Valuation and Hedging of Life Insurance Liabilities with Systematic Mortality Risk

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Abstract. This paper considers the problem of valuating and hedging life insurance contracts that are subject to systematic mortality risk in the sense that the mortality intensity of all policy-holders is affected by some underlying stochastic processes. In particular, this implies that the insurance risk cannot be eliminated by increasing the size of the portfolio and appealing to the law of large numbers. We propose to apply techniques from incomplete markets in order to hedge and valuate these contracts. We consider a special case of the affine mortality structures considered by Dahl (2004), where the underlying mortality process is driven by a time-inhomogeneous Cox-Ingersoll-Ross (CIR) model. Within this model, we study a general set of equivalent martingale measures, and determine market reserves by applying these measures. In addition, we derive risk-minimizing strategies and mean-variance indifference prices and hedging strategies for the life insurance liabilities considered. Numerical examples are included, and the use of the stochastic mortality model is compared with deterministic models.

Key words: Stochastic mortality, affine mortality structure, equivalent martingale measure, risk-minimization, mean-variance indifference pricing, Thiele's differential equation.

JEL Classification: G10.

Mathematics Subject Classification (2000): 62P05, 91B28.

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1 Introduction

During the past years, expected lifetimes have increased considerably in many countries. This has forced life insurers to adjust expectations towards the underlying mortality laws used to determine reserves. Since the future mortality is unknown, a correct description requires a stochastic model, as it has already been proposed by several authors, see e.g. Marocco and Pitacco (1998), Milevsky and Promislow (2001), Dahl (2004), Cairns, Blake and Dowd (2004), Biffis and Millossovich (2004) and references therein. For a survey on current developments in the literature and their relation to our results, we refer the reader to Section 2 in Dahl (2004). The main contribution of the present paper is not the introduction of a specific model for the mortality intensity, but rather the study of the problem of valuating and hedging life insurance liabilities that are subject to systematic changes in the underlying mortality intensity.

In Dahl (2004), a general class of Markov diffusion models are considered for the mortality intensity, and the affine mortality structures are recognized as a class with particular nice properties. Here, we study a special case of the general affine mortality structures and demonstrate how such models could be applied in practice. As starting point we take some smooth initial mortality intensity curve, which is estimated by standard methods. We then assume that the mortality intensity at a given future point in time at a given age is obtained by correcting the initial mortality intensity by the outcome of some underlying mortality improvement process, which is modeled via a time-inhomogeneous Cox-Ingersoll-Ross (CIR) model. Our model implies that the mortality intensity itself is described by a time-inhomogeneous CIR model as well. As noted by Dahl (2004), the survival probability can now be determined by using standard results for affine term structures.

Within this setting, we consider an insurance portfolio and assume that the individual lifetimes are affected by the same stochastic mortality intensity. In particular, this implies that the lifetimes are not stochastically independent. Hence, the insurance company is exposed to systematic as well as unsystematic mortality risk. Here, as in Dahl (2004), systematic mortality risk refers to the risk associated with changes in the underlying mortality intensity, whereas unsystematic mortality risk refers to the risk associated with the randomness of deaths in a portfolio with known mortality intensity. The systematic mortality risk is a non-diversifiable risk, which does not disappear when the size of the portfolio is increased, whereas the unsystematic mortality risk is diversifiable. Since the systematic mortality risk typically cannot be traded efficiently in the financial markets or in the reinsurance markets, this leaves open the problem of pricing insurance contracts. Here, we follow Dahl (2004) and apply financial theories for pricing the contracts, and study a fairly general set of martingale measures for the model. We work with a simple financial market, consisting of a savings account and a zero coupon bond and derive market reserves for general life insurance liabilities. These market reserves depend on the market's attitude towards systematic and unsystematic mortality risk. Based on an investigation of some Danish mortality data, we propose some pragmatic parameter values and calculate market reserves by solving appropriate versions of Thiele's

differential equation.

