DC Pension Plans for All: What If?

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Abstract

This study is intended to gauge the risk inherent in defined contribution (DC) pension plans on an individual and on an aggregate basis, based on United States data. Our aim is to gain insight into the consequences of a DC pension scheme becoming the predominant pillar of retirement income for an entire society. It is necessary for the primary source of retirement income to, by design, provide a sufficient pension that will offer financial security to the elderly and will facilitate the transition from employment to retirement. Due to the uncertainty in its accumulated wealth, such a requirement could not be fulfilled by a traditional DC pension plan if the pension delivery date is fixed. Therefore, rather than focus on the accumulated wealth at a specified retirement age, this study investigates the likely retirement age of DC participants if they hoped to maintain a fixed standard of living once they have retired, which will sustain them till death. Based on the simulated output of a DC flexible age of retirement model, we decide upon the optimal investment strategies. We then examine the demographic dynamics in an entire population of DC pension plan participants. The conclusions drawn demonstrate the significant role the market plays in the effectiveness of the DC pension plan scheme's success or failure. There is a high level of uncertainty in the age of retirement of each DC participant, regardless of his or her investment strategy. Furthermore, there are large retirement age discrepancies between the DC participants in different cohorts, despite their identical characteristics. We find that, even when we allow for a wide range of investment strategies amongst the members, the ratio of retirees to workers varies significantly over time. This suggests that countries dominated by DC schemes of this type may, over time, be exposed to significant risk in the size of its labour force.

Keywords: Defined contribution pension plan; Flexible retirement age; Dependency ratio; Value-at-risk

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1 Introduction

The shift from defined benefit (DB) to defined contribution (DC) pension plans is a prevailing phenomenon throughout the pensions world. Generally, the income support system for retirees is composed of three parts: personal savings, occupational pension plans, and government-provided social security. Among employer sponsored pension plans in the United States (US), a strong trend towards DC pension plans has been in effect over the past 20 years (Brown and Liu, 2001). Currently, there is also a possible threat of the US's social security program converting to a DC state pension system (Arnold et al., 1998). Consequently, the DC pension plan has grown from being a single and relatively unimportant source of pension income to becoming two significant sources, with the potential of also becoming the third and final source of income for the non-working elderly. The motivation of this study is to investigate the effects of a traditional DC pension plan acquiring such a role and monopolizing the retirees' income support system.

DC pension plan schemes are notorious for the uncertainty in the level of pension that they can provide. Research and experience has shown that, having a fixed age of retirement, it is difficult to predict accurately the pension income under a DC pension plan design. Relative to a DB pension income benchmark, the accumulated wealth in a DC plan can be extremely risky (Blake et al., 2001). If there were rigid restrictions on the worker's age of retirement, dictated by either statutory law or company policy, it would be difficult for a DC pension plan design to work on a large scale since inadequate pensions would be commonplace, rather than the exception as is the case in a well designed DB pension scheme. Chile, whose state pension plan is a DC system, affords their citizens some protection from the uncertainty in their retirement income by ensuring a minimum pension income. However, such a system can be criticized as an expensive welfare system and, to a large extent, providing a minimum pension income could expose the governmental pension sponsor to abuse and antiselection (Brown, 1999). In addition, without additional contributions, such a pension system could not be self-sustaining[1].

In order for a DC pension plan to serve as a primary pillar of retirement savings, it would need to provide an adequate level of salary replacement at retirement. By assuming the individual will lengthen or shorten their working career depending on their accumulated pension savings in relation to their expected life span, participants can delay their retirement until a sufficient pension fund is accumulated.

Our study begins by considering the simple case of a male employee who starts a DC pension plan on his 25th birthday, lives until retirement, has no dependents, and makes annual contributions of 10% of his salary at the beginning of each working year. It then goes onto investigate the demographic outcome of an entire society of DC pension plan participants. That is, if every member adopted a retirement strategy that based their retirement date on the size of the pension income that can be supported by

[1] DB state pension systems offer numerous benefits that could only be duplicated in a DC system through further additional contributions of the state. Among these include ancillary benefits such as disability and death benefits, as well as a top-up for females who will suffer an inherently lower pension income at the time of annuitization due to their longevity. DC pension plan designs are frequently applauded for their portability properties; however, this feature does not avail a state pension plan as there is usually no need for a pension to be portable, perhaps with the exception of emigrants. Under a DC pension plan, there is no redistribution of wealth. Finally, DC pension plan participants are responsible to pay onerous costs and fees, such as fund manager fees; in fact, administrative expenses are higher than in a socialized system (Brown, 1999)

their DC funds. Our study assumes that there is no age of forced retirement, either by statutory law or by company policy.

In this paper, we use stochastic simulation to investigate the range of outcomes for a variety of quantities of interest.

The results depend on capital market assumptions such as asset rate of return, salary scale rate of increase, annuity discount rate, and the relationship between each rate mentioned. We assume that the pension fund is invested across 3 assets: equities, bonds and risk free one-year bonds (cash). This study incorporates the Vasicek interest rate model (Vasicek, 1977) to generate the capital market assumptions listed above. The asset-return stochastic model is described more fully in the Appendix A. In this paper, we have taken as given the dynamics of the stock market; that is, we have not attempted to model the macroeconomics effect of the mass actions of the DC plan members such as mass demand for equities or liquidation of a particular asset.

The asset allocation strategy of the DC participants in our study is static; that is, participants maintain constant asset proportions in their portfolios throughout the accumulation phase. This results in constant portfolio rebalancing at the end of each year. Although a simplifying assumption, it was shown in (Blake et al., 2001) that a well chosen static asset allocation strategy performs substantially better than various common dynamic strategies, such as the popular lifestyle strategy.

In addition, the model also includes demographic assumptions such as mortality, contribution rate, initial age in the pension plan and the age distribution of the population's members. The contribution rate and initial age are fixed at 10% of salary and 25 years of age, respectfully, as mentioned above. Mortality assumptions and the retirement decision model are described below.

This paper is a work in progress. To date, we have looked at exaggerated scenarios and have made simplifying assumptions in order to gain insight into the general effects of our investigation. We intend to refine the model and future work will incorporate a variety of additional realistic features.

1.1 Retirement Decision Model

The possible methods to choose the date of one's retirement are infinite. Factors influencing the retirement date of a DC plan member include accumulated wealth, health, age, preference for leisure time over work, direct pressure from employer, and general peer pressure due to social customs (Brothers, 1998).

In our study, we base retirement on the level of pension income that can be provided by the DC participant's accumulated wealth. Pension income is determined by dividing the pension fund on the retirement day by the annuity factor. The annuity factor ($\ddot{a}_{x+t}(t)$) is the present value, at the time of retirement, of one unit of an annuity for the remaining life of the annuitant. The pension income, divided by the individual's salary at retirement, produces the replacement ratio, $RR(t)$:

$$RR(t) = \frac{\text{Wealth}(t)/\ddot{a}_{x+t}(t)}{\text{Salary}(t)} \tag{1}$$

where t is time and x is the age of the individual at time 0.

This study considers the retirement age as a random stopping time, when the pension purchasable exceeds two-thirds the outgoing salary. In other words, at the beginning of each working year:

$$\text{Retirement Age} = inf\left\{\ x+t:\quad RR(t) \geq \tfrac{2}{3}\ \right\}.$$

This decision rule thus incorporates the reasonable view that retirement will be deferred until such time as the member can afford retirement. The 2/3 rule does not explicitly allow for age-dependency in the decision, but age is taken into account indirectly through the annuity factor:

$$\ddot{a}_{x+t}(t) = \sum_{s=0}^{\infty} P(r(t),t,t+s)\ {}_{s}\mathrm{p}_{x+t}, \tag{2}$$

where $P(r(t),t,t+s)$ is the price at time t of a risk-free zero-coupon bond that matures at time $t+s$ and $r(t)$ is the instantaneous risk-free rate of interest at time t. The annuity factor decreases as the individual ages due to the lower number of payments expected to be made as a result of higher future expected mortality. The lower the annuity factor, the higher the pension income and the higher the likelihood of the pension income exceeding 66.67% of the outgoing salary.

Thus, we model the retirement decision based on the accumulated pension wealth of the participant as well as their expected longevity.

1.2 Mortality Model

The United States Life Tables 2002 for females and males (Arias, 2004), published by the National Center of Health Statistics, are used in this study. The data used to prepare these tables are, within the US, final numbers of deaths for year 2002, postcensal population estimates for the year 2002, and data from the Medicare program of the Centers for Medicare and Medicaid Services.

1.3 Simulation of an Entire Population

A stationary and stable population model is used to simulate the demographics of a population. That is, there is no growth in the population size and the population age distribution is identical from one period to the next (Bowers et al., 1997). The model has 76 cohorts in the population. At every point in time, the cohorts range in age from 25 to 100. The relative size of each cohort aged x is labeled l_x:

$$l_x = e^{-\int_{25}^{x} \mu_t\,\mathrm{d}t} = {}_{x-25}\mathrm{p}_{25}.$$

For example, the size of the aged 25 cohort at time t is 1, without loss of generality, and is size ${}_{s}\mathrm{p}_{25}$ at age $25+s$ at time $t+s$.

The demographic and capital assumptions continue to be the same as in the individual member scenario, with the exception that we use a mixed mortality rate of 50% female and 50% male.

From the output, the dependency ratio (ratio of the number of retirees to the number of workers) is calculated for each year of simulation.

1.4 Stochastic Simulation

The steps followed in our model construction and simulation include:

1. The asset model specification and calibration is the first step. Asset return modeling includes choosing the asset return model and defining the parameters. The derivations of the accumulation model and the parameter estimates are explained further in Appendix A.

2. We then carry out the demographic model specification and calibration. This step includes the selection of the mortality table, the age of entry of the participants, the characteristics of the population model and the design of the DC pension plan.

3. The modeling of the liability for each individual DC participant is done simultatneously and in accordance with the asset modeling. The liability is the current value of the future pension income, which depends upon the annuity factor value and the accumulated wealth. Therefore, the drivers of the liability are the fund returns, the discount rate and the age of the individual, which determines future mortality rates.

4. Another step is the selection of the asset investment strategy, where the wealth allocation between equity investments, bond investments and cash is determined[2].

5. This is followed by the generation of the pension wealth and the immediate pension for each DC participant.

6. For each DC participant, the assets and liability are calculated and carried forward for each of the simulated years until they retire, when the $RR(t) \geq 66.67\%$.

