Optimal Investment and Ruin Probability for Insurers

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ASTIN-AFIR Conference, september 2005

joint work with

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The market is composed by a bank account $S^0$ and a risky asset $S_t$, whose dynamics satisfy

\begin{align*}
S_t^0 &= S_0^0 e^{\eta t}, \quad S_0^0 = 1, \\
dS_t &= S_t (adt + \sigma dB_t), \quad S_0 = x,
\end{align*}

where $\eta \geq 0$, $a$ and $\sigma > 0$ are constants.

The risk process is based in the classical Lundberg model, using a compound Poisson process for the claims. Given the initial surplus $z$ and the constant premium rate $c$, the risk process is defined as

\begin{equation}
R_t = z + ct - \sum_{i=1}^{N_t} Y_i,
\end{equation}

where $Y_i$ represents the claim amounts.
Several authors have been studied this problem in different settings, BETWEEN OTHERS

Paulsen and Gjessing [15], Paulsen[16], Kalashnikov and Norberg [13], Frovola, Pergamischhikov, Kabanov [7]. In all these cases even if the claim size has exponential moments the ruin probability decrease only with some negative powers of the initial reserve.

Hipp and Plumb [11] minimize the the ruin probability. If the claims are exponential the ruin probability decreases exponentially.

Gaier, Grandits, Schachermayer [9] in the exponential case claims investigate whether are constants $\hat{r}$ and $c$ such that the ruin probability $\Psi(x)$ satisfies

$$\Psi(x) \leq Ce^{-\hat{r}x}$$

Browne [2] investigate the problem where the risk process is also Brownian (not a compound Poisson) and obtain a minimal bound for the ruin by an exponential function.
(i) Finite horizon problem $T > 0$ fixed.

(ii) At each time $t \in [0, T]$, the insurer divides his wealth $X_t$ between the risky and the riskless assets.

(iii) If a claim is received at time $t$, it is paid immediately.

(iv) Let $\pi_t$ be the amount of wealth invested in the risky asset at time $t$.

(v) $X_t - \pi_t$ is invested in the bank account.
If at time \( s < T \) the surplus of the company is \( x \), the wealth process satisfies the dynamics

\[
X_{t}^{s,x,\pi} = x + c(t - s) - \sum_{j=N_{s}+1}^{N_{t}} Y_{j} + \int_{s}^{t} (a - \eta)\pi_{r} dr \\
+ \int_{s}^{t} \eta X_{r}^{s,x,\pi} dr + \int_{s}^{t} \sigma \pi_{r} dB_{r}, \tag{0.1}
\]

with the convention that \( \sum_{j=1}^{0} = 0 \). When \( s = 0 \) we write for simplicity \( X_{t}^{\pi} \).

**Definition 0.1.** We say that \( \pi = \{\pi_{t}\} \) is an admissible strategy if it is a \( \mathcal{F}_{t} \)-progressively measurable process such that

\[
P[|\pi_{t}| \leq A, \ 0 \leq t \leq T] = 1,
\]

where the constant \( A \) may depend of the strategy, and the equation (0.3) has a unique solution. We denote the set of admissible strategies as \( \mathcal{A} \).

A utility function \( U : \mathbb{R} \rightarrow \mathbb{R} \in C^{2} \) strictly increasing and strictly concave.
THE OPTIMIZATION PROBLEM:

Maximize the expected utility of terminal wealth at time $T$, i.e. we are interested in the following value function

$$W(s, x) = \sup_{\pi \in A} \mathbb{E}[U(X_T^{s,x,\pi})]. \quad (0.2)$$

We say that an admissible strategy $\pi^*$ is optimal if

$$W(s, x) = \mathbb{E}[U(X_T^{s,x,\pi^*})]$$

$$X_t^{s,x,\pi} = x + c(t - s) - \sum_{j=N_s+1}^{N_t} Y_j + \int_s^t (a - \eta)\pi_r dr$$
$$+ \int_s^t \eta X_r^{s,x,\pi} dr + \int_s^t \sigma \pi_r dB_r, \quad (0.3)$$
We prove:

1. A Verification Theorem associated to HJB-equation.

2. If $U(x) = -e^{-\gamma x}$

   (i) We obtain an explicit solution.

