

**Valuation and hedging of
life insurance liabilities
with
systematic mortality risk**

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(Joint work with Mikkel Dahl,
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Methods taken from:

- ❑ Mortality modeling (Dahl, 2004)
- ❑ Affine interest rate models (e.g. Björk)
- ❑ Incomplete financial markets (Schweizer, Møller)

Some related work on mortality:

Marocco/Pitacco (1998)

Milevsky/Promislow (2001)

Cairns, Blake and Dowd (2004)

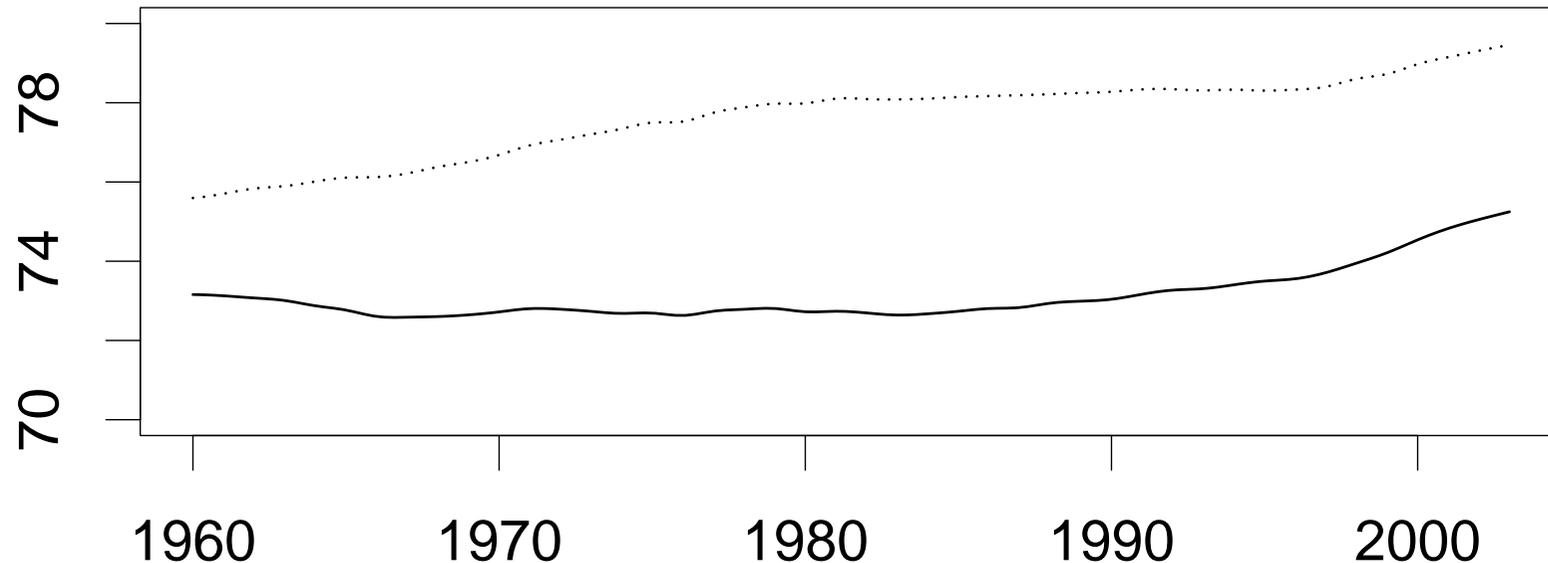
Biffis and Millossovich (2004)

Outline:

- ❑ Danish mortality data
- ❑ A stochastic model for the mortality intensity
- ❑ Insurance payment process and market reserve
- ❑ Risk-minimization and indifference pricing
- ❑ Numerical results

Empirical evidence: Danish mortality data

(See Andreev (2002) + Fledelius and Nielsen (2002))

