y(t)} \]
  where \(\alpha\), called the participation rate, is usually \(< 1\).
\[ n(t) = \text{number of fund units in the customer’s account at time } t \]

\[ n(0) = 1 \]

\[ n(t)S(t) \geq S(0)K(t) \quad \text{guarantee boundary} \]

\[ \text{e.g., } K(t) = 0.9 \ (1.03)^t \text{ for satisfying U.S. nonforfeiture laws} \]
A Typical Sample Path of the Fund Values

Guarantee Level

\[ n(t) \cdot S(t) \]

\[ S(0) \]

\[ S(0) \cdot K \]

\[ S(t) \]

\[ t \]
• What is $n(t)$?

$n(0) = 1$

$$n(t) \geq \frac{S(0)K(t)}{S(t)}$$

$n(t)$ being non-decreasing means:

For all $t \geq 0$, $n(t) \geq \max_{0 \leq \tau \leq t} n(\tau)$

Hence,

$$n(t) \geq \max \left\{ 1, \max_{0 \leq \tau \leq t} \frac{S(0)K(\tau)}{S(\tau)} \right\}$$
Therefore, the number of fund units in the customer's account at time $t$ is

$$n(t) = \max \left\{ 1, \max_{0 \leq \tau \leq t} \frac{S(0)K(\tau)}{S(\tau)} \right\}$$

The customer’s account value at time $t$ is $n(t)S(t)$. 
By the *Fundamental Theorem of Asset Pricing*, the time-0 “value” of this Dynamic Fund Protection contract is

$$E^* [e^{-rT} n(T) S(T)]$$

where * signifies that the expectation is taken with respect to the *risk-neutral* probability measure, *r* is the risk-free force of interest, and *T* is the maturity date of the contract.

This “value” should be compared with *S(0)*, the premium for one unit of the fund at time 0.
Because

$$S(T) = S(0)e^{\alpha Y(T)}$$

we have

$$E^* \left[ e^{-rT} n(T)S(T) \right]$$

$$= S(0)e^{-rT} E^* \left[ n(T)e^{\alpha Y(T)} \right]$$

What is $E^* \left[ n(T)e^{\alpha Y(T)} \right]$?
\[ E^* \left[ n(T)e^{\alpha Y(T)} \right] \]

\[ = \frac{E^* \left[ n(T)e^{\alpha Y(T)} \right]}{E^* \left[ e^{\alpha Y(T)} \right]} \ E^* \left[ e^{\alpha Y(T)} \right] \]

\[ = E^{**} \left[ n(T) \right] \ E^* \left[ e^{\alpha Y(T)} \right] \]

where ** signifies a changed probability measure (an Esscher transform).
Now, $\mathbb{E}^*[e^{\alpha Y(T)}]$ is the moment-generating function of the random variable $Y(T)$ (with respect to the risk-neutral probability measure) at the value $\alpha$.

It is assumed that $\{Y(t)\}$ is a Brownian motion with diffusion coefficient $\sigma$. Under the risk-neutral probability measure, $Y(T)$ is a normal random variable, and

$$\mathbb{E}^*[e^{\alpha Y(T)}] = \exp(\alpha \mathbb{E}^*[Y(T)] + \frac{1}{2} \alpha^2 \text{Var}^*[Y(T)])$$
where

$$E^*[Y(T)] = (r - \frac{\sigma^2}{2} - \zeta)T,$$

and

$$\text{Var}^*[Y(T)] = \text{Var}[Y(T)] = \sigma^2 T.$$

Here, \( r \) is the risk-free force of interest, and \( \zeta \) is the (constant) dividend-yield rate.