Furthermore, we investigate methods for hedging and valuating general insurance liabilities in incomplete financial markets. One possibility is to apply risk-minimization, which has been suggested by Föllmer and Sondermann (1986) and applied for the handling of insurance risks by Møller (1998, 2001a, 2001c). We demonstrate how risk-minimizing hedging strategies may be determined in the presence of systematic mortality risk. These results generalize the results in Møller (1998, 2001c), where risk-minimizing strategies were obtained without allowing for systematic mortality risk. In addition, this can be viewed as an extension of the work of Dahl (2004), where market reserves were derived in the presence of systematic mortality risk, but without considering the hedging aspect.

Utility indifference valuation and hedging has gained considerable interest over the last years as a method for valuation and hedging in incomplete markets, see e.g. Schweizer (2001b) and Becherer (2003) and references therein. These methods have been applied for the handling of insurance contracts by e.g. Becherer (2003), who worked with exponential utility functions, and by Møller (2001b, 2003a, 2003b), who worked with mean-variance indifference principles. We derive mean-variance indifference prices within our model and compare the results with the ones obtained in Møller (2001b).

The present paper is organized as follows. Section 2 contains a brief analysis of some Danish mortality data. In Section 3, we introduce the model for the underlying mortality intensity and derive the corresponding survival probabilities and forward mortality intensities. The financial market used for the calculation of market reserves, hedging strategies and indifference prices is introduced in Section 4, and the insurance portfolio is described in Section 5. Section 6 presents the combined model, the insurance payment process and the associated market reserves. Risk-minimizing hedging strategies are determined in Section 7, and mean-variance indifference prices and hedging strategies are obtained in Section 8. Numerical examples are provided in Section 9, and the Appendix contains proofs of some technical results.
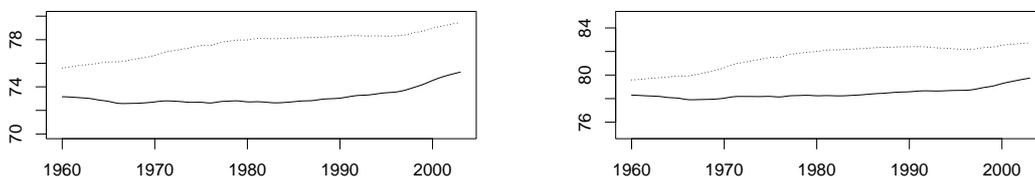

Figure 1: *To the left: Development of the expected lifetime of 30 year old females (dotted line at the top) and 30 year old males (solid line at the bottom) from year 1960 to 2003. To the right: Expected lifetimes for age 65. Estimates are based on the last 5 years of data available at calendar time.*

2 Motivation and empirical evidence

We briefly describe typical empirical findings related to the development in the mortality during the last couple of decades. The results in this section are based on
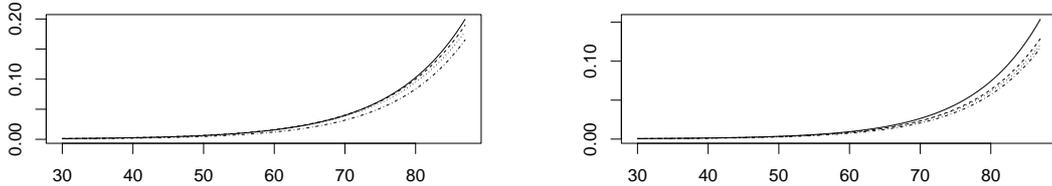

Figure 2: *Estimated mortality intensities for males (to the left) and females (to the right). Solid lines are 1970-estimates, dashed lines correspond to 1980, dotted lines 1990 and dot-dashed lines 2003.*

