On the Merger of Two Companies

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Abstract

This paper examines the merger of two stock companies under the premise, due to Bruno De Finetti, that the companies pay out dividends to their shareholders in such a way so as to maximize the expectation of the discounted dividends until (possible) ruin or bankruptcy. The aggregate net income streams of the two companies are modeled by a bivariate Wiener process. Explicit results are presented. In particular, it is shown that if for each company the product of the valuation force of interest and the square of the coefficient of variation of the aggregate net income process is less than 6.87%, the merger of the two companies would result in a gain.

Keywords: Merger, optimal dividends, barrier strategy, Wiener process

1. Introduction

This note is an application of actuarial risk theory. We study the question whether a merger of two stock companies is profitable and how much the resulting gain is. The basic premise, which is due to the Italian actuary Bruno De Finetti (1957), is that a company will pay out dividends to its shareholders in such a way so as to maximize the expectation of the discounted dividends. The dividend payments stop when the company becomes bankrupt or is ruined.

2. The Model

For the convenience of the reader, this section presents some well-known results (for example, see Gerber and Shiu 2004) that will be used in the sequel. It is assumed that the aggregate net income process (before dividend payments) of a company is a Wiener process with a positive drift μ and variance σ^2 per unit time. The optimal dividend strategy is a *barrier strategy*. A barrier strategy has a parameter b, the level of the barrier. If the capital (also called equity or surplus) of the company is less than b, no dividends are paid. Whenever the capital reaches the level b, the "overflow" is paid as dividends to the shareholders. If the initial capital exceeds b, the difference is paid immediately as dividends.

Let V(x; b) denote the expectation of the discounted dividends until ruin, if the barrier strategy corresponding to parameter b is applied. Here, x denotes the initial capital. Thus,

$$V(x; b) = x - b + V(b; b) \qquad \text{for } x > b, \tag{2.1}$$

and it can be shown that

$$\frac{\sigma^2}{2}V''(x;\, b) \;+\; \mu V'(x;\, b) \;-\; \delta V(x;\, b) \;=\; 0 \quad \text{for } 0 < x < b, \tag{2.2}$$

where $\delta > 0$ is the valuation force of interest. The differential equation (2.2) is subject to the boundary conditions

$$V(0;\, b) \;=\; 0 \tag{2.3a}$$

and

$$V'(b;\, b) \;=\; 1. \tag{2.3b}$$

It follows from (2.2), (2.3a) and (2.3b) that

$$V(x;\, b) \;=\; \frac{e^{rx} - e^{sx}}{re^{rb} - se^{sb}}, \qquad 0 \le x \le b, \tag{2.4}$$

where r and s are the roots of the characteristic equation for (2.2),

$$\frac{\sigma^2}{2}\xi^2 \;+\; \mu\xi \;-\; \delta \;=\; 0. \tag{2.5}$$

We let r denote the positive root and s the negative root,

$$r \;=\; \frac{-\mu + \sqrt{\mu^2 + 2\delta\sigma^2}}{\sigma^2}, \tag{2.6a}$$

$$s \;=\; \frac{-\mu - \sqrt{\mu^2 + 2\delta\sigma^2}}{\sigma^2}, \tag{2.6b}$$

Let b* denote the optimal value of b; that is, b* is the value of b which minimizes the denominator in (2.4). Setting the derivative of the denominator in (2.4) equal to zero, we have

$$r^2 e^{rb^*} \;-\; s^2 e^{sb^*} \;=\; 0. \tag{2.7}$$

Hence,

$$b^* \;=\; \frac{1}{r - s}\, \ln\!\left(\frac{s^2}{r^2}\right) \;=\; \frac{2}{r - s}\, \ln\!\left(\frac{-s}{r}\right). \tag{2.8}$$

Note that b*, the optimal level of the dividend barrier, does not depend on the capital x.

From (2.7) and (2.4), it follows that

$$V''(b^*; b^*) = 0. \tag{2.9}$$

Applying (2.9) and (2.3b) with b = b* to the differential equation (2.2), we obtain

$$V(b^*; b^*) = \frac{\mu}{\delta}. \tag{2.10}$$

This elegant formula will be a key for analyzing a merger.

3. Two Functions Related to the Optimal Barrier

Let

$$\zeta = \frac{\sigma}{\mu} \tag{3.1}$$

be the *coefficient of variation* of the underlying Wiener process. Then, it follows from (2.8), (2.6a) and (2.6b) that

$$b^* = \mu f(\zeta), \tag{3.2}$$

where

$$f(z) = \frac{z^2}{\sqrt{1+2\delta z^2}} \ln\left(\frac{\sqrt{1+2\delta z^2}+1}{\sqrt{1+2\delta z^2}-1}\right), \qquad z \geq 0. \tag{3.3}$$

For further discussion, it is useful to introduce the function

$$g(y) = \frac{y^2}{\sqrt{1+y^2}} \ln\left(\frac{\sqrt{1+y^2}+1}{\sqrt{1+y^2}-1}\right), \qquad y \geq 0. \tag{3.4}$$