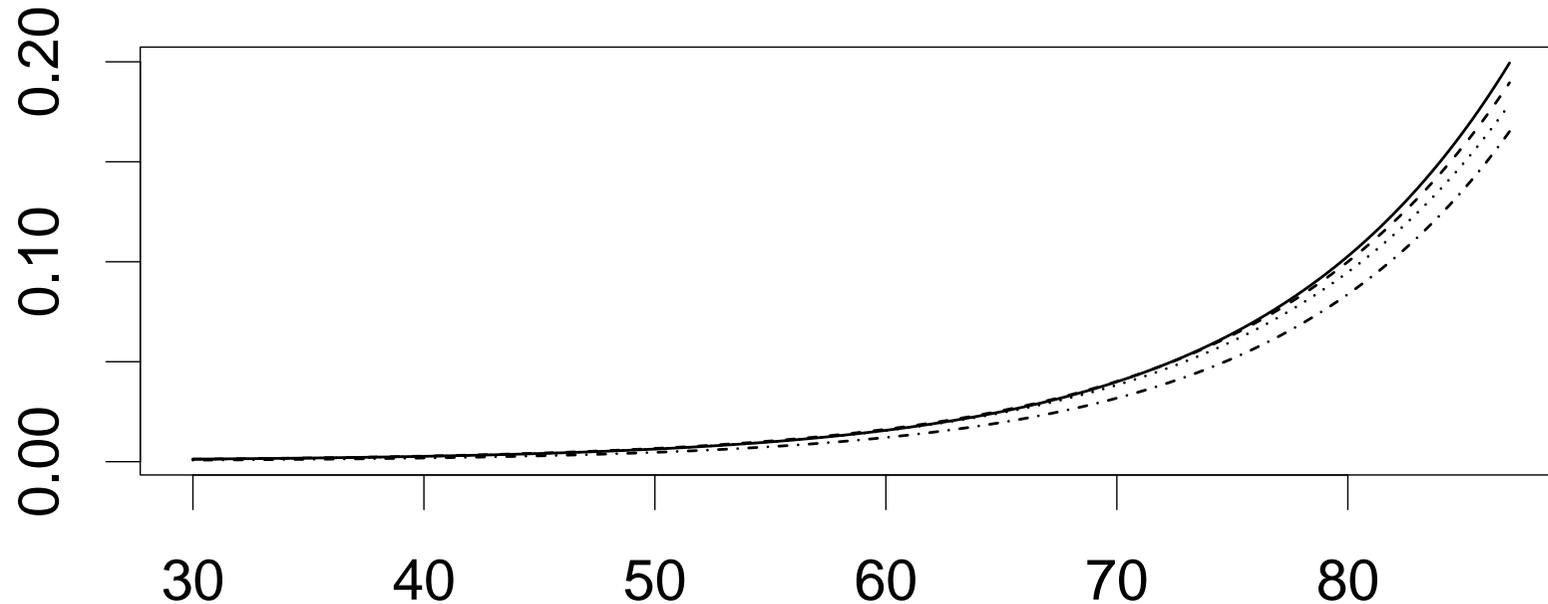


Expected lifetime of 30 year old females (dotted line) and 30 year old males (solid line).

Estimates are based on the last 5 years of data available at calendar time.

Estimated Gompertz-Makeham curves

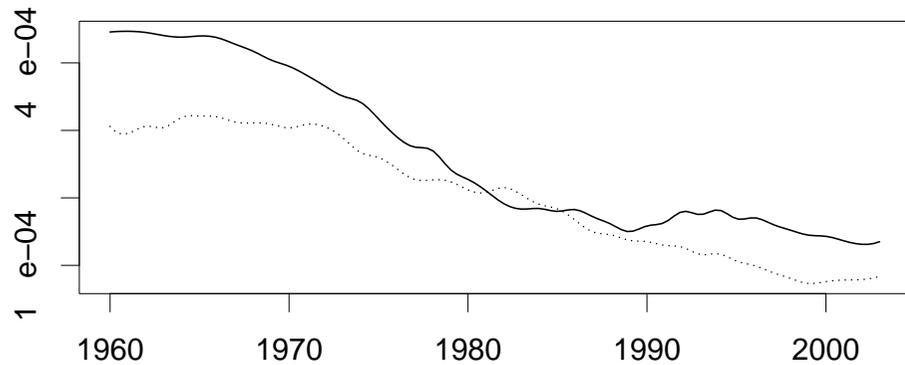
On the form $\tilde{\mu}_{y,x} = \alpha_y + \beta_y c_y^x$