   (ii) Estimate Ruin Probability (Bounds of exponential type).

   (iii) When the claims are exponential we compare the results with that of Gaier, Grandits, Schachermayer [9].

   (iv) Numerical Examples.
1 Verification Theorem

The Hamilton-Jacobi-Bellman (HJB) equation associated with the optimal stochastic control problem is given by

\[
0 = \frac{\partial V}{\partial t}(t, x) + \max_{\pi \in \mathbb{R}} \left\{ \frac{\sigma^2}{2} \pi^2 \frac{\partial^2 V}{\partial x^2}(t, x)x + (\pi(a - \eta) + \eta x) \frac{\partial V}{\partial x}(t, x) \right\} \\
+ c \frac{\partial V}{\partial x}(t, x) + \lambda \int_{\mathbb{R}} [V(t, x - y) - V(t, x)] \nu(dy),
\]

with terminal condition \( V(T, x) = U(x). \)

**Theorem 1.1.** Assume that there exists a classical solution \( V(t, x) \in C^{1,2}([0, T] \times \mathbb{R}) \) to the HJB equation (1.4) with boundary conditions \( V(T, x) = U(x). \) Assume also that for each \( \pi \in \mathcal{A} \)

\[
\int_0^T \int_{\mathbb{R}} \mathbb{E}|V(s, X_s^\pi - y) - V(s, X_s^\pi)| \nu(dy) ds < \infty,
\]

\[
\int_0^T \mathbb{E}[\pi_s^{-} \frac{\partial V}{\partial x}(s, X_s^\pi)]^2 ds < \infty.
\]

Then, for each

\( s \in [0, T], \ x \in \mathbb{R}, \)

\( V(s, x) \geq W(s, x). \)
If, in addition, there exists a measurable function $\pi^*: [0, T] \times \mathbb{R} \to \mathbb{R}$ such that

$$\pi^*(t, x) \in \arg \max_{\pi \in \mathbb{R}} \left\{ \frac{\sigma^2}{2} \pi^2 \frac{\partial^2 V}{\partial x^2}(t, x) + (\pi(a - \eta) + \eta x) \frac{\partial V}{\partial x}(t, x) \right\},$$

then $\pi^*$ defines an optimal investment strategy in feedback form if (0.3) admits a unique solution $X^\pi_{T}$ and

$$V(s, x) = W(s, x) = \mathbb{E}U[X^s_{T}, \pi^*].$$
2 Explicit solutions for exponential utility function

The utility function is of exponential type, i.e.

\[ U(x) = -e^{-\gamma x}. \]

**Theorem 2.1.** Assume that

\[ \int_{\mathbb{R}} \exp\{2\gamma ye^{\eta T}\} \nu(dy) < \infty. \]

Then, the value function defined in (0.2) has the form

\[
W(t, x) = -\exp \left\{ -\frac{1}{2} \frac{(a - \eta)^2}{\sigma^2} (T - t) + \frac{c\gamma}{\eta} [1 - e^{\eta(T-t)}] \right. \\
+ \left. \lambda \int_t^T \beta_s ds \right\} \cdot \exp \left\{ -\gamma xe^{\eta(T-t)} \right\}, \tag{2.7}
\]

and

\[ \pi^*(t, x) = \frac{a - \eta}{\gamma \sigma^2} e^{-\eta(T-t)} \]

is an optimal strategy.

In particular, when \( \eta = 0 \) we have that

\[
W(t, x) = -\exp \left\{ -\frac{1}{2} \frac{a}{\sigma^2} (T - t) + c\gamma (T - t) + \lambda \beta(T - t) \right\} e^{-\gamma x} \tag{2.8}
\]

and

\[ \pi^*(t, x) = \frac{a}{\gamma \sigma^2}. \]
3 Ruin Probability

The wealth process associated with the optimal investment strategy \( \pi^* \) is given by

\[
X_t^* = z + ct - \sum_{i=1}^{N_t} Y_j + \int_0^t \frac{(a - \eta)^2}{\gamma \sigma^2} e^{-\eta(T-r)} dr \\
+ \int_0^t \eta X_r^* dr + \int_0^t \frac{(a - \eta)}{\gamma \sigma} e^{-\eta(T-r)} dB_r, \quad \text{for } \eta > 0 \tag{3.9}
\]

and

\[
X_t^* = z + ct - \sum_{i=1}^{N_t} Y_i + \int_0^t \frac{a^2}{\gamma \sigma^2} dr + \int_0^t \frac{a}{\gamma \sigma} dB_r, \quad \text{for } \eta = 0 \tag{3.10}
\]
**Theorem 3.1.** Let us denote \( \theta = E[Y_1] \) and assume that