Danish mortality data, which have been compiled and analyzed by Andreev (2002). A more detailed statistical study is carried out in Fledelius and Nielsen (2002), who applied kernel hazard estimation. From the data material, we have determined the exposure times $W_{y,x}^*$ and number of deaths $N_{y,x}^*$ for each calendar year y, and age x, and calculated the occurrence-exposure rates $\mu_{y,x}^* = N_{y,x}^*/W_{y,x}^*$. For each fixed y, we have determined a smooth Gompertz-Makeham curve $\widetilde{\mu}_{y,x} = \alpha_y + \beta_y c_y^x$ based on the last 5 years of data available at calendar time by using standard methods as described in Norberg (2000).
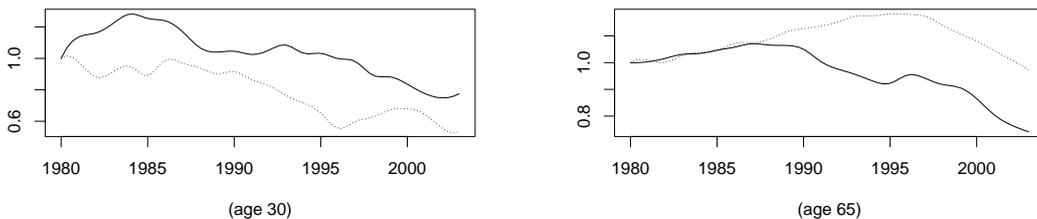

Figure 3: *Changes in the mortality intensity from 1980 to 2003 for males (solid lines) and females (dotted lines) at fixed ages. The numbers have been normalized with the 1980 mortality intensities and are based directly on the occurrence-exposure rates.*

We have visualized in Figure 1 the development in the total expected lifetime of 30- and 65-year old males and females based on the historical observations from the year 1960 to 2003. These numbers are based on raw occurrence-exposure rates. The figure shows that this method leads to an increase in the remaining lifetime from 1980 to 2003 of approximately 2.5 years for males and 1.5 years for females aged 30. Using this method, the expected lifetime in 2003 is about 75.3 years for 30 year old males and 79.5 for 30 year old females. If we alternatively use only one year of data

we see an increase from 72.5 to 75.5 for males and from 77.9 to 79.9 for females. Figure 2 contains the estimated Gompertz-Makeham mortality intensities $\widetilde{\mu}_{y,x}$ for males and females, respectively, for 1970, 1980, 1990 and 2003. These figures show how the mortality intensities have decreased during this period. A closed study of the
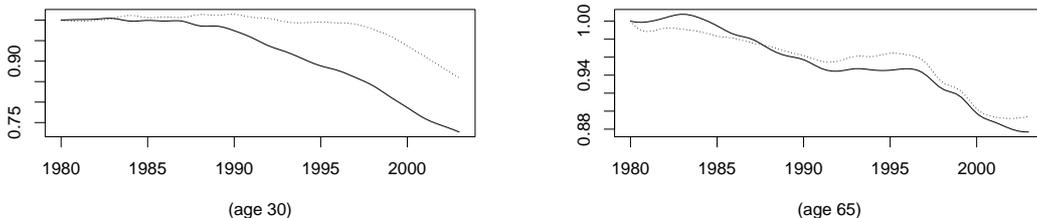

Figure 4: *Changes in the mortality intensity from 1980 to 2003 for males (solid lines) and females (dotted lines) as age increases. The numbers have been normalized with the 1980 mortality intensities and are based on the estimated Gompertz-Makeham curves.*

parameters (α_y, β_y, c_y) indicate that α_y has decreased. The estimates for β_y increase from 1960 to 1990, where the estimates for c_y decrease. In contrast, β_y decreases and c_y increases from 1990 to 2003. This approach does not involve a model that takes changes in the underlying mortality patterns into consideration. Another way to look at the mortality intensities is to consider changes in the mortality intensities at fixed ages, for example age 30 and 65, see Figure 3. For both ages, we see periods where the mortality increases and periods where it decreases. However, the general trend seems to be that mortality decreases. Moreover, we see that the mortality behaves differently for different ages and for males and females. Finally, we consider the situation, where we fix the initial age and compare the fitted Gompertz-Makeham curve for 1980 with the subsequent ones as the age increases with calendar time. This is presented in Figure 4. Again, we see periods where the ratio between the current mortality and the 1980-estimate increases and periods where it decreases.

