The First 75 Years

- Bachelier (1900)
- Levy (1925)
- Ito (1951)
- Markowitz (1952)
- Reddington (1952)
- Sprenkle (1958)
- Samuelson (1965)
- Thorpe (1969)
- Black & Scholes (73)
Black & Scholes

- Black & Scholes (1973)

- Merton (1973)

- Harrison & Kreps (1977)

Exotic Options
**Exotic Options**

- **Margrabe (1978)**
  \[ \text{PO}_T = \max(Q_1 S_1(T) - Q_2 S_2(T), 0) \]

  \[ \text{PO}_T = \max(\max(S_1, S_2) - X, 0) \]


**Deferred Annuities**

- **Margrabe (1978)**
  \[ \text{PO}_T = \max(Q_1 S_1(T) - Q_2 S_2(T), 0) \]
Deferred Annuities

- Margrabe (1978)
  \[ PO_T = \max(Q_1 S_1(T) - Q_2 S_2(T), 0) \]
  \[ GC_T = \max(g b_T a_{x+T,i(T)} - AS_T, 0) \]

Compound Options

  \[ PO_T = \max(\max(S_1, S_2) - X, 0) \]
**Compound Options**

  \[ PO_T = \max(\max(S_1, S_2) - X, 0) \]

- Useful for valuing benefits of the form
  \[ \max(S_1, S_2, X) = X + \max(\max(S_1, S_2) - X, 0) \]

- Generalises
  \[ \max(S_1, S_2, S_3, X) = \max(S_1, \max(S_2, S_3, X), 0, 0) \]

**Guaranteed Annuity Options**

  \[ PO_T = \max(\max(S_1, S_2) - X, 0) \]

  \[ Benefit = \max(AS_T, GB_T, GB_T g_T a_{s+T,T(T)}) \]

  \[ GC_T = \max(AS_T, GB_T, GB_T g_T a_{s+T,T(T)}) - AS_T \]

  \[ = GB_T + \max(\max(AS_T, GB_T g_T a_{s+T,T(T)}) - GB_T, 0) - AS_T \]

  \[ \frac{ULBenefit}{= \max(AS_T, AS_T g_T a_{s+T,T(T)})} \]

  \[ CommuteGG = \max(AS_T, gb_T a_{s+T,T(T)}, gb_T h_T) - AS_T \]
Regular Premium Business

- **Asian Options**
  Options based on the average rather than the terminal value of the underlying asset/index

- **Regular Premium**
  Policyholder has an option to pay future premium instalments—office

\[
AS_T = (1-w)^T (1-k) \left( \frac{I_{T_0}}{I_0} + \frac{I_{T_1}}{I_1} + \cdots + \frac{I_{T_{T-1}}}{I_{T-1}} \right)
\]

\[
AS_T = (1-w)^T (1-k) P[S_1 + S_2 + \cdots + S_T]
= (1-w)^T (1-k) PT\bar{S}_T
\]

\[
GC_T = \max(GB_T - AS_T, 0)
= \max(GB_T - (1-w)^T (1-k)TP\bar{S}_T, 0)
\]
**Economic Balance Sheet Unhedged**

- **Scenario**: equities down 20%, bonds unchanged
- 20% equity fall → value of guarantee increases to 25.6
- ALM capital impact: £7.6. Difference between new value of guarantees and re-valued assets that support guarantees (= 25.6–18)
- Capital injection = £3.1 to restore solvency and £7.6 to restore financial strength to perfectly matched position

**Economic Balance Sheet OTC Hedge**

- **Buy an OTC put option to match the liability guarantee**
- In theory a perfect match: no ALM impact or capital injection; but:
  - Pricing & Cash Flow
  - At mercy of market: pay implied volatility of market / counterparty
  - Other assumptions needed to make hedge perfect
  - Periodic rebalancing required, incurs high transactions costs
Replication

Put Option = (Short Position in Underlying) + Long Position in Risk Free

Position in Volatility Sensitive Asset + Position in Interest Sensitive Asset

- Delta: Protects against small immediate changes in underlying
- Gamma: Protects against small immediate changes in delta
- Vega: Protects against small immediate changes in Vol
- Rho: Protects against small immediate changes in interest

Dynamic Hedge in the Balance Sheet

<table>
<thead>
<tr>
<th></th>
<th>Equity</th>
<th>Bonds</th>
<th>Risk Free</th>
<th>Options</th>
<th>Swaps</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset Backing</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Value of Guarantee</td>
<td>20</td>
<td>(24)</td>
<td>(24)</td>
<td>60</td>
<td>5</td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- "Manufacture" the OTC derivative
- Protects against risks of:
  - Underlying assets falling
  - Volatility spiking
  - Interest rates changing
- Dynamic ➔ Technology requirements
Dynamic Hedging: Requirements

- Market consistent stochastic valuation of liability guarantees on a policy-by-policy and daily basis
- Capital markets expertise to identify hedge portfolio
- Systems capability to dynamically monitor position frequently
- Effective control systems
  - Performance and risk attribution through analysis of profit movements
  - Financial reporting requirements
  - Financial projection analysis – back testing of strategy to demonstrate effectiveness of the hedge