(a) The law of the random variables \( Y_i, i \geq 1 \) admits a (finite) Laplace transform \( L(r) \) for \( 0 < r < K \leq \infty \),

(b) If \( K < \infty \), then \( \lim_{r \to K} L(r) = \infty \).

(c) The following safety loading conditions are satisfied

\[
[e^{-\eta T} (c + \frac{(a - \eta)^2}{\gamma \sigma^2})] - \lambda \theta > 0, \quad \text{if } \eta > 0,
\]

and

\[
c + \frac{a^2}{\gamma \sigma^2} - \lambda \theta > 0, \quad \text{if } \eta > 0.
\]

Then, the ruin probability satisfies

\[
P[ \sup_{0 \leq s \leq T} -X^*_s \geq 0] \leq e^{-\delta^* z},
\]
Then, the ruin probability satisfies

\[ P\left( \sup_{0 \leq s \leq T} -X_s^* \geq 0 \right) \leq e^{-\delta^* z}, \]

where (i) If \( \eta > 0 \), then, for each \( \gamma > 0 \), \( \delta^* \) is the positive root of the equation:

\[ h(\delta, \gamma) = -\delta\{e^{-\eta T}(c+\frac{(a-\eta)^2}{\gamma \sigma^2})\} + \frac{\delta^2 (a-\eta)^2}{2 \gamma^2 \sigma^2} e^{-2\eta T} + \lambda(L(\delta)-1) = 0. \] (3.11)

(ii) If \( \eta = 0 \) then, for each \( \gamma > 0 \), \( \delta^* \) is the positive root of the equation:

\[ h^0(\delta, \gamma) = -\delta\{c + \frac{a^2}{\gamma \sigma^2}\} + \frac{\delta^2 a^2}{2 \gamma^2 \sigma^2} + \lambda(L(\delta) - 1) = 0. \] (3.12)

In addition, let \( \delta_1^1 \) be the root of the classic Cramér–Lundberg equation

\[ h^1(\delta) = -\delta c + \lambda(L(\delta) - 1) = 0. \]

If \( \eta = 0 \) and \( \frac{\delta_1^1}{2} < \gamma \), then

\[ \delta_1^1 < \delta^*. \]
In the proof we shall use a simplified version of Lemma 3.1 in [3], which we state now:

**Lemma 3.1.** Let

\[ L_t = z + \int_0^t b_s ds + \int_0^t d_s dB_s - \sum_{i=1}^{N_t} Y_i, \]

where everything is as stated in this paper, \((b_s)_{s \geq 0}\) is an adapted integrable process and \((d_s)_{s \geq 0}\) is predictable with \(\mathbb{E}[\int_0^t d_s^2 ds] < \infty\).

Assume

(i) The law of the random variables \((Y_i)_{i \geq 1}\) has finite Laplace transform \(L(r)\) for \(0 < r < K \leq \infty\).

(ii) There exist \(\delta \in (0, K)\) and a constant \(h_T(\delta) \geq 0\) such that for all \(s \in [0, T]\),

\[ \delta \int_0^s b_u du + \frac{\delta^2}{2} \int_0^s d_u^2 du + \lambda s (L(\delta) - 1) \leq h_T(\delta). \]

Then, for \(m \geq z\)

\[ \mathbb{P}[\sup_{0 < s \leq T} L_s > m] \leq \exp\{\delta(z - m) + h_T(\delta)\}. \]

The proof of this Lemma is based on maximal inequalities for martingales.
For estimate

\[ P[ \sup_{0 \leq s \leq t} -X_s^* \geq 0] \]