3 Modeling the mortality

3.1 The general model

We take as starting point an initial curve for the mortality intensity (at all ages) $\mu^{\circ,g}(x)$ for age $x \geq 0$ and gender $g =$male, female. It is assumed that $\mu^{\circ,g}(x)$ is continuously differentiable as a function of x. We neglect the gender aspect in the following, and simply write $\mu^{\circ}(x)$. For an individual aged x at time 0, the future mortality intensity is viewed as a stochastic process $\mu(x) = (\mu(x,t))_{t\in[0,T]}$ with the property that $\mu(x,0) = \mu^{\circ}(x)$. (Here, T is a fixed, finite time horizon.) In principle, one can view $\mu = (\mu(x))_{x\geq 0}$ as an infinitely dimensional process.

We model changes in the mortality intensity via a strictly positive infinite dimensional process $\zeta = (\zeta(x,t))_{x\geq 0, t\in[0,T]}$ with the property that $\zeta(x,0) = 1$ for all x. Here and in the following, we take all processes and random variables to be defined

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on some probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}(t))_{t \in [0,T]}$, which contains all available information. In addition, we work with several subfiltrations. In particular, the filtration $\mathbb{I} = (\mathcal{I}(t))_{t \in [0,T]}$ is the natural filtration of the underlying process ζ. The mortality intensity process is then modeled via:

$$\mu(x,t) = \mu^\circ(x+t)\zeta(x,t). \tag{3.1}$$

Thus, $\zeta(x,t)$ describes the change in the mortality from time 0 to t for a person of age $x + t$. The true survival probability is defined by

$$\mathcal{S}(x,t,T) = E_P\left[e^{-\int_t^T \mu(x,\tau)d\tau} \,\middle|\, \mathcal{I}(t)\right] = E_P\left[e^{-\int_t^T \mu^\circ(x+\tau)\,\zeta(x,\tau)d\tau} \,\middle|\, \mathcal{I}(t)\right], \tag{3.2}$$

and it is related to the martingale

$$\mathcal{S}^M(x,t,T) = E_P\left[e^{-\int_0^T \mu(x,\tau)d\tau} \,\middle|\, \mathcal{I}(t)\right] = e^{-\int_0^t \mu(x,\tau)d\tau}\mathcal{S}(x,t,T). \tag{3.3}$$

In general, we can consider survival probabilities under various equivalent probability measures. This is discussed in more detail in section 6.1.

3.2 Deterministic changes in mortality intensities

As a special case, assume that $\zeta(x,t) = e^{-\widetilde{\gamma}(x)t}$, where $\widetilde{\gamma}(x)$ is fixed and constant. Thus, the mortality intensity at time t of and $x+t$ year old is defined by changing the known mortality intensity at time 0 of an $x + t$-year old by the factor $\exp(-\widetilde{\gamma}(x)t)$. If $\widetilde{\gamma}(x) > 0$, this model implies that the mortality improves by the factor $\exp(-\widetilde{\gamma}(x))$ each year. In particular, taking all $\widetilde{\gamma}(x)$ equal to one fixed $\widetilde{\gamma}$ means that all intensities improve/increase by the same factor.

If μ° corresponds to a Gompertz-Makeham mortality law, i.e.

$$\mu^\circ(x+t) = \alpha + \beta\, c^{x+t}, \tag{3.4}$$

then the mortality intensity μ is given by

$$\mu(x,t) = \alpha e^{-\widetilde{\gamma}(x)t} + \beta\, c^x (c e^{-\widetilde{\gamma}(x)})^t, \tag{3.5}$$

which no longer is a Gompertz-Makeham mortality law. It now corresponds to the distribution of the minimum of two random variables following Gompertz-Makeham mortality laws with parameters $(0, \alpha, \exp(-\widetilde{\gamma}(x)))$ and $(0, \beta\, c^x, c \exp(-\widetilde{\gamma}(x)))$, respectively.