So it remains to determine \( E^{**}[n(T)] \).
To determine $E^{**}[n(T)]$, we consider the case

$$K(t) = \beta e^{g t} \quad \text{[e.g., } K(t) = 0.9(1.03)^t\text{]}$$

Then,

$$n(t) = \max \left\{ 1, \max_{0 \leq \tau \leq T} \frac{S(0)K(\tau)}{S(\tau)} \right\}$$

$$= \max \left\{ 1, \max_{0 \leq \tau \leq T} \frac{\beta e^{g \tau}}{e^{\alpha Y(\tau)}} \right\}$$

$$= \max \left\{ 1, \beta \exp \left[ \max_{0 \leq \tau \leq T} \left( g \tau - \alpha Y(\tau) \right) \right] \right\}$$
Define \( X(\tau) = g\tau - \alpha Y(\tau) \).

Let

\[
M(t) = \max_{0 \leq \tau \leq T} X(\tau)
\]

be the *running maximum* of the process \( \{X(\tau)\} \).

Then,

\[
n(t) = \max\{1, \beta e^{M(t)}\}.
\]

Thus, to determine \( E^*[n(T)] \), we need to know the probability distribution of running maximum \( M(T) \) under the ** probability measure.
For a Wiener process \( \{X(\tau)\} \) with drift \( \mu \) and diffusion coefficient \( \sigma \), it is known that

\[
Pr[M(t) \leq m] = \Phi\left( \frac{m - \mu t}{\sigma \sqrt{t}} \right)
\]

\[- e^{2m\mu/\sigma^2} \Phi\left( \frac{-m - \mu t}{\sigma \sqrt{t}} \right)\]

where \( \Phi(.) \) is the c.d.f. of the Normal \((0, 1)\) random variable. For \( X(\tau) = g\tau - \alpha Y(\tau) \), what are the drift and diffusion coefficient of \( \{X(\tau)\} \) under the ** probability measure?
Recall:

\[
\frac{\mathbb{E}^*[n(T)e^{\alpha Y(T)}]}{\mathbb{E}^*[e^{\alpha Y(T)}]} = \mathbb{E}^{**}[n(T)]
\]

Under the ** probability measure, the Brownian motion \{Y(\tau)\} has drift

\[
\mathbb{E}^*[Y(1)] + \alpha \sigma^2 = (r - \frac{\sigma^2}{2} - \zeta) + \alpha \sigma^2
\]

and (unchanged) diffusion coefficient \sigma.
Now, $X(\tau) = g\tau - \alpha Y(\tau)$. Thus, under the **probability measure, the process $\{X(\tau)\}$ has drift

$$g - \alpha [(r - \frac{\sigma^2}{2} - \zeta) + \alpha \sigma^2]$$

and diffusion coefficient $\alpha \sigma$.

Hence, $E^{**}[n(T)]$ can be evaluated, and one can then write down a closed-form formula for $E^*[e^{-rT}n(T)S(T)]$. 
Generalize to stochastic guarantee boundary:

\[ E^* \left[ e^{-rT} \max \left\{ 1, \max_{0 \leq \tau \leq T} \frac{S(0)K(\tau)}{S(\tau)} \right\} S(T) \right] \]

\[ E^* \left[ e^{-rT} \max \left\{ 1, \max_{0 \leq \tau \leq T} \frac{S_1(\tau)}{S_2(\tau)} \right\} S_2(\tau) \right] \]
A Typical Sample Path of the Fund Values

\[ n(t) \cdot S(t) \]
Earlier, \( T \) was a fixed maturity date, i.e., we were pricing “European” options.

How about “American” options?

\[
\sup_{\text{All stopping times } T} E^* \left[ e^{-rT} \max \left\{ 1, \max_{0 \leq t \leq T} \frac{S_1(t)}{S_2(t)} \right\} S_2(T) \right]
\]

Note: The payoff is \textit{path-dependent}. 
Two special cases:

\[ S_2(t) = S(t), \quad S_1(t) = S(0)(0.9)(1.03)^t \]

\[ S_1(t) = S(t), \quad S_2(t) = \text{constant} \]

The Russian option is an American *lookback* or *high watermark* option without maturity date. Its time-0 price is:

$$\sup_T \mathbb{E}^* \left[ e^{-rT} \max_{0 \leq t \leq T} \left\{ k, \max S(t) \right\} \right]$$

The constant $k$ can be viewed as the historical maximum of the stock prices (the maximum before time 0).
• Let $S_j(t) = S_j(0) \ e^{X_j(t)}$, $j = 1, 2$, and we assume that $\{X_1(t), X_2(t)\}$ is a bivariate Brownian motion.