7. Given the various assumptions, the program generates an empirical distribution of possible retirement ages (for individual investors) or dependency ratios (for entire populations) corresponding to each particular investment strategy. The results are ranked; hence, allowing us to produce confidence intervals, investigate relevant percentiles and examine dispersion.

8. Particular investment strategies are chosen and the results are examined using histograms, cumulative distribution functions, time series plots and scatterplots. Such plots provide insight into the consequences of DC pension plan schemes.

The retirement ages, their range and the consequent overall effect on a population's dependency ratio are the focus of this study.

2 Results for an Individual DC Participant

This section contains the simulated results of an individual DC pension plan member and is followed by the simulated results of an entire population where every member holds a DC pension plan.

It is first necessary to decide upon an appropriate investment strategy for the individual and for the aggregate group of DC pension plan participants. Therefore, we commence by describing the risk measures chosen to evaluate and compare the investment strategies. A risk measure is a statistical summary of the amount of the potential loss, whether it be monetary or the timing of retirement. A single measure of risk is required as it facilitates the comparison among many asset mixes. These tools summarize the probability distribution constructed by the uncertain events, referred to as random variables. There may be several preferred risk measures since the random variables carry various consequences to the retiree and each consequence should be measured

[2]The investment strategies are, of course, chosen without any knowledge of the simulated asset returns. For computational efficiency and to aid comparisons between different strategies, we simulate the asset returns once and use this same set of sample paths for each of the investment strategies.

for risk accordingly. The risk measure associated with each investment strategy is useful in deciding between assets for investment; nonetheless, once efficient portfolios are decided upon, it is useful to look at the whole of the distribution under that particular investment strategy, such as histograms, cumulative distribution functions (CDFs), time series plots and scatterplots. It is from these plots that the effectiveness and impact of the DC pension plan design can be understood.

2.1 Individual Member Simulation

When saving for retirement, it is supposed that the individual wishes that the level of annual pension at retirement to achieve some percentage of pre-retirement salary, known as the replacement ratio (RR) benchmark target; thus, enabling the individual to maintain a particular living standard. Having the RR fixed at a suitable target (66.67%), it bears little risk; however, it is the age of retirement that becomes the random variable and is required to be measured for risk.

The traditional variance risk measure, on its own, may not necessarily accomplish this purpose since an investment strategy that produces a tightly distributed range of retirement ages will appear desirable despite the fact that they may all fall above a suitable target age. Therefore, the mean age of retirement should also be considered. In our case, the variance of the age of retirement is an acceptable measure of risk for the following reasons:

- The distribution of the age at retirement is relatively symmetric, as will be seen in later graphs (Figure 2 in Section 2.2).

- Long tails are not possible as the retirement age is always contained between the initial age and the limiting age of the mortality table. Realizing that the annuity factor, defined by equation (2) in Section 1.1, approaches one as the individual approaches the limiting age of the mortality table, it is almost impossible for the individual to not retire prior to the limiting age. Not only can the retirement age not exceed the limiting age of the mortality table, age 100, individuals are more likely to retire as they age, as explained in Section 1.1.

- Due to the age of retirement being always contained between the initial age and the limiting age of the mortality table, the tails of the distribution are either short or inexistent. It is for this reason that investment strategies which produce a lower (higher) average age of retirement will not necessarily cause a significant decrease (increase) in the dispersion of the distribution; in fact, the effect may only be a shift of the entire curve to the left or to the right. Therefore, it is sensible to examine the variance, along with the mean, when determining the risk inherent in an investment portfolio on the age of retirement. For that reason, standard deviation is chosen as one of the retirement age risk measures.

This retirement model assumes that the individual, having decided on a fixed RR upon retirement, has some flexibility in their date of retirement. Notwithstanding, it is reasonable to assume that there is a limitation on this flexibility and that the individual will eventually want or need to retire, due to reasons such as illness and desire for leisure. Such an individual would reasonably want to know what is the latest age that they can expect to retire under each particular investment strategy, where the probability of retiring prior to this time is quite high. The Value-at-Risk (VaR) measure answers this question by indicating the pth quantile of the distribution. For example, the VaR 95% value is the age at which there is a 95% probability that the individual will retire prior to this time, given that the model and data are correct. Using the pth percentile of the retirement age distribution, a VaR(p%) is calculated from the formula:

$$Pr(x+t < \text{VaR(p\%)}) = \text{p\%}$$

where $x+t$ is the actual age of retirement (a random variable).

There are many benefits to using VaR as a risk measure, including the following (Panning, 1999):

- Intelligible - VaR is an estimate of the maximum age of retirement that would be necessary under extreme scenarios for all but a specified percentage of possible scenarios.

- Comparable - The extreme retirement ages can be compared across alternative investment strategies.

- Useful - VaR answers important questions. If one investment alternative has a higher VaR than another, there should also be an earlier mean retirement age for the member to justifiably select that alternative.

In the simulations, 66 different investment strategies are executed, each containing a different combination of bond, cash and equity exposure. Each asset is tried in 10% increments, giving rise to a total of 66 investment portfolio strategies to consider.

The simulated mean age of retirement for each investment strategy tested is plotted against the corresponding age of retirement risk measures. The plots give an impression of the opportunity set (OS) for each risk measure, traced out by the 66 investment strategies. From the OS, we can infer the minimum risk portfolios (MRPs), which are the portfolios that carry the lowest risk for each given level of return. Three OS plots of the results are produced where the return measure for each plot is the mean age of retirement that the RR achieves 66.67%, while the 3 risk measures are the standard deviation of the retirement ages and the VaR measures, which will denote the 90% and the 95% quantiles of the retirement age distributions.

The resulting OSs are shown in Figure 1. Prior to examining these results, it is first beneficial to make some broad comments regarding the VaR OS plots.

- The equity and the bond content of each plotted portfolio can be deduced from the graph. The larger, plotted end points are each marked by their bond content, and one end point is marked by the equity content, which applies to both end points and all points between. When moving from left to right between the two end points, the proportion of the bond content changes by increments of 10%, while the proportion of the equity content remains constant. The cash content of each portfolio consists of the proportion unoccupied by equities or bonds.

- The indicator of success is a low age of retirement; therefore, lower values on the y-axis and on the x-axis are superior. For that reason, in the VaR plots, points in the OS nearer to the bottom left are good.

- Within the model, possible ages of retirement are integer ages. This manifests itself in a stepped boundary to the OS, horizontally separated by integer years.

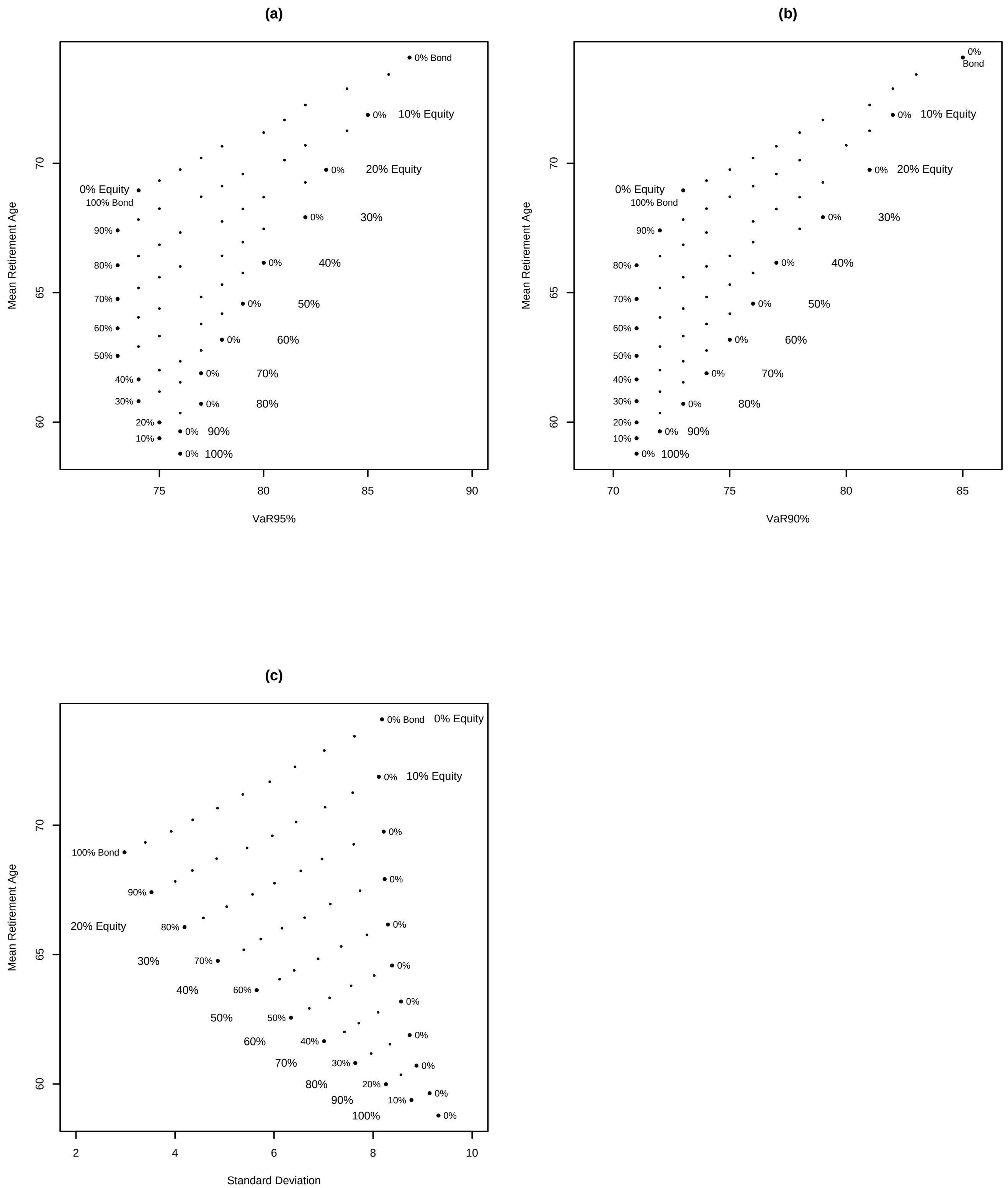

Figure 1: Opportunity sets for an individual DC participant. (a) The VaR 90% is plotted against the mean age of retirement. (b) The VaR 95% is plotted against the mean age of retirement. (c) The standard deviation is plotted against the mean age of retirement.