- The bottom line: a very complex task requiring specialist actuarial, capital market and systems expertise / investment
Overview of Hedging Program

for Insurance Guarantees

MG-Hedge System
structure reflects market risk management framework

Actuarial Liability Valuation
- Per policy seriadam
- Stochastic

Trade Positioning System
- Live market data
- Asset management interface

Financial Reporting System
- Financial control
- Profit measurement analysis and projections

IT Interface
In force Data
Asset & Market Data
Goal of Hedging
Replicate Embedded Option So That:

\[ \text{End of Period Guarantee Value} \]
\[ \text{Changes due to market movements} \]
\[ \text{Guarantee Premiums} \]
\[ \text{Claims} \]
\[ \text{Interest} \]
\[ \text{Beginning of Period Guarantee Value} \]

\[ \text{EOP Guarantee Value} \]
\[ \text{EOP Hedge Asset Value} \]

\[ \text{Net Gain (Loss)} \]

Balance Sheet Volatility
Allowing for Demographic & Path Dependencies

Hedge Report
Income Statements & Projections

<table>
<thead>
<tr>
<th>Income Statement Projection</th>
<th>Projection Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Income</td>
<td></td>
<td>$12,274.0</td>
<td>$14,331.1</td>
<td>$96,028.6</td>
<td>$124,441.3</td>
<td>$7,796.7</td>
</tr>
<tr>
<td>Premium Income</td>
<td></td>
<td>$68.4</td>
<td>$1,724.0</td>
<td>$3,463.3</td>
<td>$2,991.5</td>
<td>$2,609.6</td>
</tr>
<tr>
<td>Investment Income</td>
<td></td>
<td>$11,695.6</td>
<td>$45,227.0</td>
<td>$96,667.5</td>
<td>$121,778.0</td>
<td>$5,188.1</td>
</tr>
<tr>
<td>Fixed Income Portfolio</td>
<td></td>
<td>$244.0</td>
<td>$1,470.2</td>
<td>$4,728.2</td>
<td>$10,155.3</td>
<td>$13,586.1</td>
</tr>
<tr>
<td>Futures</td>
<td></td>
<td>$1,964.6</td>
<td>$4,064.0</td>
<td>$1,422.0</td>
<td>$1,200.0</td>
<td>$730.3</td>
</tr>
<tr>
<td>Options &amp; Swaps</td>
<td></td>
<td>$6.075.0</td>
<td>$75,433.6</td>
<td>$103,457.1</td>
<td>$135,164.7</td>
<td>$16,712.3</td>
</tr>
<tr>
<td>Total Expenses</td>
<td></td>
<td>$9,268.8</td>
<td>$39,441.3</td>
<td>$91,891.3</td>
<td>$125,880.0</td>
<td>$11,270.7</td>
</tr>
<tr>
<td>Increase in Fair Value Liability</td>
<td></td>
<td>$3,738.1</td>
<td>$16,291.6</td>
<td>$19,087.1</td>
<td>$125,877.3</td>
<td>$11,270.7</td>
</tr>
<tr>
<td>Interest on Debt</td>
<td></td>
<td>$6.6</td>
<td>$10.7</td>
<td>$12.8</td>
<td>$2.7</td>
<td>$2.7</td>
</tr>
<tr>
<td>Pre-Tax Income</td>
<td></td>
<td>$2,607.2</td>
<td>$7,131.8</td>
<td>$6,319.7</td>
<td>$(1,289.7)</td>
<td>$(3,474.0)</td>
</tr>
<tr>
<td>Equity Market Return</td>
<td></td>
<td>$25%</td>
<td>$25%</td>
<td>$25%</td>
<td>$25%</td>
<td>$15%</td>
</tr>
<tr>
<td>10 Year Interest Rate</td>
<td></td>
<td>$3%</td>
<td>$9%</td>
<td>$6%</td>
<td>$4%</td>
<td>$3%</td>
</tr>
</tbody>
</table>
Hedge Reporting
P&L - Unhedged vs. Hedged

Quarterly P&L Volatility Unhedged
90th Percentile
75th Percentile
10th Percentile
25th Percentile
Median

Quarterly P&L Volatility Hedged
90th Percentile
75th Percentile
10th Percentile
25th Percentile
Median

Embedded Value Reporting
### Replication

<table>
<thead>
<tr>
<th>Reddington Immunisation</th>
<th>Black Scholes Dynamic Hedging</th>
</tr>
</thead>
<tbody>
<tr>
<td>Immunises</td>
<td>Immunises</td>
</tr>
<tr>
<td>Small immediate changes</td>
<td>Small Immediate Changes</td>
</tr>
<tr>
<td>Frequent Rebalancing</td>
<td>Frequent Rebalancing</td>
</tr>
<tr>
<td>Duration Matching</td>
<td>Delta (rate of change of value</td>
</tr>
<tr>
<td></td>
<td>with respect to underlying)</td>
</tr>
<tr>
<td>Convexity Matching</td>
<td>Gamma (second derivative)</td>
</tr>
<tr>
<td>(second derivative)</td>
<td>ADD: Vega, Rho etc</td>
</tr>
</tbody>
</table>

### Contacts

For further information, please contact

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