3.3 Time-inhomogeneous CIR models

The empirical findings in Section 3.1 indicate that the deterministic type of model considered above is too simple to capture the true nature of the mortality. We propose instead to model the underlying mortality improvement process via

$$d\zeta(x,t) = (\gamma(x,t) - \delta(x,t)\zeta(x,t))dt + \sigma(x,t)\sqrt{\zeta(x,t)}dW^\mu(t), \tag{3.6}$$

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where $W^\mu(t)$ is a standard Brownian motion under P. This is similar to a so-called time-inhomogeneous CIR model, originally proposed by Hull and White (1990) as an extension of the short rate model in Cox, Ingersoll and Ross (1985), see also Rogers (1995). We assume that $2\gamma(x,t) \geq (\sigma(x,t))^2$ such that ζ is strictly positive, see Maghsoodi (1996). Here, γ, δ and σ are assumed to be known, continuous functions. It now follows via Itô's formula that

$$d\mu(x,t) = (\gamma^\mu(x,t) - \delta^\mu(x,t)\mu(x,t))dt + \sigma^\mu(x,t)\sqrt{\mu(x,t)}dW^\mu(t), \qquad (3.7)$$

where

$$\gamma^\mu(x,t) = \gamma(x,t)\mu^\circ(x+t), \qquad (3.8)$$

$$\delta^\mu(x,t) = \delta(x,t) - \frac{\frac{d}{dt}\mu^\circ(x+t)}{\mu^\circ(x+t)}, \qquad (3.9)$$

$$\sigma^\mu(x,t) = \sigma(x,t)\sqrt{\mu^\circ(x+t)}. \qquad (3.10)$$

This shows that μ also follows an time-inhomogeneous CIR model, a property which was also noted by Rogers (1995). In particular, we note that $\gamma^\mu(x,t)/(\sigma^\mu(x,t))^2 = \gamma(x,t)/(\sigma(x,t))^2$, such that μ is strictly positive as well. If $\gamma(x,t)/(\sigma(x,t))^2$, and thus $\gamma^\mu(x,t)/(\sigma^\mu(x,t))^2$, is independent of t, then numerical calculations can be simplified considerably, see Jamshidian (1995). The following proposition regarding the survival probability follows e.g. from Björk (2004, Proposition 22.2); see also Dahl (2004).

Proposition 3.1 *(Affine mortality structure)*
The survival probability $\mathcal{S}(x,t,T)$ is given by

$$\mathcal{S}(x,t,T) = e^{A^\mu(x,t,T) - B^\mu(x,t,T)\mu(x,t)},$$

where

$$\frac{\partial}{\partial t}B^\mu(x,t,T) = \delta^\mu(x,t)B^\mu(x,t,T) + \frac{1}{2}(\sigma^\mu(x,t))^2(B^\mu(x,t,T))^2 - 1, \qquad (3.11)$$

$$\frac{\partial}{\partial t}A^\mu(x,t,T) = \gamma^\mu(x,t)B^\mu(x,t,T), \qquad (3.12)$$

with $B^\mu(x,T,T) = 0$ and $A^\mu(x,T,T) = 0$. The dynamics of the survival probability are given by

$$d\mathcal{S}(x,t,T) = \mathcal{S}(x,t,T)\left(\mu(x,t)dt - \sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)dW^\mu(t)\right).$$

Forward mortality intensities
Inspired by interest rate theory, Dahl (2004) introduced the concept of forward mortality intensities. In an affine setting, the forward mortality intensities are given by

$$f^\mu(x,t,T) = -\frac{\partial}{\partial T}\log\mathcal{S}(x,t,T) = \mu(x,t)\frac{\partial}{\partial T}B^\mu(x,t,T) - \frac{\partial}{\partial T}A^\mu(x,t,T). \qquad (3.13)$$