2.2 Results

The OSs in Figure 1, suggests that the ideal portfolio has the majority of the funds in equities and the remainder in bonds. According to the plots, a pure equity DC account will allow for a mean retirement age of 58.8. Moreover, the results indicate the individual will retire prior to age 71 with a 90% probability and age 76 with a 95% probability. The VaR 95% risk measure indicates that the DC participant should distribute at least 50% of their funds in equities with the remainder in bonds. Holding cash in the portfolio is shown to be inefficient.

Overall, an efficient investor should allocate a large proportion of their funds in equities. This is despite the fact that the standard deviation plot indicates that there is less precision as the proportion of equities increases. These conclusions are further supported by Figure 2, where the empirical distributions of the ages of retirement resulting from a pure bond, pure equity and pure cash portfolio reveal the simulated retirement age behavior under each asset strategy. The right tails of the bond and equity histograms, shown in graph (a) and graph (c), indicate that the worst case scenarios are similar between a bond and an equity portfolio.

The preference for an equity portfolio can be understood further by examining the CDFs in Figure 3, where the worst-case scenarios of the empirical CDFs of graph (d) in Figure 2 are enlarged[3]. On the right tail of the equity portfolio CDF, after crossing the bond portfolio CDF, is where it is more likely that the equity portfolio will deliver higher ages of retirement. The high crossover point in Figure 2(d), made clear in Figure 3, indicates that equities are favored by most risk averse investors. The cash portfolio CDF does not cross the equity portfolio CDF, again reaffirming its undesirable performance.

Investors who require higher assurance that they will retire prior to a particular date (that is, a 95% probability rather than a 90% probability) and are hence concerned with extreme worst-case scenarios, should increase their exposure to bonds; thus, decreasing the equity content. The driving explanation behind the poorer performance of the equity investment under the VaR 95% risk measure compared to the VaR 90% risk measure is its relatively larger dispersion of results, which pushes the VaR 95% to a comparatively higher additional value. This is illustrated in the standard deviation plot of Figure 1, graph (c), where the equity investment strategy is ranked as the most dispersed of the portfolios.

Overall, although investing the majority of funds in equities creates less certainty in the age of retirement, it is still the most superior investment strategy since there is more opportunity for early retirement and, in the worst-case scenarios, the individual will still most likely retire before or at the age that a bond investment strategy would have permitted. The ability of a worker to increase their risky asset allocation as a result of having a flexible retirement date has been concluded in previous studies[4]

Figure 1 and Figure 2 also illustrate the impact of a DC pension plan design on an individual DC pension plan participant. When targeting to maintain an adequate

[3]Each point on the CDF curve shows the probability that the age of retirement will fall below a particular level. If one curve lies more to the right than the other curve, then that particular investment strategy is likely to deliver higher ages of retirement than the other investment strategy.

[4]Lachance (Lachance, 2003) examines how a worker's optimal portfolio choice is influenced by their capacity to adjust their retirement date as a function of market fluctuations. Deriving and utilizing a closed-form solution of a model for the optimal consumption and portfolio choice when a worker's retirement is flexible, she concludes that more investment risk can be assumed if a worker's retirement date is flexible instead of being fixed.

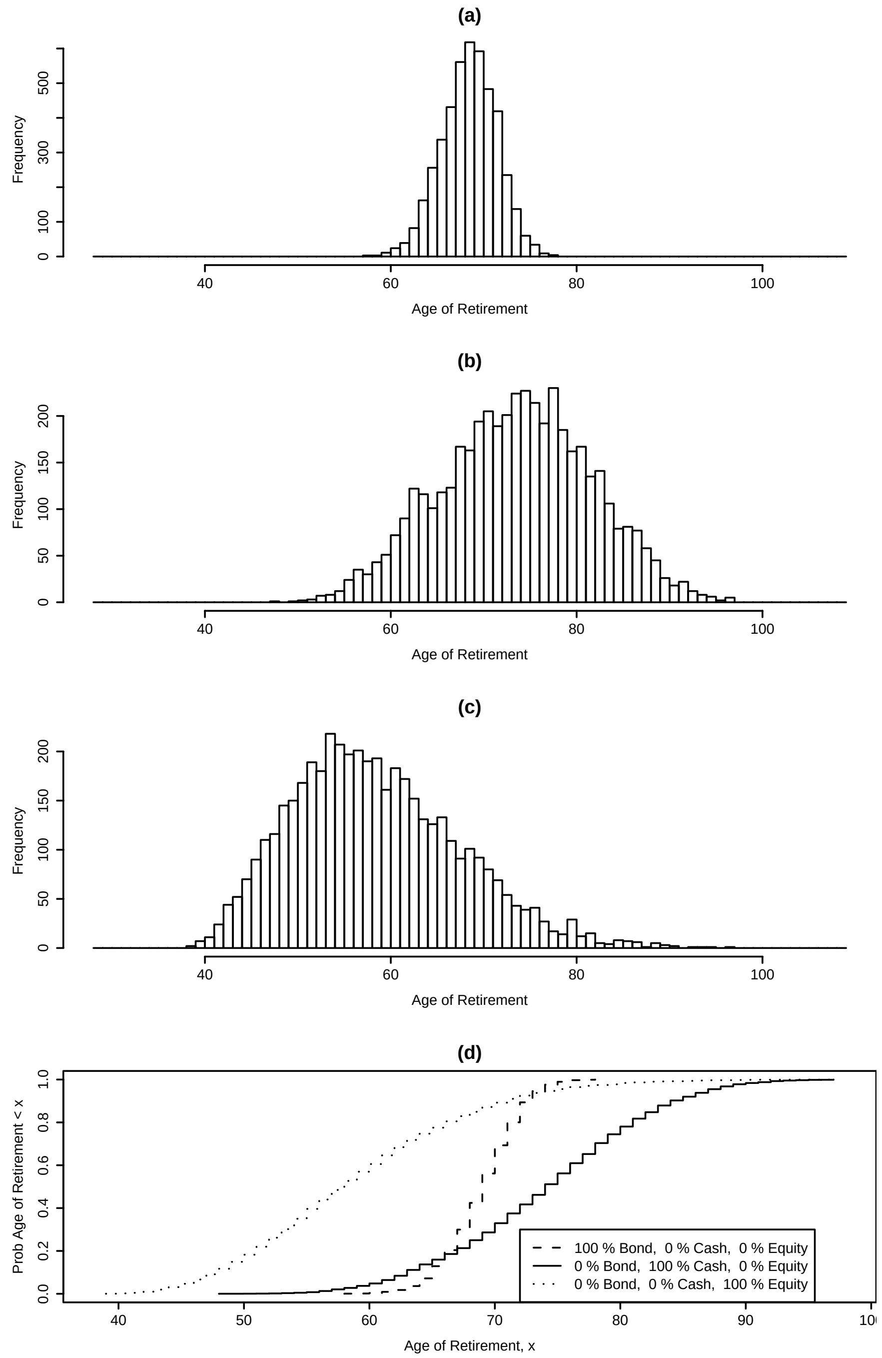

Figure 2: Empirical distribution of the ages of retirement, using an: (a) all bond, (b) all cash and (c) all equity investment strategy. (d) Cumulative distribution functions of an all bond, all cash and all equity investment strategy.

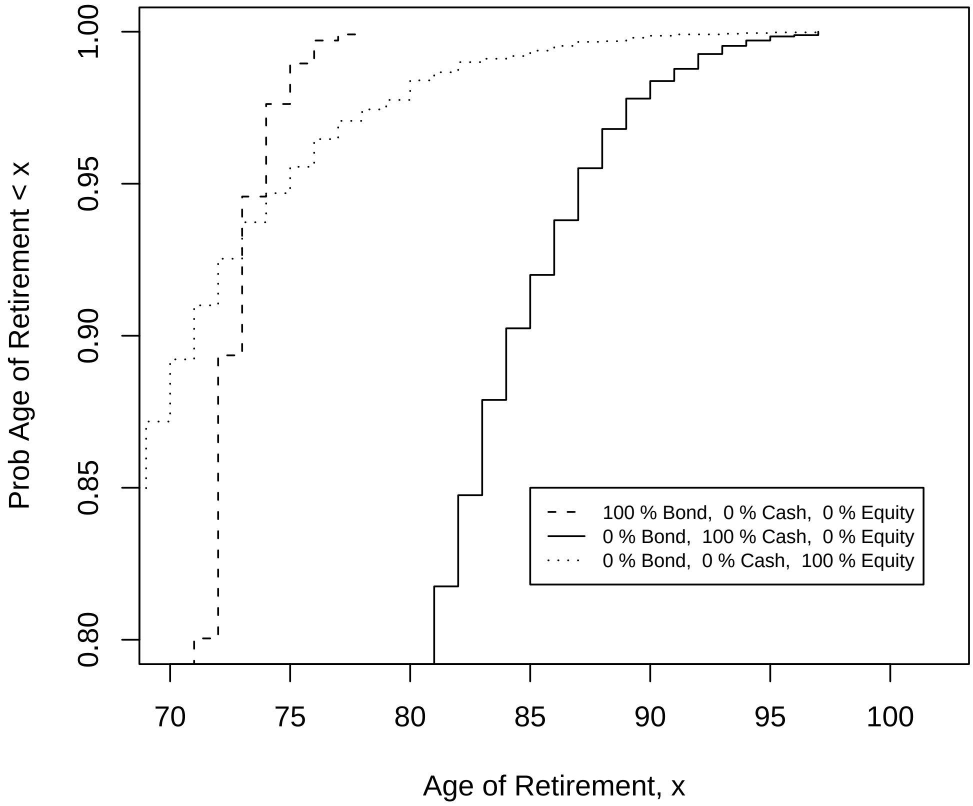

Figure 3: Upper tails of the cumulative distribution functions (Figure 2(d)) of the ages of retirement resulting from an all bond, all cash and all equity investment strategy.

standard of living, there is minimal certainty in a DC pension participant's age of retirement. Investing in a pure bond portfolio delivers the greatest precision; nevertheless, the range of retirement ages still spans over 10 years, from age 63 to age 74, with 95% probability. Investing in a pure equity portfolio, the most efficient portfolio according to both VaR risk measures, results in a possible retirement age anywhere between age 44 and age 79, with 95% probability.