• Assume that each stock (or stock index) pays dividends continuously at a rate proportional to its price. That is, for $j=1, 2$, there is a constant $\zeta_j > 0$, such that stock $j$ pays dividends of amount $\zeta_j S_j(t)dt$ between time $t$ and time $t+dt$. 
• Then, under the risk-neutral probability measure, \( \left\{ e^{-\left( r - \zeta_j \right) t} S_j(t) \right\}, \ j = 1, 2, \) are martingales.

• Again, \( r \) is the risk-free force of interest.
• Then, under the risk-neutral probability measure, \( \left\{ e^{-(r-\zeta_j)t} S_j(t) \right\}, \ j = 1, 2, \) are martingales.

Address of the Society of Actuaries is:
475 North Martingale Road
Schaumburg, Illinois, U.S.A.
• Address of the Institute of Actuaries of Australia is:

    4 Martin Place, Sydney

But ....
• Address of the Institute of Actuaries of Australia is:

  4 Martin Place, Sydney

But its current president is

  Andrew Gale
• How about the Swiss Association of Actuaries?
• How about the Swiss Association of Actuaries?

Under the risk-neutral probability measure,

\[ \left\{ e^{-(r - \xi_j) t} S_j(t) \right\}, \ j = 1, 2, \]

are martingales. Also, there are two martingales of the form

\[ \left\{ e^{-rt} \left[ S_1(t) \right]^\theta \left[ S_2(t) \right]^{1-\theta} \right\}. \]

The martingale condition is

\[ e^{-rt} \mathbb{E}^* \left[ e^{\theta X_1(t) + (1-\theta)X_2(t)} \right] = 1. \]
This leads to the quadratic equation
\[ -r + E^*\left[ \theta X_1(1) + (1 - \theta)X_2(1) \right] + \frac{1}{2} \text{Var}\left[ \theta X_1(1) + (1 - \theta)X_2(1) \right] = 0. \]

Its solutions are \( \theta_1 < 0 \) and \( \theta_2 > 1 \). Thus, the two processes,
\[ \left\{ e^{-rt} \left[ S_1(t) \right]^\theta_j \left[ S_2(t) \right]^{1-\theta_j} \right\}, \quad j = 1, 2, \]
are martingales under the risk-neutral probability measure.
Let \( n(T) = \max \left\{ 1, \max_{0 \leq \tau \leq T} \frac{S_1(\tau)}{S_2(\tau)} \right\} \)

For \( s_1 > 0, s_2 > 0 \), define

\[
V(s_1, s_2) = \sup_{T} E^{*} \left[ e^{-rT} n(T)S_2(T) | S_1(0) = s_1, S_2(0) = s_2 \right]
\]

The supremum is taken over all stopping times \( T \). There is no fixed expiry date.

This is the price of the perpetual *dynamic protection option*. 
\[ h(s) = (\theta_2 - 1)s^{\theta_1} + (1 - \theta_1)s^{\theta_2}, \quad s > 0 \]

\[ \tilde{\varphi} = \left( \frac{-\theta_1(\theta_2 - 1)}{(1 - \theta_1)\theta_2} \right)^{\frac{1}{\theta_2 - \theta_1}} \]

\[ 0 < \tilde{\varphi} < 1 \]

\[ V(s_1, s_2) = \begin{cases} \frac{h(s_1 / s_2)}{h(\tilde{\varphi})} s_2 & \text{if } \tilde{\varphi} < \frac{s_1}{s_2} \leq 1 \\ s_2 & \text{if } 0 < \frac{s_1}{s_2} \leq \tilde{\varphi} \end{cases} \]
Instead of \( n(T) = \max \left\{ 1, \max_{0 \leq t \leq T} \frac{S_1(t)}{S_2(t)} \right\} \),

we now consider \( n(T) = \max \left\{ 1, \frac{S_1(T)}{S_2(T)} \right\} \).