Then,

$$f(z) = \frac{1}{2\delta} g(\sqrt{2\delta}\, z), \qquad z \geq 0. \tag{3.5}$$

The function g(y), $y \geq 0$, is an increasing function, with $g(0) = 0$ and $g(\propto) = 2$. We shall be interested in the convexity of f, hence in the convexity of g. This property can be readily examined by mathematical software such as Maple or Mathematica. It is found that there exists a number $\tilde{\xi} = 0.3708175\ldots$ such that

$$g''(y) > 0 \qquad \text{for } 0 \leq y < \tilde{\xi}, \tag{3.6a}$$

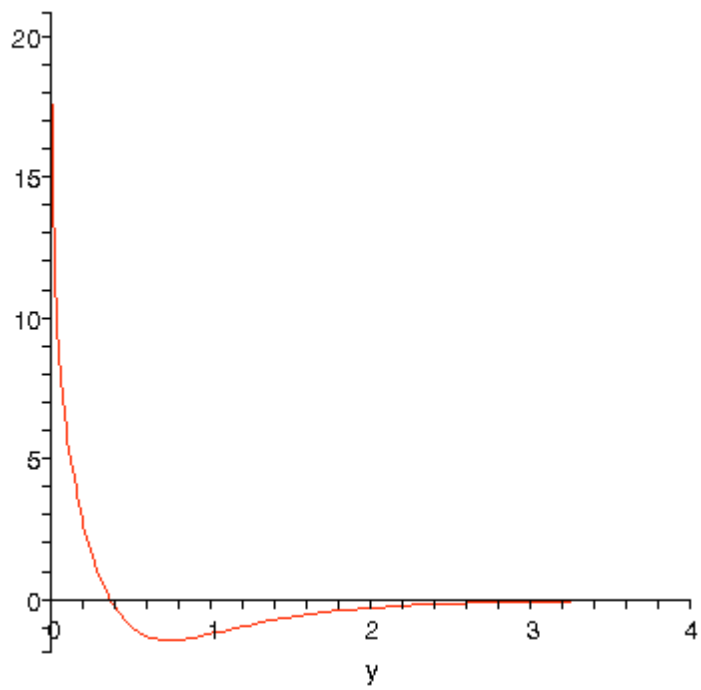
and

$$g''(y) < 0 \qquad \text{for } \tilde{\xi} < y < \propto. \tag{3.6b}$$

The graph of the second derivative g"(y) is displayed in Figure 1.

Figure 1

The second derivative g"(y)



4. The Situation Before and After the Merger

We consider two stock companies, labeled 1 and 2. We assume that the aggregate net income process (before dividend payments) of company j is a Wiener process with positive drift μ_j and variance σ_j^2 per unit time. The optimal barrier for company j is

$$b_j^* = \mu_j f(\zeta_j), \tag{4.1}$$

with f defined by (3.3) and

$$\zeta_j = \frac{\sigma_j}{\mu_j}. \tag{4.2}$$

Let $V_j(x; b)$ denote the expectation of the company's discounted dividends until ruin, if x is its capital and the barrier strategy corresponding to parameter b is applied.

Furthermore, we assume that joint aggregate net income process (before dividend payments) is a bivariate Wiener process with correlation coefficient ρ. Hence, after the merger, the resulting aggregate net income process (before dividend payments) is a Wiener process, with parameters

$$\mu_m = \mu_1 + \mu_2 \tag{4.3}$$

and

$$\sigma_m^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2. \tag{4.4}$$

Thus, the optimal barrier for the merged company is

$$b_m^* = (\mu_1 + \mu_2) f(\zeta_m), \tag{4.5}$$

with

$$\zeta_m = \frac{\sigma_m}{\mu_m} = \frac{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}}{\mu_1 + \mu_2}. \tag{4.6}$$

Let $V_m(x; b)$ denote the expectation of the merged company's discounted dividends until ruin. Then, the potential gain upon merging the two companies is

$$V_m(x_1 + x_2;\ b_m^*) - [V_1(x_1;\ b_1^*) + V_2(x_2;\ b_2^*)], \tag{4.7}$$

where x_j is the current capital of company j, j =1, 2, and $x_1 + x_2$ is the capital of the merged company. Each of the three terms in (4.7) can be calculated by applying the appropriate version of (2.4), (2.6) and (2.8). A merger is profitable if expression (4.7) is positive.

There is a situation where the sign of expression (4.7) can be readily identified. It follows from (2.10) and (4.3) that

$$V_m(b_m^*; b_m^*) = V_1(b_1^*;\ b_1^*) + V_2(b_2^*;\ b_2^*). \tag{4.8}$$

Therefore, let us assume that the current capital of company j is b_j^*, j = 1, 2. Hence, $(b_1^* + b_2^*)$ is the capital of the merged company. If

$$b_m^* < b_1^* + b_2^*, \tag{4.9}$$

we see from (2.1) and (4.8) that

$$\begin{aligned} V_m(b_1^* + b_2^*;\ b_m^*) &= (b_1^* + b_2^* - b_m^*) + V_m(b_m^*;\ b_m^*) \\ &= (b_1^* + b_2^* - b_m^*) + V_1(b_1^*;\ b_1^*) + V_2(b_2^*;\ b_2^*). \end{aligned} \tag{4.10}$$

Therefore, the merger yields an immediate profit of $(b_1^* + b_2^* - b_m^*)$. On the other hand, if

$$b_m^* > b_1^* + b_2^*, \tag{4.11}$$

then

$$V_m(b_1^* + b_2^*;\ b_m^*) < V_m(b_m^*;\ b_m^*) = V_1(b_1^*;\ b_1^*) + V_2(b_2^*;\ b_2^*) \tag{4.12}$$

by (4.8), and the merger does not make economic sense.