In reality, an individual may be unable to wait until an elderly age to retire and will need to exit the workforce prematurely; whereupon, their DC pension fund will not sustain an acceptable standard of living. The age of retirement's ambiguity could also cause significant anxiety to the DC pension participant in their retirement planning.

3 Results of an Entire Population

When appraising the value of a pension scheme in a population, a suitable indicator of success or failure is the population's resulting dependency ratio for each year of simulation. The dependency ratio is defined here as the ratio of the number of retirees to the number of workers. Normally, a high dependency ratio raises concern since it indicates an aging population, which inevitably increases the cost of economic programs such as social security and health care. Yet, considering that the distribution of ages is unchanging within our model, an increasing dependency ratio simply indicates that individuals are financially able to retire earlier. A high dependency does not automatically produce dependency costs, which are costs incurred to the working population to financially provide for the needs of the retirees. In fact, "dependency ratio" may be a misnomer in our study since only those workers with a financially secured retirement become pensioners; consequently not requiring the financial support of the working population.

Our simulation assumes that the entire working population adopts a DC pension plan design; therefore, we are considering only the working or previously employed individuals when measuring the dependency ratio.

The model we have used in this study makes the initial assumption that all members pay the same contribution rate, and adopt the same investment and retirement strategies. Although a state pension system could easily offer such an extreme degree of uniformity, we realize that real DC pensions systems will contain significant heterogeneity. Even if all occupational pension schemes became of the DC variety[5], the workers whose employers provide minimal occupational pension benefits, or none at all, and whose personal savings are scarce will still need to be considered[6]. However, our purpose in investigating first this extreme case is not to put this forward as a realistic representation of the risks we face in the future. Instead we present it here as a potential worst-case scenario. Specifically, we will see that this extreme system results in a considerable degree of variability in the size of the working population. We can then work away from this worst case to introduce heterogeneity into the system and analyze its effect.

[5]Such a shift within occupational pension plan scheme designs from defined benefit (DB) to DC is not unrealistic. This shift has begun to emerge in the US and is understood to occur when tax legislation and pension regulation increasingly favor DC plan designs over DB plan designs (Brown and Liu 2001).

[6]Such individuals will likely retire prior to accumulating a sufficient state pension income, thus causing the dependency ratio value to increase. More importantly, the dependency costs to state-funded social programs to support such individuals will rise as these workers will likely rely on the working population in their retirement.

Based on the assumption that the DC pension plan design is adopted completely uniformly across all workers within the working population and that retirement is triggered by a worker's accumulated pension income, a low dependency ratio raises concern, as this indicates that elderly workers are financially unable to retire due to the insufficiency of their DC pension fund account. If such a case did occur in reality, factors other than finances may force retirement, such as illness or disability, thus causing insufficient pensions and hardship for such retirees. In other words, the DC pension scheme will fail these elderly participants.

The investment strategies of the members within the population will have a large impact on the retirement success of each individual. Based on the previous section's results, it seems that a majority equity portfolio, with the remainder invested in bonds, is the best investment strategy for an individual DC pension plan member. Nonetheless, the *Fallacy of Composition* (Brown, 1997) argues that what may be good for the individual may not be good in aggregate. That is, although an investment portfolio may be optimal for an individual DC participant over their lifetime, this same investment strategy may not be the optimal solution for an entire population over many lifetimes.

3.1 Everyone Follows the Herd

To get an impression of the most successful portfolios when dealing with the dynamics of an entire population, we will look at the performance of each portfolio, assuming the extreme scenario that everyone in the population follows an identical asset allocation strategy.

We will examine the effects of a range of investment strategies on the population dynamics by investigating how the dependency ratio varies over time. Similar to the individual member simulation, the simulated mean dependency ratio is plotted against the dependency ratio risk measures corresponding to each of the asset strategies. Three OS plots of the results are produced, each traced out by 66 investment strategies. The return measure for each plot is the mean dependency ratio across 4500 years of simulation, while the 3 risk measures are the standard deviation of the dependency ratios and the VaR measures, which indicates the 90% and the 95% quantiles of the dependency ratio distributions.

The resulting OSs are shown in Figure 4. It is again useful to note several general aspects of the VaR OS plots.

- The method of marking the portfolio content of each plotted point is the same as in Figure 1.
- The indicators of success are high dependency ratios; therefore, higher values on the y-axis and on the x-axis are the preferable portfolios. Specifically, the points in the VaR OSs nearer to the top right are good.
- Due to the homogeneous investment strategy of every member within the population, there are no 'gaps' in the ages of the retirees. More specifically, for each year of simulation, there will be a single age in which everyone at or above is retired and everyone below is working. Therefore, every dependency ratio level will have an equivalent age that indicates the age of the prevailing youngest retiree. For example, a dependency ratio of 40.52% indicates that everyone at or above the age of 65 is retired. The following table illustrates the equivalent ages for various levels of dependency ratio:

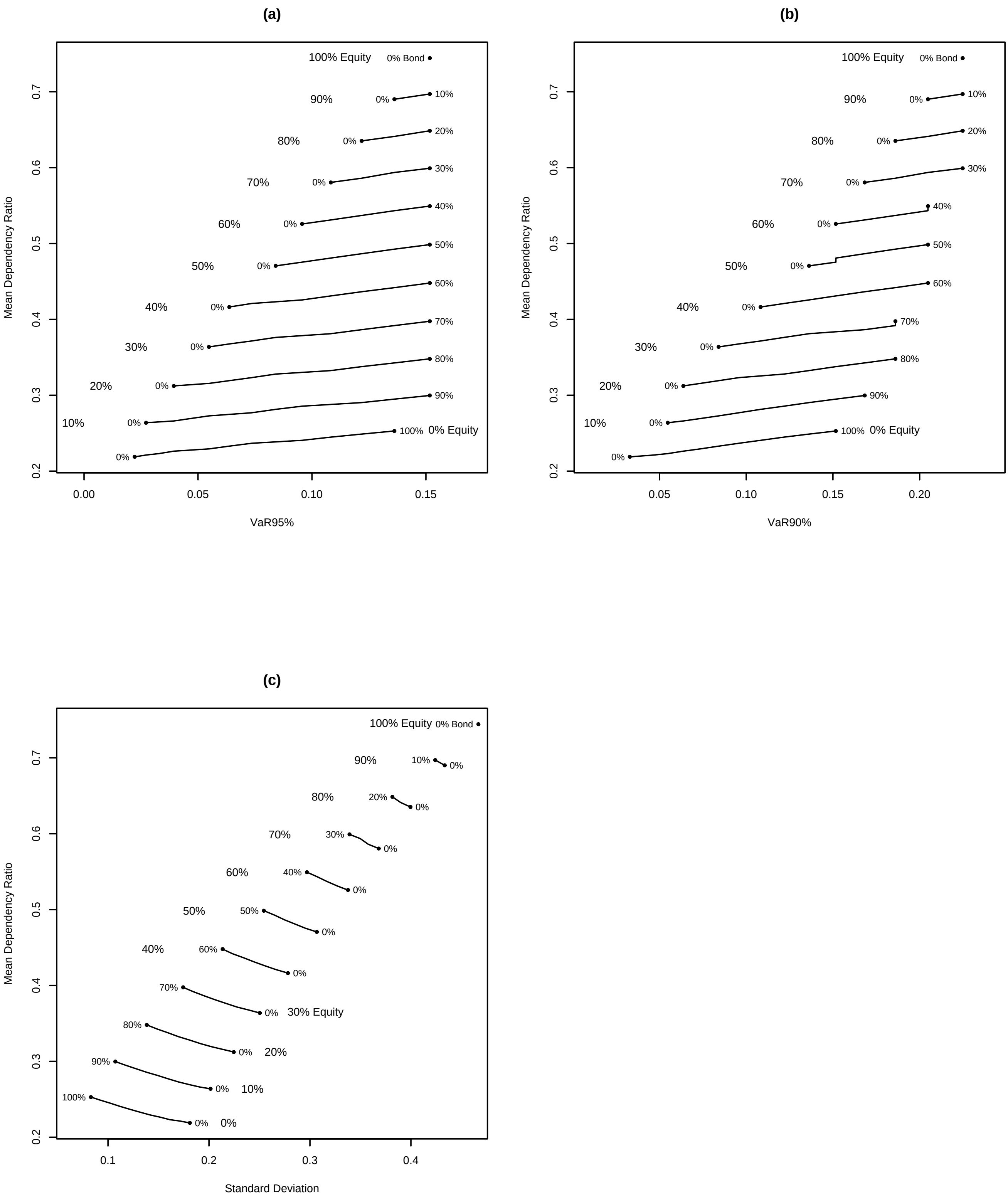

Figure 4: Opportunity sets for an entire population of DC members who homogeneously allocate their funds in the specified investment portfolio, simulated for 66 different asset allocation strategies. (a) The VaR 90% is plotted against the mean dependency ratio. (b) The VaR 95% is plotted against the mean dependency ratio. (c) The standard deviation is plotted against the mean dependency ratio.

Dependency Ratio	0.8313	0.4032	0.1683
Equivalent Age of Youngest Retiree	55	65	75

- Since the population is discretized and all members follow the same strategy, the dependency ratio is restricted to a discrete set of values, which are determined by the youngest retiree at any given time.

The OSs in Figure 4 continues to suggest that, if all individuals were realistically able to choose a homogeneous investment strategy, the ideal investment strategy would be to allocate the majority of the funds in equities, with the remainder in bonds and avoiding cash. According to both the VaR 95% and the VaR 90% risk measures, the performance of a pure equity portfolio exceeds all other asset allocation strategies. The mean dependency ratio of 74.4% is considerably high, and the equivalent mean age of the youngest retiree is between ages 56 and 57.