We can not apply directly Lemma 3.1 to \(-X^*\), observe that

\[ P[ \sup_{0 \leq s \leq t} -X_s^* \geq 0] = P[ \sup_{0 \leq s \leq t} -Z_s \geq 0]. \]

where \( Z_t = X_t^* e^{-\eta t} \)

\[ Z_t = z + \int_0^t e^{-\eta r} cdr - \sum_{j=1}^{N_t} e^{-\eta j} Y_j + \int_0^t \frac{(a - \eta)^2}{\gamma \sigma^2} e^{-\eta T} dr \]

\[ + \int_0^t \frac{a - \eta}{\gamma \sigma} e^{-\eta T} dB_r. \]

Let \(-Z_1^1\) be as follows

\[ -Z_t^1 = -z - cte^{-\eta T} + \sum_{i=1}^{N_t} Y_i - t(a - \eta)^2 \gamma \sigma^2 e^{-\eta T} - \int_0^t \frac{a - \eta}{\gamma \sigma} e^{-\eta T} dB_r. \]

Then it is clear that

\[ -Z_s^1 \geq -Z_s = -X_s^* e^{-\eta s}. \]

and

\[ P[ \sup_{0 \leq s \leq t} -Z_s \geq 0] = P[ \sup_{0 \leq s \leq t} -Z_s^1 \geq 0] \]

We apply the Lemma to \(-Z_s^1\) with \( h_T(\delta) = Th(\delta) \). The existence of the positive root is guaranteed by the safety loading condition.
Proposition 3.1. We assume that the random variables $Y_i, \ i \geq 1$ are exponential with mean $\theta$ and

$$0 < \gamma < \frac{e^{-\eta T}}{\theta}. \quad (3.13)$$

Then

$$W(t, x) = -\exp \left\{ -\frac{1}{2} \frac{a - \eta}{\sigma^2} (T - t) + \frac{c\gamma}{\eta} [1 - e^{\eta(T-t)}] 
- \frac{\lambda}{\eta} \log \left( \frac{1 - \gamma \theta}{1 - \gamma \theta e^{\eta(T-t)}} \right) \right\} \cdot \exp \left\{ -\gamma x e^{\eta(T-t)} \right\}. $$

In particular, if $\eta = 0$,

$$0 < \gamma < \frac{1}{\theta},$$

and

$$W(t, x) = -\exp \left\{ -\frac{1}{2} \frac{a}{\sigma^2} (T - t) + c\gamma(T - t) - \lambda \frac{\gamma \theta}{1 - \gamma \theta} (T - t) \right\} e^{-\gamma x}. $$
Compare with Gaier, Grandits, Schahermayer [9]:

For each $\gamma$ we obtain a positive root $\delta_\gamma$ of $h$, using the implicit theorem, it can be shown that $\delta_\gamma$ is maximum when $\delta_\gamma = \gamma$. Gaier, Grandits and Schachermayer, see [9], obtain the strategy $\pi_t$ that guarantees that the ruin probability is optimal: $\pi_t = \frac{a}{r\sigma}$, where $\hat{r}$ is the solution of the following equation

$$\lambda(L[r] - 1) = \frac{a^2}{2\sigma^2} + cr.$$  \hspace{1cm} (3.14)

It can be easily shown that $\delta_{\hat{r}} = \hat{r}$. So if we chose as $\gamma = \hat{r}$ we get the strategy that is optimal for the exponential utility function and also that has the less ruin probability.
In the exponential case, for $\eta = 0$, $h(\delta, \gamma)$ becomes

$$h(\delta, \gamma) = \frac{a^2\theta}{2\gamma^2\sigma^2} \delta^2 - ((c + \frac{a^2}{\gamma\sigma^2})\theta + \frac{a^2}{2\gamma^2\sigma^2})\delta + (c + \frac{a^2}{\gamma^2\sigma^2} - \theta\lambda). \quad (3.15)$$

For each $\gamma \in (0, 1/\theta)$ we obtain a positive root $\delta_\gamma$ of $h$ of the form

$$\delta_\gamma = \left(\frac{1}{2\theta} + \frac{c + Ka}{K^2\sigma^2}\right) + \sqrt{\left(\frac{c + Ka}{K^2\sigma^2}\right)^2 + \frac{1}{4\theta^2} - \frac{c + Ka - \lambda\theta}{\theta K^2\sigma^2}}.$$

In the exponential case equation (3.13) becomes

$$f(r) = c\theta r^2 + \left(\frac{a^2\theta}{2\sigma^2} + \lambda\theta - c\right)r - \frac{a^2}{2\sigma^2} \quad (3.16)$$

whose solution satisfies $\delta_\hat{r} = \hat{r}$. 