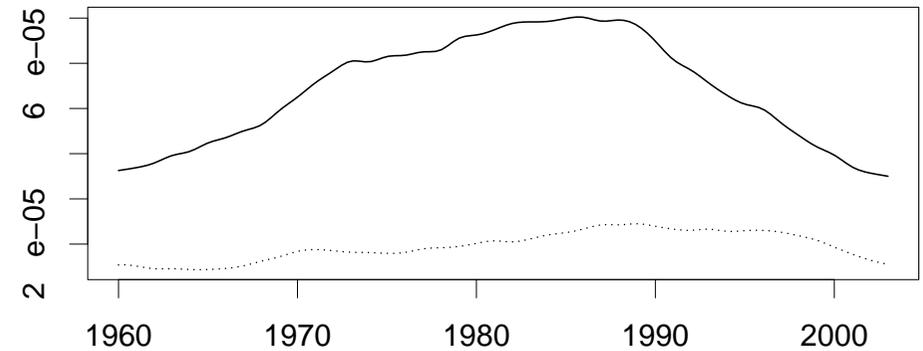
Estimates for males. Similar patterns for females.

(Solid lines are 1970-estimates, dashed lines 1980, dotted lines 1990 and dot-dashed lines 2003)

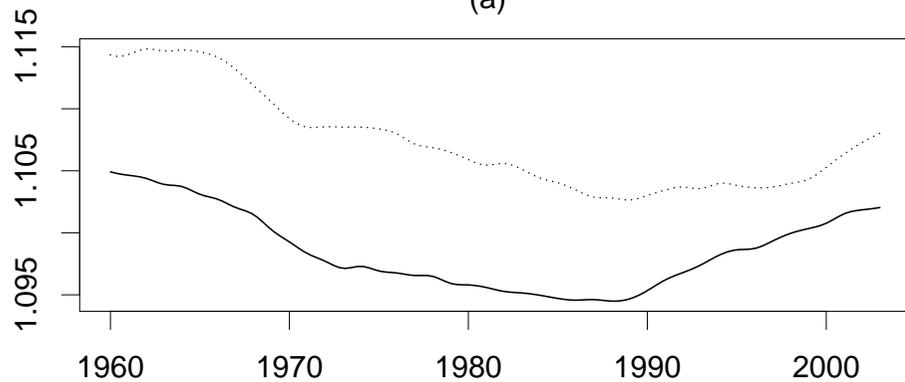
Estimated GM-parameters



(a)



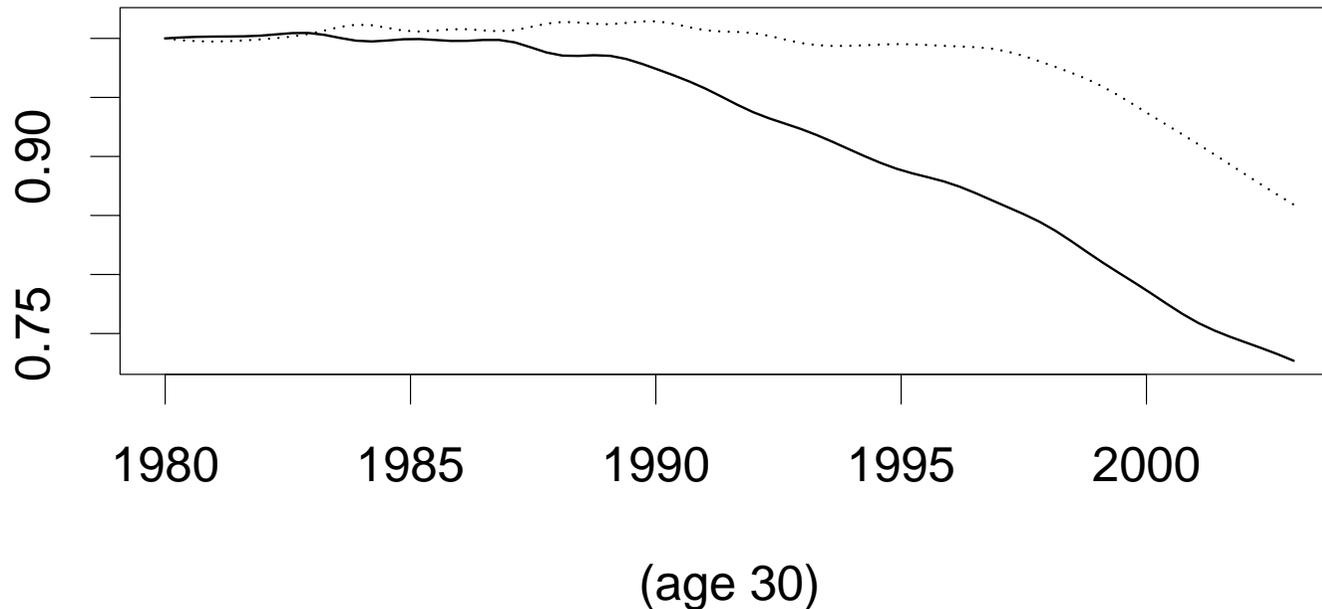
(b)