The superiority of the equity investment strategy can be more clearly understood by examining the distribution of the dependency ratios under a pure equity, pure bond and pure cash portfolio in Figure 5. This graphs the empirical CDF of an all equity portfolio, all bond portfolio and all cash portfolio. On the left tail of the equity portfolio CDF, prior to crossing the bond portfolio CDF, is where it is more likely that the equity portfolio will deliver a lower dependency ratio. The plot indicates that it is slightly more likely that dependency ratios from an equity portfolio will fall below very low benchmarks than are the dependency ratios from a bond portfolio. More specifically, the probability of both a bond and an equity portfolio to produce a dependency ratio below 14%, which is the equivalent to having the youngest retiree being a minimum of age 77, is less than 5%. However, the probability that a bond portfolio will produce a dependency ratio under 10% is less than 1%, while the same probability with an equity portfolio is approximately 2.5%. Participants would need to retire at a minimum of age 80 in order to produce a dependency ratio less than 10%. The crossover point is relatively low and suggests that, in the majority of cases, equities outperform bonds.

Interestingly, these results are consistent with current asset allocation trends of DC pension plans participants in the US. DC pension plan investors are increasingly moving to equity investments. Between 1983 and 1996, US members have increased the proportion of their DC pension plan assets in equities from 27% to 60% (Mitchell, 1998).

3.2 Variety in the Population's Investment Strategy

We now start to move towards a more realistic scenario. Having established the boundaries of an efficient investment strategy for the DC members of the population that is in line with the current market, our next step is to examine the inner dynamics of the retirement patterns within the population.

To add some realism to the simulation, we make the assumption that the members of the population choose one of three investment strategies that are centered on what we have determined to be a realistic and well-performing investment strategy. We assume that one third of each age group chooses one of the following portfolios:

Portfolio A: 20% Equities, 80% Bonds;

Portfolio B: 60% Equities, 40% Bonds;

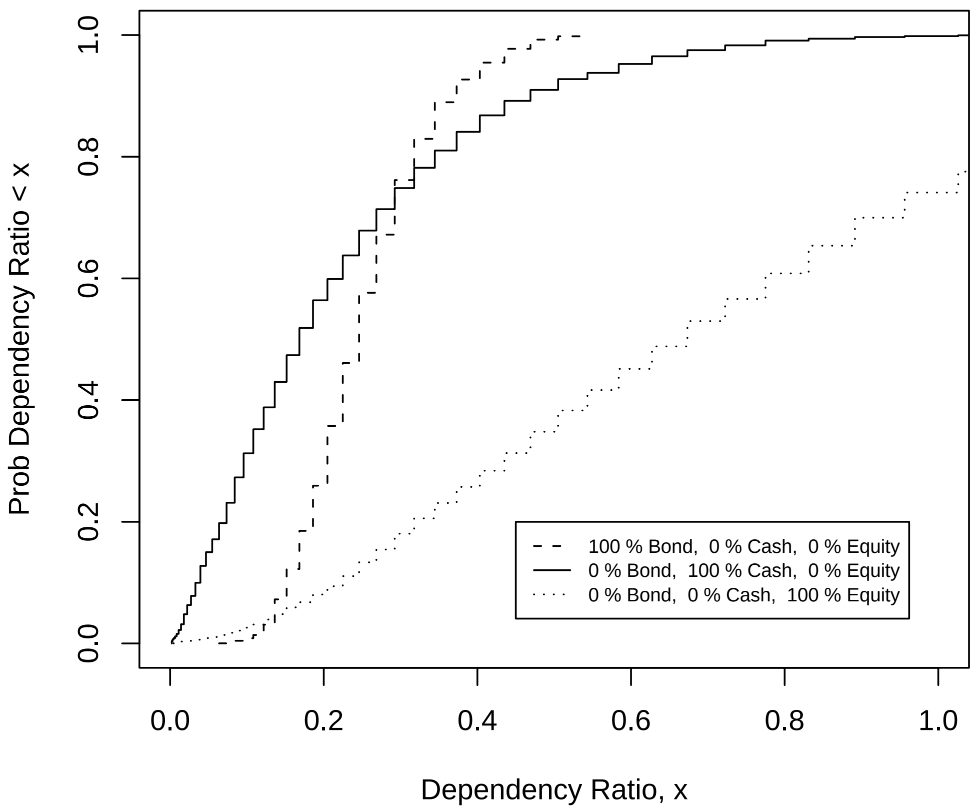

Figure 5: Empirical cumulative distribution functions of the dependency ratios, resulting from a population of DC participants who homogeneously allocate their funds in an all bond, all cash and all equity investment strategy.

Portfolio C: 100% Equities, 0% Bonds.

Once Portfolios A, B and C are allocated to their respective one third of new entrants, those members will maintain the same proportion of assets over their entire working life, classified earlier as a static asset allocation strategy.

A 60% equity and 40% bond portfolio is chosen to be the population's mean portfolio for the following reasons:

- It is consistent with the current market average for DC schemes, which allows for some diversification between equities and bonds.

- Allocating a proportion of funds in up to 50% bonds was seen as efficient for an individual participant by the VaR 95% risk measure in Figure 1.

There may also exist investors with even higher levels of risk aversion, whose concerns lie in the 99% percentile of the age of retirement distribution. As was mentioned in Section 2.2, investors who require elevated levels of assurance that they will retire prior to a particular age should increase their exposure to bonds since portfolios with a high bond weighting deliver better results in the worst case scenarios. Portfolio A satisfies the needs of such investors, as well as inserting additional diversity to the portfolio selection of the population.

We examine various results over the span of 300 simulated years. When graphing the results, we exponentially smooth the average empirical investment performance of the total pension fund; thus, assigning a greater weight to the more recent fund returns. We chose a smoothing coefficient of 5 %, which assigns the last year's rate of return a weight of 5%, which is added to the 95% weight of the previous year's average[7]. Displaying the asset performance using this smoothing technique reflects the increasing importance of the most current fund returns on the DC participants' accumulated wealth. The total pension fund of an entire population with a heterogeneous investment strategy (composed of portfolios A, B and C) is labeled 'Portfolio ABC'. Similarly, 'Portfolio B' signifies the aggregated fund of the portion of the population who invest their funds in portfolio B.

It is worth noting that since the time span of the simulation is relatively short, the time series plots will appear quite a bit different from one simulation trial to the next. Therefore, we will focus on observations that are consistent across all the trials executed.

The results demonstrate that the most significant demographic effect is the correlation between the dependency ratio and the smoothed investment performance, as seen in Figure 6. Unsurprisingly, during bull markets, members are able to retire earlier; thus, increasing the dependency ratio. Likewise, the dependency ratio decreases during a bear market. Their harmonious movement is displayed in Figure 6(a), where a double y-axis facilitates their comparison. The scatterplot in graph (b) illustrates the correlation coefficient between the dependency ratio and the smoothed interest rate, which is quite dramatic at over 90%. The correlation coefficient exceeded 90% in every trial attempted.

In Figure 7(a), we plot the dependency ratio over time resulting from the same simulation run for the homogeneous population strategy B and the heterogeneous strategy

[7] This coefficient of 5% was found to be approximately optimal in terms of maximizing the correlation with the dependency ratio

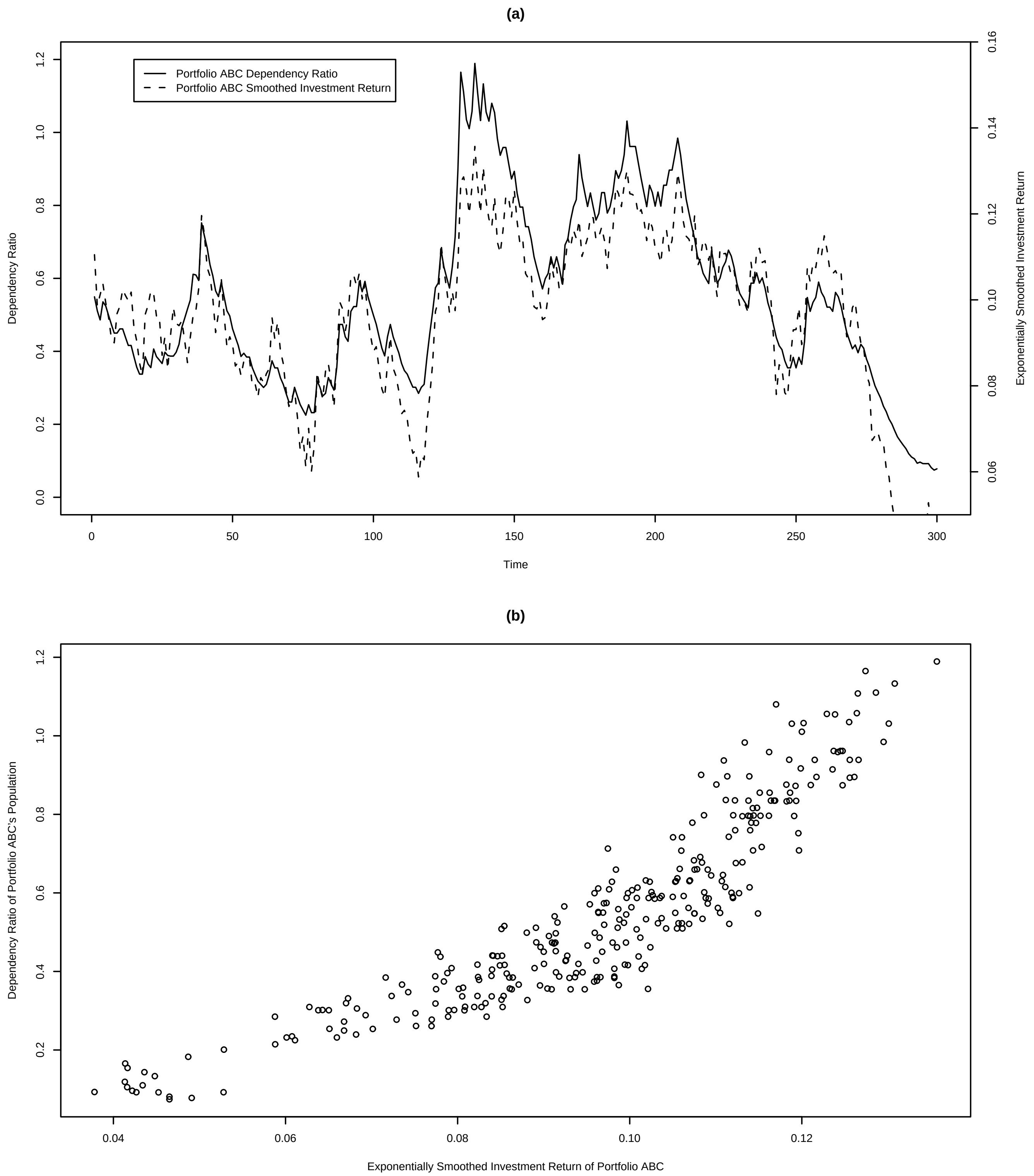

Figure 6: Time series plots and a scatter-plot of the dependency ratios and the exponentially smoothed empirical investment performance: (a) Time series plots of the dependency ratios (left-handed scale) for an entire population who heterogeneously invest their funds and the smoothed investment return (right-handed scale) of Portfolio ABC. Exponential smoothing coefficient = 5%. (b) Scatterplot of the dependency ratios for an entire population with a heterogeneous investment strategy versus the exponentially smoothed investment performance of Portfolio ABC.