Then, \( n(T)S_2(T) = \max\{S_1(T), S_2(T)\} \),

which is the payoff of the maximum option (also called alternative option or greater-of option). This is a simpler option since the payoff is not path-dependent.
The price of the American maximum option without a fixed expiry date is:

\[ W(s_1, s_2) \]

\[ = \sup_T E^* \left[ e^{-rT} \max\{S_1(T), S_2(T)\} \mid S_1(0)=s_1, S_2(0)=s_2 \right], \quad s_1 > 0 \text{ and } s_2 > 0. \]

This option has been evaluated in the paper Gerber and Shiu, “Martingale Approach to Pricing Perpetual American Options on Two Stocks,” *Mathematical Finance*, Vol 6 (1996).
\[ \sup_{T} E^* [e^{-rT} n(T) S_2(T) \mid S_1(0) = s_1, S_2(0) = s_2] \]

For \( V(s_1, s_2) \), \( n(T) = \max \left\{ 1, \max_{0 \leq t \leq T} \frac{S_1(t)}{S_2(t)} \right\} \).

For \( W(s_1, s_2) \), \( n(T) = \max \left\{ 1, \frac{S_1(T)}{S_2(T)} \right\} \).

Obviously, \( V(s_1, s_2) \geq W(s_1, s_2) \).
\[
\sup_{T} E^{*} \left[ e^{-r T} n(T) S_2(T) \middle| S_1(0) = s_1, S_2(0) = s_2 \right]
\]

For \(V(s_1, s_2)\), \(n(T) = \max \left\{ 1, \max_{0 \leq t \leq T} \frac{S_1(t)}{S_2(t)} \right\}\).

For \(W(s_1, s_2)\), \(n(T) = \max \left\{ 1, \frac{S_1(T)}{S_2(T)} \right\}\).

Obviously, \(V(s_1, s_2) \geq W(s_1, s_2)\).

Surprisingly, there is a constant \(\tilde{c} > 1\), such that
\[
V(s_1, s_2) = W(\tilde{c} s_1, s_2).
\]
\[ \tilde{b} = \left( \frac{-\theta_1}{1-\theta_1} \right)^{(1-\theta_1)/(\theta_2-\theta_1)} \left( \frac{\theta_2}{\theta_2-1} \right)^{(\theta_2-1)/(\theta_2-\theta_1)} \]

\[ \tilde{c} = \left( \frac{-\theta_1}{1-\theta_1} \right)^{-\theta_1/(\theta_2-\theta_1)} \left( \frac{\theta_2}{\theta_2-1} \right)^{\theta_2/(\theta_2-\theta_1)} \]

\[ k(x) = \frac{\theta_2 (x/\tilde{b})^{\theta_1} - \theta_1 (x/\tilde{b})^{\theta_2}}{\theta_2 - \theta_1} \quad x > 0 \]
\[ W(s_1, s_2) = \begin{cases} 
  s_2 & \text{if } \frac{s_1}{s_2} \leq \tilde{b} \\
  s_2 \cdot k\left(\frac{s_1}{s_2}\right) & \text{if } \tilde{b} < \frac{s_1}{s_2} < \tilde{c} \\
  s_1 & \text{if } \frac{s_1}{s_2} \geq \tilde{c} 
\end{cases} \]
$s_1 = \tilde{b} s_2$