(c)

Gompertz-Makeham parameters α (figure (a)), β (figure (b)) and c (figure (c)) for females (the dotted lines) and males (solid lines) from year 1960 to 2003. Estimates based on data available at calendar time and 5 years back in time.

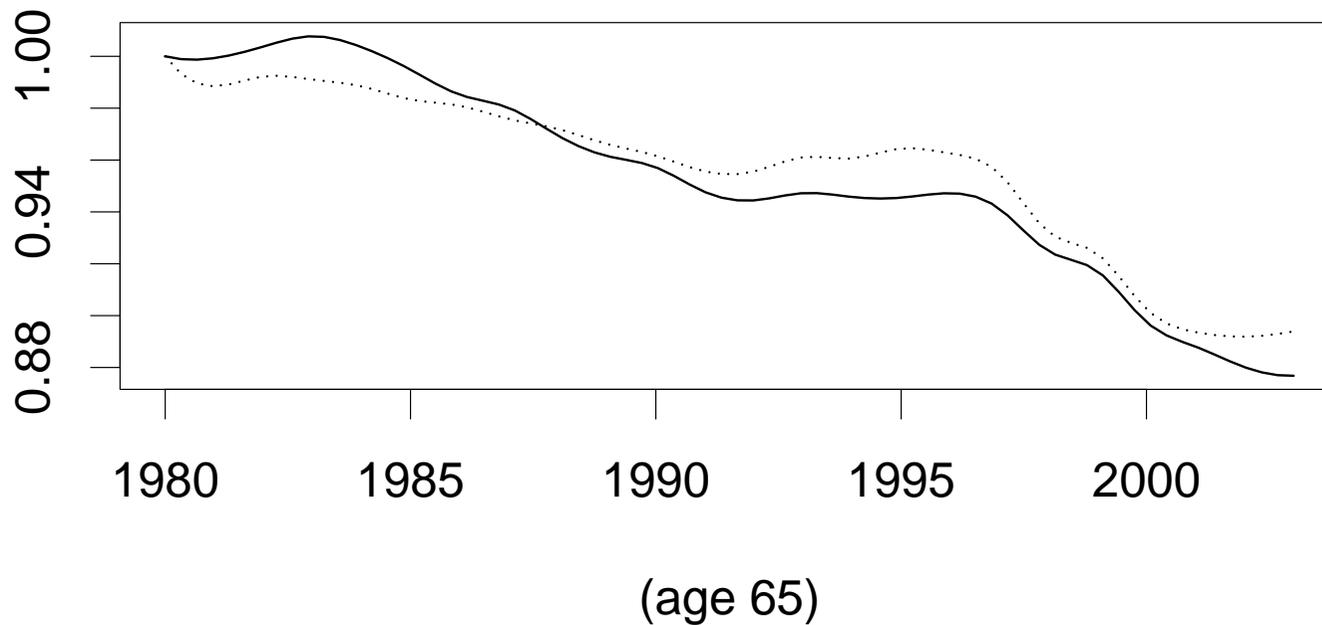
Mortality trends, start age 30



Decrease in the mortality intensity from 1980 to 2003 for males (solid lines) and females (dotted lines) as age increases.

Numbers are normalized with the 1980-mortality intensities and based on the estimated Gompertz-Makeham curves.