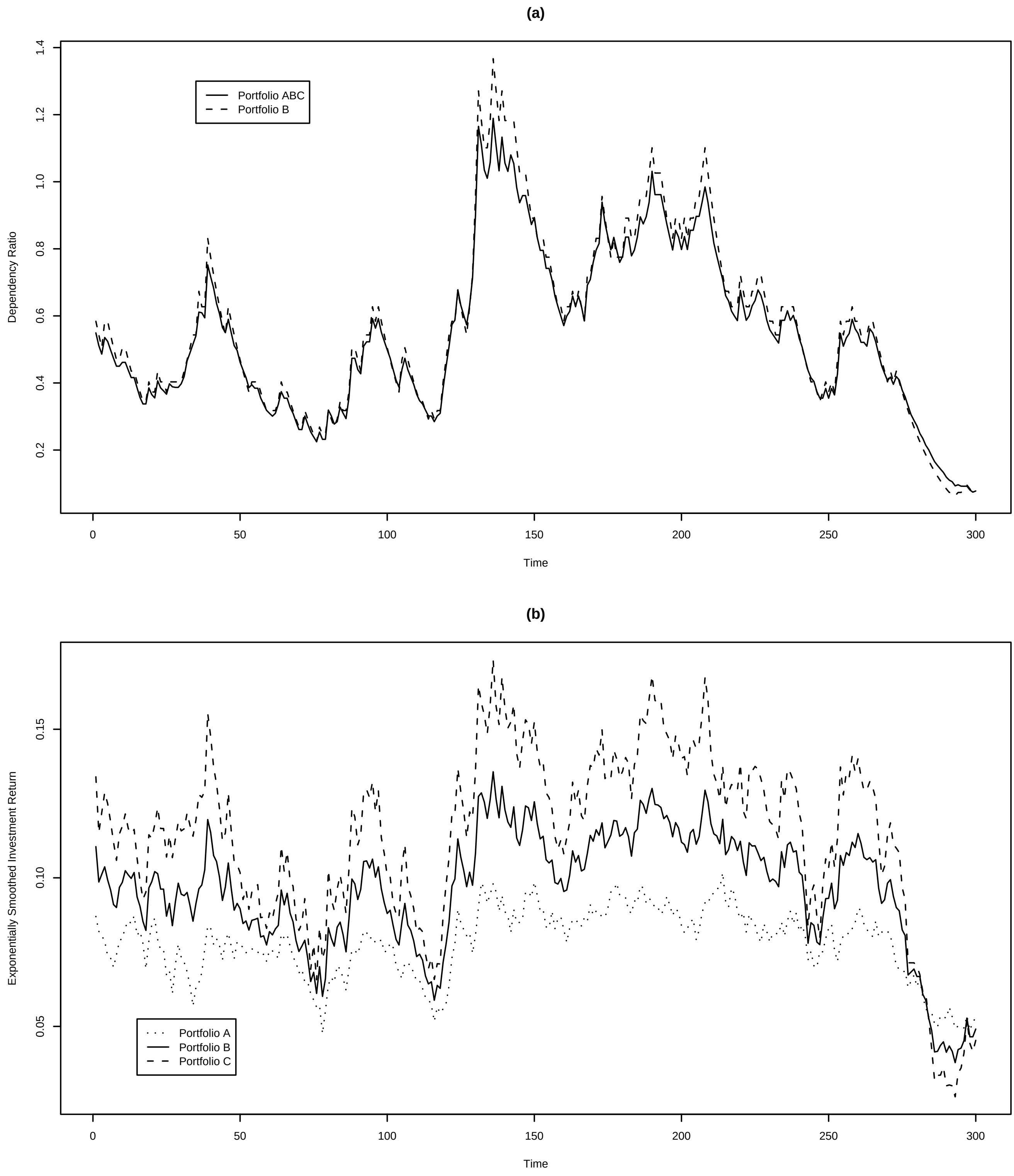

Figure 7: (a) Time series plots of the dependency ratios for an entire population with a heterogeneous investment strategy and for the members of the population with a homogeneous investment strategy in Portfolio B. (b) Smoothed investment return plots of Portfolio A, Portfolio B and Portfolio C. Exponential smoothing coefficient = 5%.

Portfolio	Portfolio A	Portfolio B	Portfolio C
Portfolio A	1	0.88	0.79
Portfolio B	0.88	1	0.99
Portfolio C	0.79	0.99	1

Table 1: Long-term correlation between the smoothed investment return of each portfolio.

ABC. The striking conclusion from this is that diversity in the chosen investment strategies does little to reduce the significant fluctuation over time of the dependency ratio. This could be due to the fact that the smoothed investment return of each portfolio have similar movement, as seen in Figure 7(b). Simulating over an extended time period reveals that the fluctuation of the smoothed rates are highly correlated, as shown in Table 1. This is despite the fact that the correlation between annual log returns on bonds and equities is as low as 6.6%.

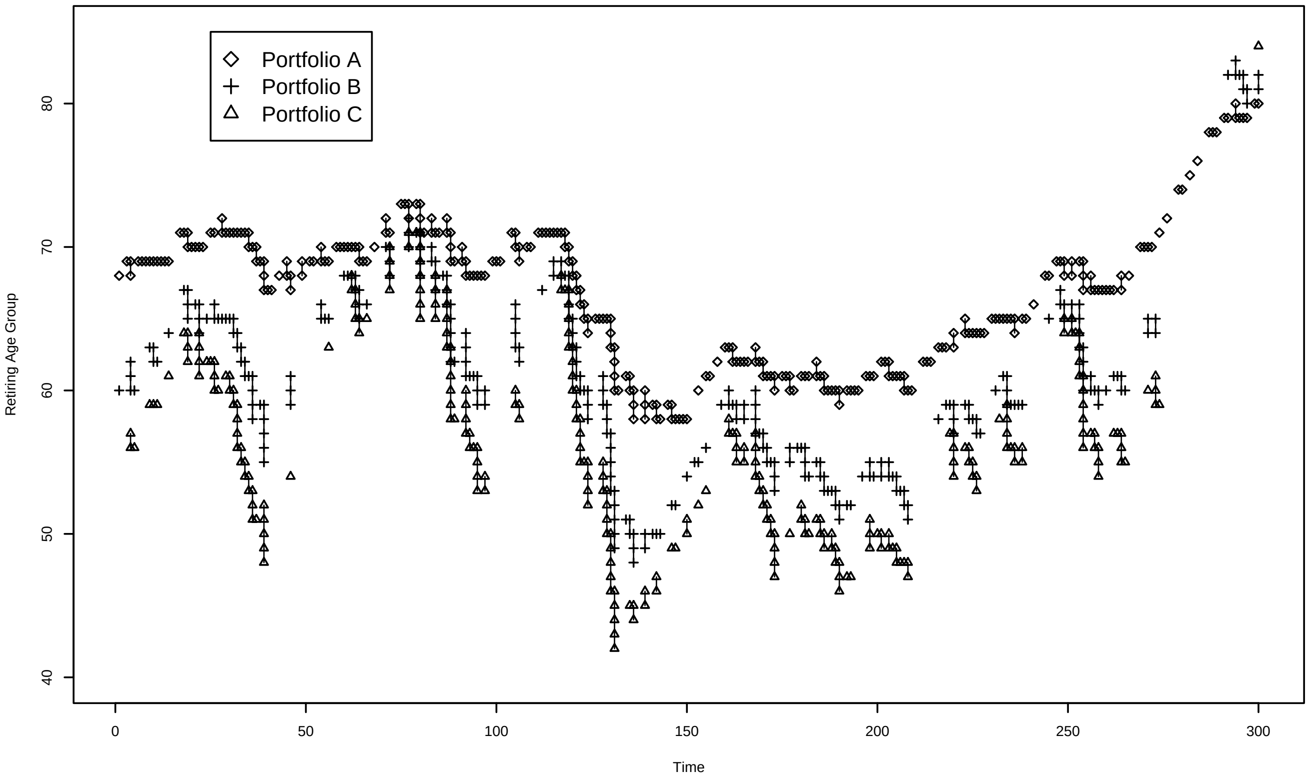

Figure 8: Time series plot of the ages of the new retirees for each year of simulation. The retirements are plotted by age group and are divided according to their investment strategy. To facilitate viewing, the age groups with the same investment strategy and who retire in the same year are connected by a thin line

Figure 8 displays the ages of the newly retired members within each year. The superior performance of the equity investment is apparent. The age plot of the portfolio with the least amount of equities is indeed more stable (diamonds), but it consistently produces the highest ages of retirement, causing the stability to be a poor tradeoff for the near certainty of a delayed retirement. In contrast, the pure equity portfolio produces ages of retirement (triangles) that are much more variable, but consistently lower than the other investment strategies. The inconsistency in the ages of retirement from one year to the next within every investment strategy is shocking when the identical nature of each member is considered; explicitly, each retiree has identical investment portfolios, employment histories, characteristics and levels of contribution. Recall that it was stated that the simulation of the asset returns do not consider the interrelations among the sectors of the economy and their effect on asset prices. However, the pattern of retirements revealed by Figure 8 indicate that market dynamics might be influenced by the mass liquidation of pension assets.

Therefore, although a variety of asset allocation decisions among the participants will offer great benefits to some of the members of the population, it will do little to improve the fluctuation of the population's dependency ratio, which is controlled primarily by the unpredictable performance of the market. The lack of stability of the dependency ratio is worrisome and could have far reaching effects, as is the DC pension system's inability to retire the participants at systematic and reasonable ages.

4 Implications to the Individual DC Participant within the Population

The individuals within the population are identical in their mortality, DC account contributions, and employment history. If participating in a DB pension plan, these employees would earn identical pension benefits. In addition, the model is designed so that all the DC participants have their funds invested in sensible portfolios. It has already been established that there is uncertainty and variety in the ages of retirement; we will next investigate how much this will affect individual employees.