$\tilde{c} s_2$

Continuation Region

EXERCISE

EXERCISE

$s_1$

$s_2$
It can be readily checked that

\[ \tilde{\varphi} = \frac{\tilde{b}}{\tilde{c}}. \]

From this, we realized that

\[ V(s_1, s_2) = W(\tilde{c}s_1, s_2) \]

But why is this formula true?
Y K Kwok and C C Chu wrote a discussion on “Pricing Perpetual Fund Protection with Withdrawal Protection,” North American Actuarial Journal, Vol 7 (2), 2003. They introduced the concept of a perpetual option with “up to n resets”. When the number of possible resets n becomes \( \infty \), we have

\[
V(s_1, s_2) = W(\tilde{c}s_1, s_2)
\]
$S_2(t)$

$S_1(t)$

Reset
For \( n = 1, 2, 3, \ldots \), and let \( V_n(s_1, s_2) \) denote the price of the option with \textit{up to} \( n \) resets, where \( s_1 = S_1(0) > 0 \) and \( s_2 = S_2(0) > 0 \). The option has no fixed expiry date. Thus,

\[
V_{n+1}(s_1, s_2)
= \sup \mathbb{E}^*[e^{-rT} \max \{V_n(S_1(T), S_1(T)), S_2(T)\} \mid S_1(0) = s_1, S_2(0) = s_2].
\]

Because \( V_n(s_1, s_2) \) is a homogeneous function of degree 1,

\[
V_n(S_1(T), S_1(T)) = V_n(1, 1)S_1(T)
\]
Define

\[ \kappa_n = V_n(1, 1). \]

Then,

\[
V_{n+1}(s_1, s_2) = \sup_{T} E^* \left[ e^{-rT} \max \{ \kappa_n S_1(T), S_2(T) \} \right] \bigg| S_1(0) = s_1, S_2(0) = s_2
\]

\[
= \sup_{T} E^* \left[ e^{-rT} \max \{ S_1(T), S_2(T) \} \right] \bigg| S_1(0) = \kappa_n s_1, S_2(0) = s_2
\]

\[
= W(\kappa_n s_1, s_2).
\]
Withdraw with $s_2$

$\frac{s_1}{s_2} = \frac{\tilde{b}}{\kappa_n}$

Reset and the option is worth $V_n(s_1, s_1)$

$V_{n+1}(s_1, s_2) = W(\kappa_n s_1, s_2)$
\[ V_{n+1}(s_1, s_2) = W(\kappa_n s_1, s_2) \]

Put \( s_1 = s_2 = 1 \). Then
\[ V_{n+1}(1, 1) = W(\kappa_n, 1), \]
or
\[ \kappa_{n+1} = W(\kappa_n, 1) = k(\kappa_n), \]

where
\[ k(x) = \frac{\theta_2 (x/\tilde{b})^{\theta_1} - \theta_1 (x/\tilde{b})^{\theta_2}}{\theta_2 - \theta_1} \quad x > 0. \]
Smooth Pasting Condition (High Contact Condition)
κ₁ = W(1, 1) = k(1)
κ₂ = k(κ₁)
κ₃ = k(κ₂)
...
κₙ₊₁ = k(κₙ),
...
\[ V_1(s_1, s_2) = W(s_1, s_2) \]
\[ V_2(s_1, s_2) = W(\kappa_1 s_1, s_2) \]
\[ V_3(s_1, s_2) = W(\kappa_2 s_1, s_2) \]
\( V_4(s_1, s_2) = W(\kappa_3 s_1, s_2) \)
\[ V(s_1, s_2) = W(\kappa_\infty s_1, s_2) \]

\[ \frac{s_1}{s_2} = \frac{\tilde{b}}{\kappa_\infty} = \tilde{\phi} \]

\[ \frac{s_1}{s_2} = \frac{\tilde{c}}{\kappa_\infty} = 1 \]


Thank you for your patience

Time for lunch