Mortality trends, start age 65



Same tendency as age 30

However, more moderate decrease

Modeling the mortality intensity:

Known at time 0:

$\mu^\circ(x + t)$ is mortality intensity “today” at all ages $x + t$

Unknown at time 0:

$\zeta(t, x)$ is relative change in the mortality from 0 to t , age x

Mortality intensity:

$$\mu(x, t) = \mu^\circ(x + t)\zeta(x, t)$$

(In general, a stochastic process)

True survival probability from t to T given information $\mathcal{I}(t)$:

$$\mathcal{S}(x, t, T) = E_P \left[e^{-\int_t^T \mu(x, \tau) d\tau} \middle| \mathcal{I}(t) \right]$$

A specific model:

Time-inhomogeneous CIR model known from finance:

$$d\zeta(x, t) = (\gamma(x, t) - \delta(x, t)\zeta(x, t))dt + \sigma(x, t)\sqrt{\zeta(x, t)}dW^\mu(t)$$

Proposition (Affine mortality structure, Dahl, 2004)

The survival probability $\mathcal{S}(x, t, T)$ is

$$\mathcal{S}(x, t, T) = e^{A^\mu(x, t, T) - B^\mu(x, t, T)\mu(x, t)}$$

where

$$\frac{\partial}{\partial t} B^\mu(x, t, T) = \delta^\mu(x, t)B^\mu(x, t, T) + \frac{1}{2}(\sigma^\mu(x, t))^2 (B^\mu(x, t, T))^2 - 1$$

$$\frac{\partial}{\partial t} A^\mu(x, t, T) = \gamma^\mu(x, t)B^\mu(x, t, T)$$

with $B^\mu(x, T, T) = 0$ and $A^\mu(x, T, T) = 0$

Forward mortality intensity

$$f^\mu(x, t, T) = -\frac{\partial}{\partial T} \log S(x, t, T) = \mu(x, t) \frac{\partial}{\partial T} B^\mu(x, t, T) - \frac{\partial}{\partial T} A^\mu(x, t, T)$$

Survival probability

$$S(x, t, T) = e^{-\int_t^T f^\mu(x, t, u) du} \neq e^{-\int_t^T \mu(x, u) du}$$

Change of measure for mortality and financial market

Equivalent measure Q

Financial market

Standard affine model for short rate:

$$dr(t) = \left(\gamma^{r, \alpha, Q} - \delta^{r, \alpha, Q} r(t) \right) dt + \sqrt{\gamma^{r, \sigma} + \delta^{r, \sigma} r(t)} dW^{r, Q}(t)$$

Portfolio of insured lives

T_1, \dots, T_n are i.i.d. given ζ with

$$P(T_1 > t | \mathcal{I}(T)) = e^{-\int_0^t \mu(x,s) ds}$$

Counting process

$$N(x, t) = \sum_{i=1}^n \mathbf{1}_{(T_i \leq t)},$$

Insurance payment process (Benefits – premiums)

$$\begin{aligned} dA(t) &= (n - N(x, \bar{T})) \Delta A_0(\bar{T}) dI_{\{t \geq \bar{T}\}} \\ &\quad + a_0(t)(n - N(x, t)) dt \\ &\quad + a_1(t) dN(x, t) \end{aligned}$$

(a_i, A_0 deterministic functions)

Market reserves

$$\begin{aligned}\tilde{V}^Q(t) &= E_Q \left[\int_{(t,T]} e^{-\int_t^\tau r(u)du} dA(\tau) \middle| \mathcal{F}(t) \right] \\ &= (n - N(x, t)) V^Q(t, r(t), \mu(x, t))\end{aligned}$$

where

$$\begin{aligned}V^Q(t, r(t), \mu(x, t)) &= \int_t^T P(t, \tau) \mathcal{S}^Q(x, t, \tau) \left(a_0(\tau) + a_1(\tau) f^{\mu, Q}(x, t, \tau) \right) d\tau \\ &\quad + P(t, \bar{T}) \mathcal{S}^Q(x, t, \bar{T}) \Delta A_0(\bar{T})\end{aligned}$$

Other principles applied in the paper

Risk-minimization

Mean-variance indifference pricing

↪ loadings for systematic mortality risk

Mean-variance indifference pricing (Schweizer, Møller)