Consider the case of two employees, beginning their employment only one year apart. One would expect that if they were identical in respect of their salary, their pension account contributions, their portfolio selection and their mortality, they would retire at the same age (i.e. one year apart). However, Figure 9 reveals that this is not always the case. In a simulation of 1000 trials, 676 trials result in different ages of retirement. The points which have a low value on the x-axis, but a relatively high value on the y-axis, illustrates a situation where a coworker, beginning employment only one year later, retires at a significantly later age. For example, in graph (a), there is a simulation that results in the first employee retiring at age 65, while the second employee is required to wait until age 78 to retire. From this it can be concluded that the market performance, which is out of the control of the individual DC participants, plays a tremendous role in the DC member's age of retirement and the retirement dynamics in a population of DC pension plan participants.

Moreover, the discrepancy in the ages of retirement is more pronounced in graph (b). In this scenario, the two employees begin employment on the same date, but their portfolio choice differs. The first employee chooses a pure equity portfolio and the second employee alters their portfolio to contain 50% bonds and 50% equities. The two workers retire at the same age in only 17% of the trials. In an extreme simulation, the first employee retires at age 47, while the second employee does not retire until age 71. It was seen in Figure 6 that diversifying the investment strategies of the members of the population does little to improve the sporadic movement of the dependency ratios from year to year. Notwithstanding, having varying investment strategies has a huge impact on the size of the gap in retirement ages between the individual members.

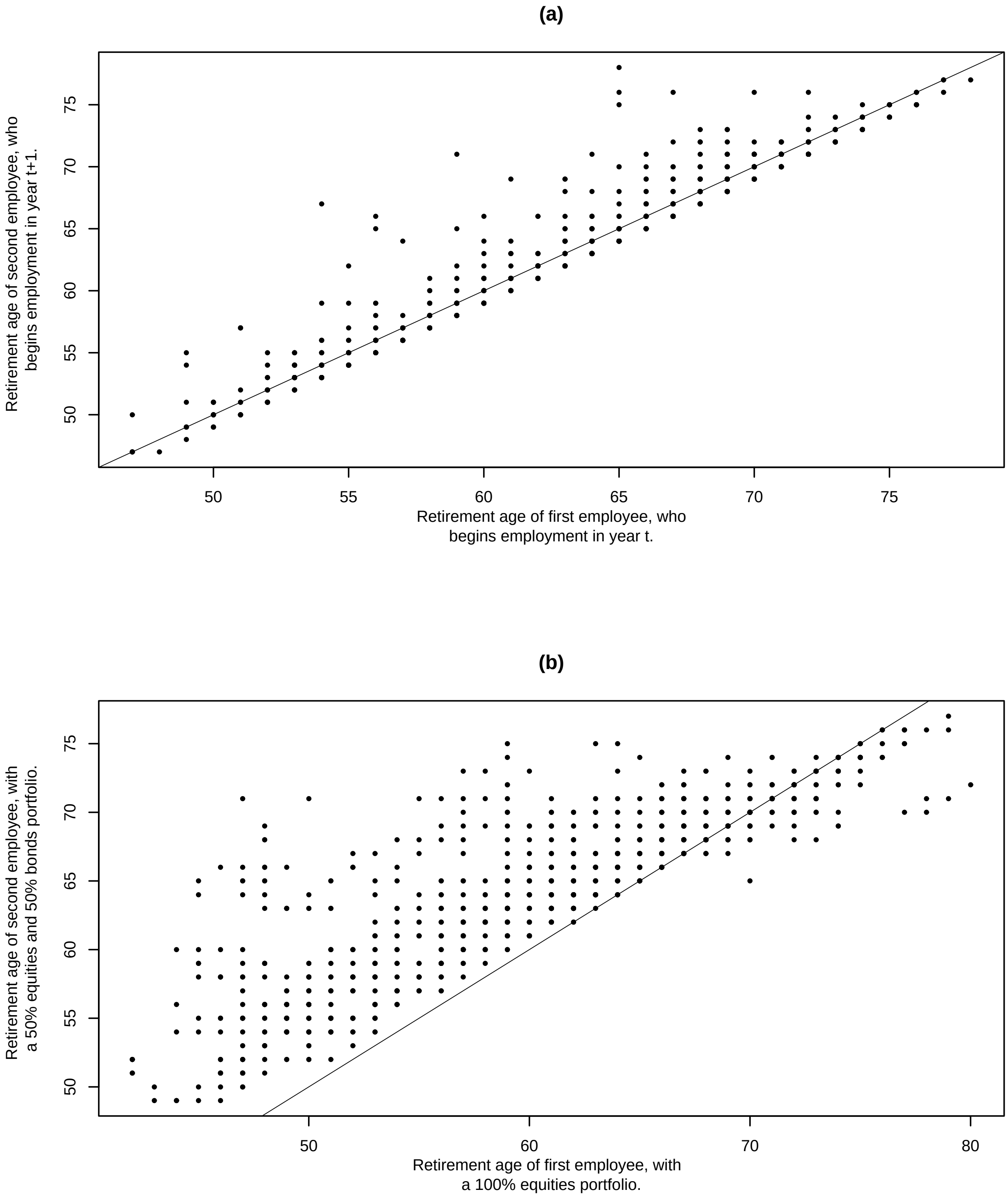

Figure 9: Scatterplots of ages of retirement of two individuals: (a) who begin employment one year apart, with identical investment strategies of 60% equities and 40% bonds, and (b) who begin employment in the same year, but with different investment strategies of 100% equities for the first member, and 50% equities and 50% bonds for the second member. Both plots also show the line $x = y$ to facilitate interpretation.

5 Conclusion

According to our flexible age of retirement model, if a DC pension system is introduced to an entire society to serve as their principal salary replacement in retirement, the financial market will have an exceptional impact on the proportions of retirees and workers from one year to the next. Hence, as there is significant fluctuation in the market's performance, so too will there be corresponding swings in the population's workforce demographics. Additionally, a successful market will generate the retirement of the masses, causing an enormous liquidation of assets which could potentially cause market equilibrium upset.

The success of the market, exasperated by the chosen investment strategy, causes a huge disparity in the ages of retirement of the individuals, despite their analogous retirement related characteristics. The inequities in the ages of retirement affect the majority of participants, including those whose identical characteristics would have earned them an identical defined benefit pension and who began employment within a close time frame. Further to the detriment in the society's labor force structure, the unpredictability of the financial markets produces ambiguous and unmanageable retirement ages, which could lead to personal hardship and anxiety for the individual DC member.

Acknowledgements

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Appendix

A Accumulation Model

A.1 Vasicek Model

In this section, we describe the arbitrage-free stochastic model used for modeling the dynamics of the equity, bond and cash prices in this study. The specific model chosen to simulate the instantaneous risk-free rate of return and the equity returns is the Vasicek model (Vasicek, 1977). This is a one-factor model for the term structure of interest rates within a continuous-time framework. It offers the benefits of simplicity and closed form solutions for bond prices. In addition, it is an autoregressive model, meaning the item being modeled cannot drift off to plus or minus infinity or to zero, but will eventually be pulled back to some long-term target. This model also has the advantage of being arbitrage free, but it also has drawbacks. One disadvantage in the Vasicek model is that it allows for the instantaneous risk-free rate of interest to become negative, which is somewhat unrealistic. Secondly, as a one-factor interest-rate model, there is only a single source of uncertainty in yield curve movements. Our model is also simplified through the assumption that salaries grow at a fixed rate of 5%.

The Vasicek model contains the following parameters:

- μ_r: the real-world long-term mean risk-free rate,
  - $\tilde{\mu}_r$: the risk-neutral long-term mean risk-free rate,
  - $\mu_r = \tilde{\mu}_r + \frac{\sigma_r \delta_1}{\alpha_r}$, where δ_1 is the market price of risk associated with the source of risk $W_1(t)$.
  - Long-bond data is introduced in the parameter estimation procedure to incorporate the appropriate risk premium on the interest rate process; specifically, estimate δ_1. This procedure is described in Section A.3.
- α_r: the rate at which the risk free rate of interest reverts back to its long-term mean, μ_r,
- σ_r: the local volatility of short-term interest rates,

- σ_{s1}: the equity volatility which is systematic with the interest rate market,
- σ_{s2}: the equity volatility which is idiosyncratic with the interest rate market,
- risk premium on equities: represented by $\sigma_{s1}\delta_1+\sigma_{s2}\delta_2$, this is the expected long-term excess return on an equity investment over the risk-free rate of interest,
  - δ_1 and δ_2 are the market prices of risk associated with the two sources of risk $W_1(t)$ and $W_2(t)$.

Following the assumption that the instantaneous risk-free rate of interest, $r(t)$, follows the Vasicek model, its stochastic differential equation (SDE) under the real-world measure is as follows:

$$\begin{aligned} \mathrm{d}r(t) &= \alpha_r(\tilde{\mu}_r - r(t))\,\mathrm{d}t + \sigma_r\,\mathrm{d}\tilde{W}_1(t) \\ &= \alpha_r(\tilde{\mu}_r - r(t))\,\mathrm{d}t + \sigma_r(\,\mathrm{d}W_1(t) + \delta_1\,\mathrm{d}t) \\ &= \alpha_r(\mu_r - r(t))\,\mathrm{d}t + \sigma_r\,\mathrm{d}W_1(t) \end{aligned}$$

where $\tilde{W}_1(t)$ is a standard Brownian Motion under the risk-neutral probability measure and $W_1(t)$ is a standard Brownian Motion under the real-world probability measure.

Similarly, in order to derive the equity share (net of dividend tax) total return, the assumption is that the SDE for equities, $S(t)$, is as follows:

$$\mathrm{d}S(t) = S(t)\,[r(t)+\sigma_{s1}\delta_1+\sigma_{s2}\delta_2]\,\mathrm{d}t + S(t)\sigma_{s1}\,\mathrm{d}W_1(t) + S(t)\sigma_{s2}\,\mathrm{d}W_2(t),$$

where $W_2(t)$ is a second standard Brownian Motion under the real-world probability measure, independent of $W_1(t)$ (Cairns, 2004).

