BSDE with enlarged filtration

*Option hedging of an insider trader*

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Problem Setting

Market Model

BSDE and enlarged filtration

Results

Influent investor and FBSDE
Insider Trading = Additional Information on the market

Most often treated point of view: wealth optimization with asymmetrical information
(ex: Grorud, Pontier, Amendinger, Becherer, Schweizer, Imkeller, Föllmer among others)

Different point of view: hedging problem

In the present model, prices are driven by both a Brownian motion, and Jump processes
(Poisson point processes)

Two different cases studied: small investors (→BSDE), and also large investors (influent insider trader) (→FBSDE)
**Basic Example**:

- \( L = S_T \) the insider knows final stock price \( S_T \).
- He wants to hedge a digital option \( 1_{S_T \leq K} \).
- He has 2 possible investments: invest on risky asset if \( S_T \leq K \) and do nothing otherwise.
- He has an obvious different strategy from the non insider trader, and even an arbitrage opportunity.

**Natural questions arise**:

- Is the hedging strategy identical to the non informed trader?
- Is it unique? Are there more hedging strategies?
- Market completeness/incompleteness? Arbitrage opportunities?
Financial Problem

- Option Hedging, represented by a payoff $\xi$ to reach at maturity $T$

- Transcription: portfolio duplication, look for initial wealth $X_0$ and the portfolio $\pi$ such that final wealth $X_T = \xi$.

- One agent has an information $L$ at time 0 concerning time $T' > T$.

- Will he have different investment from the uninformed agent?

- How does the market differ from a market with symmetrical information?
Main Tools of the Model

- Introducing an insider in a well-known market model
- Comparing insider and non informed strategies
- To model financial strategies of the agents: BSDE
- To model the additional information of the insider: Enlargement of Filtration
Market model

• Prices driven by both $W$ Brownian motion and $\mu$ Poisson measure,

\[(\mathcal{F}_t)_{t \in [0,T]} \text{ natural filtration of } (W, \tilde{N}),\]

• $k$ risky assets, 1 riskless asset,

• no Arbitrage opportunities (AOA)

An insider in the Market

• **Strong initial information**: insider trader has at time 0 the information $L$, unknown from the common agent.

• $L \in \mathcal{F}_{T'}$, with $T' > T$: it will be public at time $T'$.

• There are 2 different spaces: the non insider space, and the insider space, with $L$ added.

• New **enlarged filtration**: the smallest right-continuous filtration that contains initial filtration and information $L$:

\[\mathcal{Y}_t = \bigcap_{s > t} (\mathcal{F}_t \vee \sigma(L))\]
Initially enlarged filtration: Usual hypotheses

- Adding $L$ to the initial filtration:
  \[ \mathcal{Y}_t = \bigcap_{s > t} (\mathcal{F}_t \vee \sigma(L)) \]

  - **Hypothesis** ($H_3$): There exists a probability $Q$ equivalent to $P$ under which $\mathcal{F}_t$ and $\sigma(L)$ are independent, $\forall t < T'$.

- Fundamental Properties:
  - Under new probability $Q$, $W_t$ is a $(\mathcal{Y}, Q)$-Brownian motion, and $\tilde{N}_t$ a $(\mathcal{Y}, Q)$-compensated Poisson process.
  - Martingale Representation Theorem under $(\mathcal{Y}, Q)$. 

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Translating the problem to a BSDE (1)

- Hedging problem, payoff \( \xi \) to be reached at maturity \( T < T' \).

- Wealth equation, standard self-financing hypothesis:

\[
dX_t = X_tr_t dt + (\pi_t, b_t - r_t 1) dt + (\pi_t, \sigma_t dW_t) + \int_E (\pi_t -, \phi(t, e)) \mu(dt, de)
\]

- And integrating from \( t \) to \( T \), it follows:

\[
X_t = X_T - \int_t^T \left[ (X_s r_s - c_s) + (\pi_s, b_s - r_s 1) \right] ds - \int_t^T (\sigma^*_s \pi_s, dW_s) - f(s, X_s, Z_s, U_s) ds - \int_t^T \int_E (\pi_s - , \phi(s, e)) \tilde{\mu}(ds, de)
\]