In order to illustrate the behavior of the ruin probability for infinite horizon when the optimal strategy of investment \( \pi_t = \frac{a}{\gamma \sigma} \) is applied, we present some numerical results for the exponential case, with data used by Hipp and Plum, see [11], for different values of \( \gamma \in (0, 1) \). The parameters have the following values: \( a = \sigma = \theta = \lambda = 1 \), \( c = 2 \), and \( \eta = 0 \).

Graph 1

Graph 1 shows how the root \( \delta(\gamma) \) of \( h(\delta) \) varies for different values of \( \gamma \). For our data the root of (3.16) is \( \hat{\delta} = 0.640388 \) and the Lundberg parameter for the classical case is 0.5. As it was expected the maximum value of \( \delta \) is obtained at 0.640388 and for \( \gamma \in [.25, .9] \) the root is larger than 0.5.

Graph 2 shows how \( K \) decreases as \( \gamma \) increases. This has the advantage that the ruin probability is almost the same as in the optimal case without needing a large sum of money to invest in the risk asset.

Let

\[
S_t = \sum_{i=1}^{N_t} Y_i - ct - \int_0^t \frac{a^2}{\gamma \sigma^2} dr - \int_0^t \frac{a}{\gamma \sigma} dB_r,
\]

(3.17) denote the surplus; observe that \( S_t = z - X_t \). Let \( \tau(z) = \inf_{0 \leq t < \infty} \{ t > \)
\( 0 \mid S_\tau > z \), we are interested on estimating

\[
P[\tau(z) < \infty] = E(1_{\tau(z) < \infty}).
\]

We use a Monte-Carlo method with importance sampling to estimate the ruin probability. Importance sampling is applied to overcome several difficulties:

1. Given that the horizon is infinite, a stopping time \( T \) must be defined for the simulation which introduce an error difficult to estimate.
2. When the probability is small, less than $10^{-3}$, which is the case for our data when $z > 7$, we are simulating a rare event. In order to do it well we have to generate an impractical number of paths.

3. When a crude Monte-Carlo method is used the relative error increases as $z$ becomes large.

These problems can be handle if we change the probability measure to one that increases the probability of occurrence of $\{\tau(z) < \infty\}$. Asmussen [1] propose to use an exponential change of measure. Let $P^*$ be the equivalent probability of $P$ given by the Radon-Nykodin derivative

$$\frac{dP^*}{dP} = e^{\delta S_{\tau(z)} - \tau(z) h(\delta)},$$

where $h(\delta)$ is given by 3.15. If we chose as $\delta$ the root $\delta^*$ of $h$ we have that the calculation of the ruin probability reduces to

$$E(1 \mathbb{I}[\tau(z) < \infty]) = E^*(e^{-\delta^* S_{\tau(z)} \mathbb{I}[\tau(z) < \infty]}).$$

As $P^*(\tau(z) < \infty) = 1$, we don’t have to worry about the stopping time. We also obtain a considerable reduction of the variance which implies a lesser number of paths for Monte-Carlo. When $\delta = \delta^*$ the estimation is optimal, in an asymptotic sense, for variance reduction; the variance is bounded by $e^{-2\delta^* z}$ which tends to zero when $z$ goes to infinity.

Graph 3 compares the probability of survival, equal to $1$ minus the ruin probability, for values of $z \in [0, 6]$ for $\gamma = .9$, $\gamma = 0.640388$ and when there is no investment. As it can be seen the ruin probability is almost the same for the first two cases even when we need to invest for $\gamma = .9$ a smaller amount of money.
References