Decomposition for discounted payments

$$H = \tilde{E}[H] + g^H + N^H$$

$\tilde{E}[H]$: expected value of H under specific mg measure \tilde{P}

$\tilde{E}[H] + g^H$: projection in L^2 of H on “ \mathcal{R} + space of trading gains”

N^H : “orthogonal term”

Fair premiums: $v_i(H) = \tilde{E}[H] + a_i (\text{Var}_P[N^H])^{\beta_i}$

Variance principle: $i = 1, \beta_1 = 1$

Standard deviation principle: $i = 2, \beta_2 = 1/2$

Result:

$$\text{Var}_P[N^H] = n \int_0^T \gamma_1(t) \gamma_2(t) dt + n^2 \int_0^T \gamma_1(t) \gamma_3(t) dt$$

Important: Systematic risk leads to n^2 -term

Auxiliary function

$$\hat{V}^Q(t, t') = \int_t^T e^{-\int_t^\tau (f^r(t', u) + f^{\mu, Q}(x, t', u)) du} (a_0(\tau) + a_1(\tau) f^{\mu, Q}(x, t', \tau)) d\tau \\ + e^{-\int_t^T (f^r(t', u) + f^{\mu, Q}(x, t', u)) du} \Delta A_0(\bar{T})$$

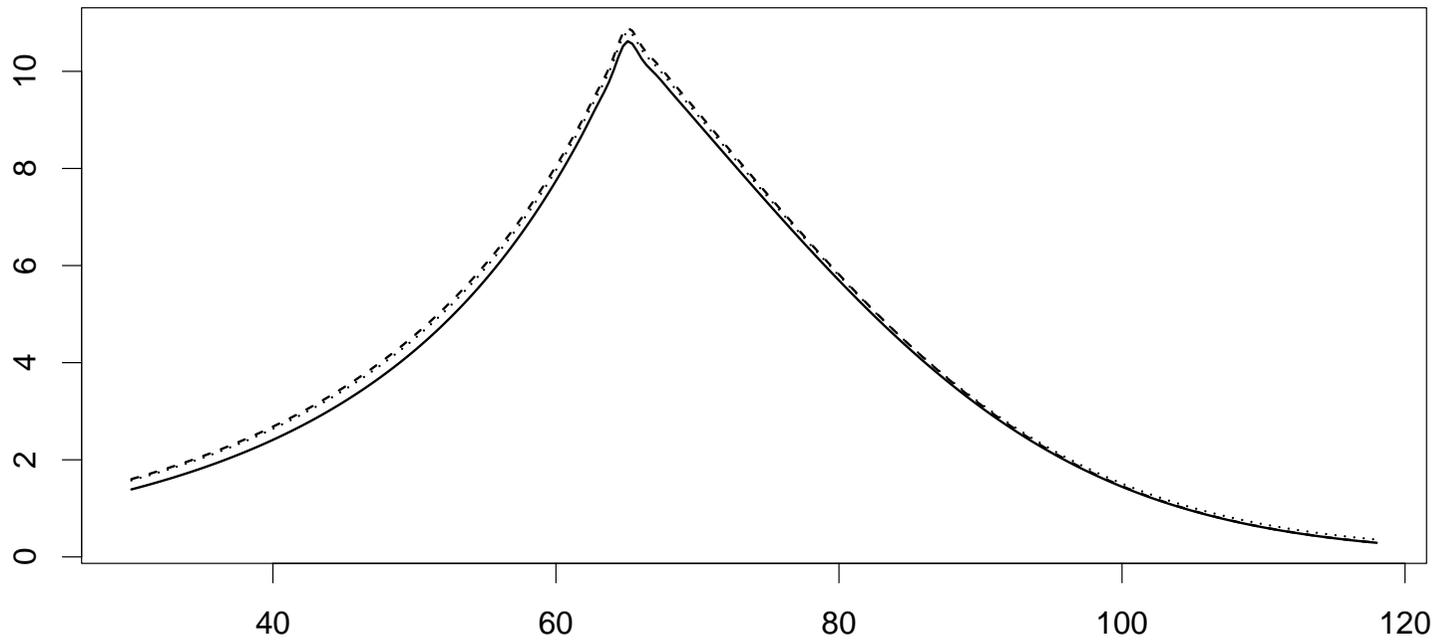
Thiele's differential equation

$$\frac{\partial}{\partial t} \hat{V}^Q(t, t') = (f^r(t', t) + f^{\mu, Q}(x, t', t)) \hat{V}^Q(t, t') - a_0(t) - a_1(t) f^{\mu, Q}(x, t', t)$$