- Solving the hedging problem means finding \( (X, Z, U) \) solution of the BSDE:

\[
X_t = \xi + \int_t^T f(s, X_s, Z_s, U_s) ds - \int_t^T (Z_s, dW_s) - \int_t^T \int_E U_s(e) \tilde{\mu}(ds, de)
\]
Results under $H_3$

\[ X_t = \xi + \int_t^T f(s, X_s, Z_s, U_s) \, ds - \int_t^T (Z_s, dW_s) - \int_t^T \int_E U_s(e) \tilde{\mu}(ds, de) \]

- From Barles, Buckdahn and Pardoux (1997) and Tang and Li (1994), if $f$ is globally Lipschitz in $x$, $z$ and $u$, then there exists a unique triplet $(X, Z, U)$ solution of the BSDE.

- The existence and uniqueness theorem can be adapted in the enlarged space.

**Existence and Uniqueness Theorem** Our BSDE in the enlarged space has a unique solution $(X', Z', U')$.

- Thanks to Jacod and Shiryaev, we prove a martingale representation theorem under the enlarged filtration (Using independence of $\mathcal{F}$ and $\sigma(L)$ under $Q$ to state a martingale representation property for $(W, N)$ IIP on $(\mathcal{Y}, Q)$).
- Constructing a strict contraction to obtain a unique solution.
Viability and completeness of the insider market: Brownian case

- If $\sigma$ invertible, direct consequence of the existence and uniqueness result: the insider trader has a unique admissible strategy.

- Comparison of the strategies: the hedging strategy is the same for both agents.

- Complete non insider market
  - Insider market is viable: information $L$ does not create any arbitrage opportunities.
  - Insider market may have several risk-neutral probabilities (as in Grorud 1998), but all prices computed under different risk-neutral probabilities are the same.
FBSDE for a large investor

**Limit**: In the previous model, the insider is a small investor, whose investment strategy does not influence asset prices.

Not always realistic. → Large/Influent investor hypothesis.

→ **FBSDE** to solve in the enlarged space.

\[
\begin{aligned}
P_t &= P_0 + \int_0^t b(s, P_s, X_s, Z_s) ds + \int_0^t < \sigma(s, P_s, X_s, Z_s), dW_s > \\
X_t &= X_T - \int_t^T f(s, P_s, X_s, Z_s) ds - \int_t^T < Z_s, dW_s > 
\end{aligned}
\]  

(1)

**Influence hypothesis**: the informed investor may influence asset prices.

- It is a **large** investor: his wealth $X$ may influence prices
- and he is **influent**: his portfolio $\pi$ influences prices
Existence and Uniqueness of solution

- Under Lipschitz, linear growth and integrability conditions on $b, \sigma, f$

- 3 cases where Pardoux and Tang obtained results:
  - *Weak influence*: $b$ and $\sigma$ weakly depend on $X$ and $Z$
  - The agent wants to hedge a finite value a.s. : $g$ constant
  - The portfolio does not influence volatility of prices : $\sigma$ indépendent of $Z$.

- Our Result under $(H_3)$, in complete market
  - Using our theorem on BSDE under enlarged filtration, we state that the enlarged FBSDE has a unique solution, under the same hypotheses as Pardoux and Tang.
  - The influent agent, in one of the 3 influence cases, has a unique admissible strategy.
  - Complete insider market, as previously.
Incomplete market for the non informed agent

Consider a non informed agent investing on this market. For a given contingent claim, his wealth and portfolio will satisfy a BSDE on $(\Omega, \mathcal{F}^P, P)$.

There is no necessarily exact hedging solution. And if there is a solution, it will be a priori different from the insider one.

The non informed agent will invest on a market with an incomplete information: his information is represented by the filtration generated by asset prices. New filtration to deal with.

Hedging under Incomplete information (Föllmer and Schweizer 91)
Conclusion

- Incomplete market from a non insider point of view. Less information. Incomplete information. Different strategy $F^P$-adapted.
  - Study of the strategy of the non informed agent.
  - Quantifying the "loss" of the non informed agent, due to the lack of information.
- Model with jumps (incomplete markets).
References


[8] Grorud 2000, Asymmetric information in a financial market with jumps

[9] Grorud, Pontier, 2001 Asymmetrical information and incomplete markets