5. A Sufficient Condition for Merger

As a function of ρ, the optimal barrier b_m^*, given by (4.5), is an increasing function. To see this, note that the coefficient of variation ζ_m, given by (4.6), is an increasing function of ρ, and that the function f is an increasing function (because the function g is an increasing function). Hence, if (4.9) holds for $\rho = 1$, it holds for all $\rho \in [-1, 1]$.

We now examine the case $\rho = 1$. It follows from (4.6) that

$$\zeta_m = \frac{\sigma_1 + \sigma_2}{\mu_1 + \mu_2}, \tag{5.1}$$

which can be written as a weighted average of ζ_1 and ζ_2:

$$\zeta_m = \frac{\mu_1}{\mu_m}\zeta_1 + \frac{\mu_2}{\mu_m}\zeta_2. \tag{5.2}$$

Thus, if $\zeta_1 = \zeta_2$, then ζ_m has the same value, and $b_m^* = b_1^* + b_2^*$. If $\zeta_1 \neq \zeta_2$, then condition (4.9) is now the condition

$$f\left(\frac{\mu_1}{\mu_m}\zeta_1 + \frac{\mu_2}{\mu_m}\zeta_2\right) < \frac{\mu_1}{\mu_m} f(\zeta_1) + \frac{\mu_2}{\mu_m} f(\zeta_2) \tag{5.3}$$

because of (4.5) and (4.1). A sufficient condition for (5.3) to hold is that the graph of the function f(z) between $z = \zeta_1$ and $z = \zeta_2$ is below the secant. Now we recall from (3.6a) that $g''(y) > 0$ for $0 \le y < \tilde{\zeta} = 0.3708175\ldots$ It thus follows from (3.5) that if

$$\sqrt{2\delta}\,\zeta_j < \tilde{\zeta} \tag{5.4}$$

for j = 1 and j = 2, we can be sure that inequality (4.9) holds.

In conclusion, if for both company 1 and company 2 we have

$$\frac{\delta\sigma_j^2}{\mu_j^2} < \frac{\tilde{\xi}^2}{2} = 0.0687528\ldots, \tag{5.5}$$

and if the current capital of each company is at its optimal dividend barrier, then a merger would be profitable. This result holds for every correlation coefficient ρ.

6. A Generalization

Cai, Gerber and Yang (2005) consider the model in Section 2. But they assume that the surplus earns investment income at a rate $\iota < \delta$. They show that

$$V(b^*;\ b^*) = \frac{\mu + \iota b^*}{\delta}, \tag{6.1}$$

which generalizes (2.10). Some of the ideas in Section 4 can be repeated in this model.

Consider two companies with optimal barriers b_1^* and b_2^*, respectively. We assume $x_1 = b_1^*$ and $x_2 = b_2^*$. Let b_m^* denote the optimal barrier of the merged company.

Suppose that $b_m^* < b_1^* + b_2^*$. The sum of the expectation of the discounted optimal dividends is

$$V_1(b_1^*;\ b_1^*) + V_2(b_2^*;\ b_2^*) = \frac{\mu_m + \iota(b_1^* + b_2^*)}{\delta}, \tag{6.2}$$

and the expectation of the discounted optimal dividends after merger is

$$\begin{aligned} V_m(b_1^* + b_2^*;\ b_m^*) &= (b_1^* + b_2^* - b_m^*) + V_m(b_m^*;\ b_m^*) \\ &= (b_1^* + b_2^* - b_m^*) + \frac{\mu_m + \iota b_m^*}{\delta}. \end{aligned} \tag{6.3}$$

Therefore, the gain due to the merger, (6.3) minus (6.2), is $\left(1 - \frac{\iota}{\delta}\right)(b_1^* + b_2^* - b_m^*)$.

Now, suppose that $b_m^* > b_1^* + b_2^*$. From $V_m'(x;\ b_m^*) > 1$ for $x < b_m^*$, it follows that the expectation of the discounted optimal dividends after merger is

$$V_m(b_1^* + b_2^*;\ b_m^*) < V_m(b_m^*;\ b_m^*) - (b_m^* - b_1^* - b_2^*)$$

$$= \frac{\mu_m + \iota b_m^*}{\delta} - (b_m^* - b_1^* - b_2^*). \qquad (6.4)$$

Therefore, the loss due to a merger, (6.2) minus (6.4), is at least $\left(1 - \frac{\iota}{\delta}\right)(b_m^* - b_1^* - b_2^*)$.

Hence a merger does not make economic sense.

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