Steps:

- ❑ Determine forward mortality intensity and forward rate
- ❑ Solve differential equation

Market reserves



Reserves for a life annuity starting at age 65.

Reserve based on 2003-estimate for males (solid line), exponentially corrected mortality intensities (dashed line) and forward mortalities (dotted line)

Numerical examples

Recall:

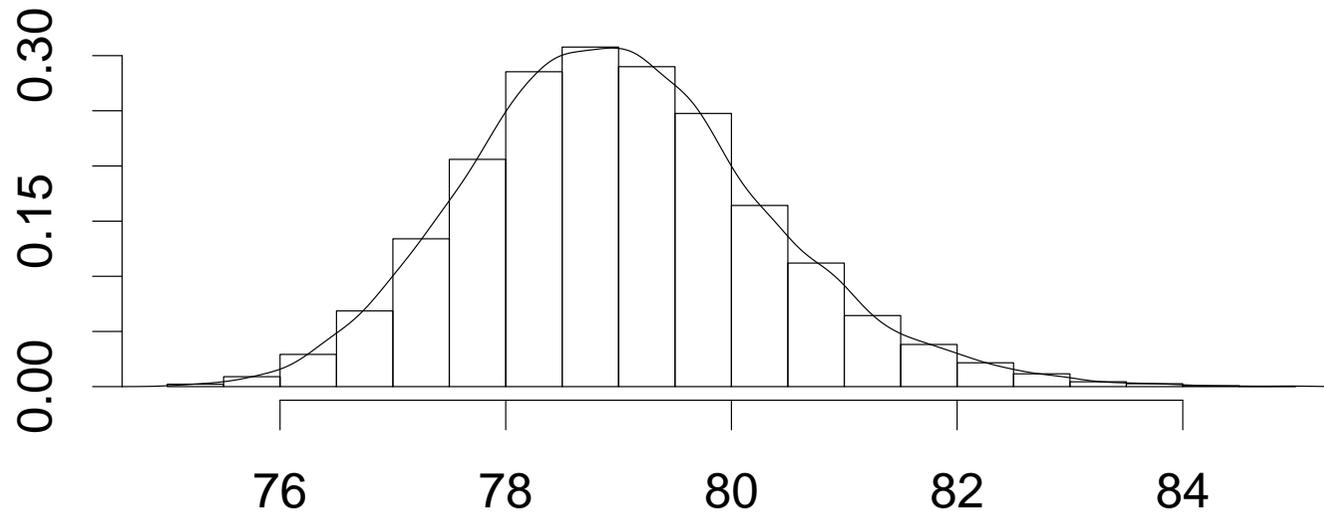
$$d\zeta(x, t) = (\gamma(x, t) - \delta(x, t)\zeta(x, t))dt + \sigma(x, t)\sqrt{\zeta(x, t)}dW^\mu(t)$$

	$\delta(x, t)$	$\gamma(x, t)$	$\sigma(x, t)$
Case I	$\tilde{\delta}$	$\tilde{\delta}e^{-\tilde{\gamma}t}$	$\tilde{\sigma}$
Case II	$\tilde{\gamma}$	$\frac{1}{2}\tilde{\sigma}^2$	$\tilde{\sigma}$

Quantiles, time horizon 20 years:

	$\tilde{\delta}$	$\tilde{\gamma}$	$\tilde{\sigma}$	5%	25%	50%	75%	95%
Case I	0.2	0.008	0.03	0.814	0.856	0.886	0.917	0.962
Case II		0.008	0.02	0.726	0.801	0.854	0.909	0.990

Simulation for expected lifetimes



Histogram for the **expected lifetime** for a policy-holder aged 30 for Case II

With 2003-estimate, expected lifetime is 75.8 years