According to the Vasicek model, the price at time t of a risk-free zero-coupon bond that matures at time T is given by

$$P(r(t),t,T) = e^{[A(t,T)-B(t,T)r(t)]}, \tag{3}$$

where

$$B(t,T) = \frac{1-e^{-\alpha_r(T-t)}}{\alpha_r},$$

$$A(t,T) = (B(t,T)-(T-t))\left(\tilde{\mu}_r - \frac{\sigma_r^2}{2\alpha_r^2}\right) - \frac{\sigma_r^2 B(t,T)^2}{4\alpha_r}.$$

The logarithmic return between time t and $t+1$ on a risk-free perpetuity, which pays 1 at the end of each future year without risk and is otherwise known as bonds in this study, is given by:

$$\log\frac{1+\sum_{T=2}^{\infty}P(r(t+1),t+1,T)}{\sum_{T=1}^{\infty}P(r(t),t,T)}.$$

The log return between time t and $t+1$ for the risk-free cash account, which is a one year risk free bond, is given by:

$$\log\frac{1}{P(r(t),t,t+1)}.$$

Additionally, the annuity factor, for retirement age $x+t$ at time t, ignoring expenses, is given by:

$$\ddot{a}_{x+t}(t) = \sum_{s=0}^{\infty} P(r(t),t,t+s)\;{}_{s}\mathrm{p}_{x+t}.$$

A.2 Derivation of Asset Model Parameter Estimates

To utilize the Vasicek model to derive parameter estimates which will lead to an arbitrage free model for the assets, we use the SDE moments directly to derive the conditional distribution function of $r(t)$ and $S(t)$; whereupon, the resulting maximum likelihood function is maximized with current and relevant data to produce parameter estimates.

Deriving the conditional moments for the risk free rate over T year(s) period $(r(T))$ and for the log return on equities over T years $\left(\log \frac{S(T)}{S(0)}\right)$, $\left(\log \frac{S(T)}{S(0)} \mid r(0)\ ,\ r(T) \mid r(0)\right)$ has a bivariate normal distribution with:

$$\begin{aligned} E\left(\log \frac{S(T)}{S(0)} \mid r(0)\right) &= \mu_r T + \left[\frac{1-e^{-\alpha_r T}}{\alpha_r}\right][r(0)-\mu_r] \\ &\quad + T\left[\sigma_{s1}\delta_1 + \sigma_{s2}\delta_2 - \frac{1}{2}(\sigma_{s2}^2 + \sigma_{s1}^2)\right], \end{aligned}$$

$$E\left(r(T) \mid r(0)\right) = \mu_r + e^{-\alpha_r T}\left(r(0)-\mu_r\right),$$

$$\begin{aligned} Var\left(\log \frac{S(T)}{S(0)} \mid r(0)\right) &= \left(\frac{\sigma_r}{\alpha_r} + \sigma_{s1}\right)^2 T - \frac{2\sigma_r}{\alpha_r^2}\left(\frac{\sigma_r}{\alpha_r} + \sigma_{s1}\right)\left(1-e^{-\alpha_r T}\right) \\ &\quad + \left(\frac{\sigma_r}{\alpha_r}\right)^2 \frac{1-e^{-2\alpha_r T}}{2\alpha_r} + \sigma_{s2}^2 T, \end{aligned}$$

$$Var\left(r(T) \mid r(0)\right) = \sigma_r^2 \frac{1-e^{-2\alpha_r T}}{2\alpha_r},$$

$$Cov\left(\log \frac{S(T)}{S(0)}, r(T) \mid r(0)\right) = \left(\frac{\sigma_r}{\alpha_r} + \sigma_{s1}\right)\sigma_r \frac{1-e^{-\alpha_r T}}{\alpha_r} - \frac{\sigma_r^2\left(1-e^{-2\alpha_r}\right)}{2\alpha_r^2}.$$

To Simulate $r(t)$ and $\log \frac{S(t)}{S(t-1)}$, let $Z_1, Z_2 \sim iidN(0,1)$:

$$r(t) = E\left(r(t) \mid r(t-1)\right) + C_{11}Z_1 + C_{12}Z_2,$$

$$\log \frac{S(t)}{S(t-1)} = E\left(\log \frac{S(t)}{S(t-1)} \mid r(t-1)\right) + C_{21}Z_1 + C_{22}Z_2.$$

Z_1 and Z_2 are simulated by applying the Box-Muller transformation to pseudo random numbers.

We choose the Cholesky decomposition for C:

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{Var(r(T))} & 0 \\ \frac{Cov(X,r(T))}{\sqrt{Var(r(T))}} & \sqrt{Var(X) - \frac{Cov(X,r(T))^2}{Var(r(T))}} \end{bmatrix}.$$

The maximum likelihood estimators of $(\mu_r,\ \sigma_r,\ \alpha_r,\ \sigma_{s1},\ \delta_1,\ \sigma_{s2},\ \delta_2)$ are derived by maximizing the bivariate normal probability distribution function of $\left(\log \frac{S(T)}{S(0)} \mid r(0)\ ,\ r(T) \mid r(0)\right)$.

Using daily US Government Federal Reserve historical 3-month Treasury Constant Maturities data from 1970 to 2004[8] and incorporating Vasicek model assumptions, historical $r(t)$ data can be produced. In addition, $S(t)$ is represented with monthly US Equity Share TRI historical data from 1970 to 2004[9]. From this data, the following estimates are calculated to maximize the log likelihood function:

Parameter	μ_r	α_r	σ_r	$\sigma_{s1}\delta_1 + \sigma_{s2}\delta_2$	σ_{s1}	σ_{s2}
Estimates	0.051	0.15	0.0185	0.053	-0.01	0.156

The risk premium and long-term mean risk free rate estimates are consistent with historical data and in line with market views on these quantities. $\hat{\alpha}_r$ represents a moderate level of mean reversion. Examining the volatility parameters, $\hat{\sigma}_{s1}$ indicates that equity returns have a very low, negative correlation with changes in interest rates. A possible explanation is that equities and risk-free interest rates are somewhat negatively correlated; that is, unexpected rising interest rates will cause equity values to decrease due to a higher discount rate applied to the equity's future expected payments.

A.3 Determination of the Yield Curve Risk Premium

The importance of determining the yield curve risk premium in our model is because, with all else being equal, bonds with longer terms will typically yield higher returns. This is a result of its heightened sensitivity to the prevailing risk-free interest rate, which will produce additional volatility in its price. This premium is incorporated in the Vasicek model by the use of the $\tilde{\mu}_r$ in the price at time t of a risk-free zero-coupon bond that matures at time T, as seen in equation 3.

Likewise to $S(t)$ and $r(t)$, in order to model the long bond total return, we follow the assumption that the value of $C(t)$, an irredeemable bond fund with reinvestment, follows the Vasicek model. Its stochastic differential equation (SDE) under the real-world measure is as follows:

$$\begin{aligned} \mathrm{d}C(t) &= C(t)\left[r(t)\,\mathrm{d}t + \sigma_c\left(\mathrm{d}W_1(t) + \delta_1\,\mathrm{d}t\right)\right] \\ &= C(t)\left[\left(r(t) + \sigma_c\delta_1\right)\mathrm{d}t + \sigma_c\,\mathrm{d}W_1(t)\right], \end{aligned}$$

where σ_c is the volatility of the excess return of irredeemable bond over cash. We make the simplifying assumption that σ_c is constant, although in reality there is evidence that the volatility of excess return of long-term bonds over cash is affected by the bond's term to maturity. Following from this assumption, the SDE at time t for a risk-free zero-coupon bond that matures at time T is as follows:

$$\mathrm{d}P(r(t),t,T) = P(r(t),t,T)\left[\left(r(t) + \sigma_p(t,T)\delta_1\right)\mathrm{d}t + \sigma_p(t,T)\,\mathrm{d}W_1(t)\right],$$

where $\sigma_p(t,T)$ is the volatility of the zero-coupon bond. For a long-dated zero coupon bond (that is, $T - t \to \infty$), $\sigma_p(t,T)$ will be approximately equal to $-\sigma_r/\alpha_r$. Therefore,

[8] http://www.federalreserve.gov/releases/h15/data.htm

[9] The US equity return data was kindly provided by David Wilkie. The data was constructed using several series of data covering different time periods.

with the assumption of constant volatility, the value of the yield curve premium on a long-dated zero-coupon bond is approximated by $-\delta_1\sigma_r/\alpha_r$.

Accordingly, all the parameters that are needed to determine $\tilde{\mu}_r$ have been estimated, with the exception of δ_1. To estimate δ_1 and calculate the risk premium on the interest rate process, we examine the historical annual total return for long-term bonds, $l(t)$ (the total return assumes reinvestment of dividends). The data used is the daily US Government Federal Reserve historical 20 and 30 year Treasury Constant Maturities data from 1970 to 2004.

The logarithmic return between time t and $t+s$ on a risk-free perpetuity, which pays 1 at the end of each time interval s without risk, is given by:

$$\log \frac{1+\frac{1}{(1+l(t+s))^{1/s}-1}}{\frac{1}{(1+l(t))^{1/s}-1}}. \tag{4}$$

The log return between time t and $t+s$ for the risk-free cash account, which is a risk free bond with a term of s years, is given by:

$$\log \frac{1}{P(r(t),t,t+s)}. \tag{5}$$

By letting $s=\frac{1}{250}$ and utilizing the historical long bond data in equations (4) and (5), we estimate the log of the excess return of irredeemable bond over cash to be set equal to the average difference between (4) and (5). With the assumption of a constant bond volatility of σ_c, this results in a risk premium of $\sigma_c\delta_1$. We assume in the current version of our work that the risk premium on $C(t)$ is the same as that on a long-dated zero-coupon bond: that is, $-\delta_1\sigma_r/\alpha_r$. We will relax this assumption in future work to refine our estimate for δ_1. Having estimates for σ_r and α_r, an estimate for δ_1 is thus calculated, from which we derive an estimate for $\tilde{\mu}_r$.

Given that (4) is dependent on the δ_1 estimate, this is a recursive process which terminates once the $\hat{\delta_1}$ used in (4) and the calculated δ_1 estimate are identical.

The annual parameter estimates are as follows:

Parameter	Average Annualized Excess	δ_1	$\tilde{\mu}_r$
Estimates	0.02022	-0.1642	0.07123

A 2% yield curve premium is reasonable and in line with market view on this quantity.