The importance of forward mortality intensities is underlined by writing the survival probability on the form

$$\mathcal{S}(x,t,T) = e^{-\int_t^T f^\mu(x,t,u)du}.$$

4 The financial market

In this section, we introduce the financial market used for the calculations in the following sections. The financial market is essentially assumed to exist of two traded assets: A savings account and a zero coupon bond with maturity T. The price processes are given by B and $P(\cdot, T)$, respectively. The uncertainty in the financial market is described via a time-homogeneous affine model for the short rate. Hence, the short rate dynamics under P are

$$dr(t) = \alpha^r(r(t))dt + \sigma^r(r(t))dW^r(t), \tag{4.1}$$

where

$$\alpha^r(r(t)) = \gamma^{r,\alpha} - \delta^{r,\alpha}r(t),$$
$$\sigma^r(r(t)) = \sqrt{\gamma^{r,\sigma} + \delta^{r,\sigma}r(t)}.$$

Here, W^r is a standard Brownian motion under P and $\gamma^{r,\alpha}$, $\delta^{r,\alpha}$, $\gamma^{r,\sigma}$ and $\delta^{r,\sigma}$ are constants. Denote by $\mathcal{G} = (\mathcal{G}(t))_{t\in[0,T]}$ the natural filtration generated by W^r. The dynamics under P of the price processes are given by

$$dB(t) = r(t)B(t)dt, \tag{4.2}$$
$$dP(t,T) = (r(t) + \rho(t,r(t)))P(t,T)dt + \sigma^p(t,r(t))P(t,T)dW^r(t), \tag{4.3}$$

where

$$\rho(t,r(t)) = \sigma^p(t,r(t))\left(\frac{\widetilde{c}}{\sigma^r(r(t))} + c\sigma^r(r(t))\right). \tag{4.4}$$

Here, c and $\widetilde{c}$ are constants satisfying certain conditions given in Remark 4.1. With this choice of ρ, σ^p is uniquely determined from standard theory for affine short rate models, see (4.11).

If we restrict the model to the filtration $\mathcal{G}$, the unique equivalent martingale measure for the financial market is

$$\frac{dQ}{dP} = \widehat{\Lambda}(T), \tag{4.5}$$

where $d\widehat{\Lambda}(t) = \widehat{\Lambda}(t)h^r(t)dW^r(t)$, $\widehat{\Lambda}(0) = 1$, and where

$$h^r(t) = -\frac{\rho(t,r(t))}{\sigma^p(t,r(t))} = -\left(\frac{\widetilde{c}}{\sigma^r(r(t))} + c\sigma^r(r(t))\right). \tag{4.6}$$

Under Q given by (4.5) the dynamics of the short rate are given by

$$dr(t) = \left(\gamma^{r,\alpha,Q} - \delta^{r,\alpha,Q}r(t)\right)dt + \sqrt{\gamma^{r,\sigma} + \delta^{r,\sigma}r(t)}dW^{r,Q}(t), \tag{4.7}$$

where $W^{r,Q}$ is a standard Brownian motion under Q and

$$\gamma^{r,\alpha,Q} = \gamma^{r,\alpha} - c\gamma^{r,\sigma} - \widetilde{c},$$
$$\delta^{r,\alpha,Q} = \delta^{r,\alpha} + c\delta^{r,\sigma}.$$

Since the drift and squared diffusion terms in (4.7) are affine in r, we have an affine term structure, see Björk (2004, Proposition 22.2). Thus, the bond price is given by

$$P(t,T) = e^{A^r(t,T) - B^r(t,T)r(t)},$$

where $A^r(t,T)$ and $B^r(t,T)$ solves

$$\frac{\partial}{\partial t}B^r(t,T) = \delta^{r,\alpha,Q}B^r(t,T) + \frac{1}{2}\delta^{r,\sigma}(B^r(t,T))^2 - 1, \qquad (4.8)$$

$$\frac{\partial}{\partial t}A^r(t,T) = \gamma^{r,\alpha,Q}B^r(t,T) - \frac{1}{2}\gamma^{r,\sigma}(B^r(t,T))^2, \qquad (4.9)$$

with $B^r(T,T) = 0$ and $A^r(T,T) = 0$. The bond price dynamics under Q can be determined by applying Itô's formula:

$$dP(t,T) = r(t)P(t,T)dt - \sigma^r(r(t))B^r(t,T)P(t,T)dW^{r,Q}(t), \qquad (4.10)$$

which in turn gives that

$$\sigma^p(t,r(t)) = -\sigma^r(r(t))B^r(t,T). \qquad (4.11)$$

Remark 4.1 Recall that if $\delta^{r,\sigma} \neq 0$ and

$$\frac{\gamma^{r,\alpha}}{\delta^{r,\sigma}} + \frac{\delta^{r,\alpha}\gamma^{r,\sigma}}{(\delta^{r,\sigma})^2} < \frac{1}{2}, \qquad (4.12)$$

then $P(r(t) = 0) > 0$. Hence, we immediately get from (4.4) that $\widetilde{c} = 0$ in this case. If $\delta^{r,\sigma} \neq 0$ and (4.12) does not hold, then, exploiting the results of Cheridito, Filipović and Kimmel (2003), gives that (4.5) defines an equivalent martingale measure if

$$\widetilde{c} \leq \gamma^{r,\alpha} + \frac{\delta^{r,\alpha}\gamma^{r,\sigma}}{\delta^{r,\sigma}} - \frac{\delta^{r,\sigma}}{2}. \qquad (4.13)$$

No restrictions apply to c in any case or to $\widetilde{c}$ if $\delta^{r,\sigma} = 0$.

Remark 4.2 If $\delta^{r,\sigma} = 0$ the short rate is described by a Vasiček model, see Vasiček (1977). In this case the functions A^r and B^r are given by

$$B^r(t,T) = \frac{1}{\delta^{r,\alpha,Q}}\left(1 - e^{-\delta^{r,\alpha,Q}(T-t)}\right),$$

$$A^r(t,T) = \frac{(B^r(t,T) - T + t)(\gamma^{r,\alpha,Q}\delta^{r,\alpha,Q} - \frac{1}{2}\gamma^{r,\sigma})}{(\delta^{r,\alpha,Q})^2} - \frac{\gamma^{r,\sigma}(B^r(t,T))^2}{4\delta^{r,\alpha,Q}}.$$

Letting $\gamma^{r,\sigma} = 0$, we get a time-homogeneous CIR model (for the short rate), see Cox et al. (1985), which gives the following expressions for A^r and B^r

$$B^r(t,T) = \frac{2\left(e^{\xi^{r,Q}(T-t)} - 1\right)}{(\xi^{r,Q} + \delta^{r,\alpha,Q})(e^{\xi^{r,Q}(T-t)} - 1) + 2\xi^{r,Q}},$$

$$A^r(t,T) = \frac{2\gamma^{r,\alpha,Q}}{\delta^{r,\sigma}}\log\left(\frac{2\xi^{r,Q}e^{(\xi^{r,Q}+\delta^{r,\alpha,Q})\frac{T-t}{2}}}{(\xi^{r,Q} + \delta^{r,\alpha,Q})\left(e^{\xi^{r,Q}(T-t)} - 1\right) + 2\xi^{r,Q}}\right),$$

where $\xi^{r,Q} = \sqrt{(\delta^{r,\alpha,Q})^2 + 2\delta^{r,\sigma}}$. For both models, the functions A^r and B^r depend on t and T via the difference $T - t$, only.

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5 The insurance portfolio

Consider an insurance portfolio consisting of n insured lives of the same age x. We assume that the individual remaining lifetimes at time 0 of the insured are described by a sequence $T_1, \ldots, T_n$ of identically distributed non-negative random variables. Moreover, we assume that

$$P(T_1 > t | \mathcal{I}(T)) = e^{-\int_0^t \mu(x,s)ds}, \ \ 0 \leq t \leq T,$$

and that the censored lifetimes $T_i^* = T_i 1_{\{T_i \leq T\}} + T 1_{\{T_i > T\}}$, $i = 1, \ldots, n$, are i.i.d. given $\mathcal{I}(T)$. Thus, given the development of the underlying process ζ, the mortality intensity at time s is simply $\mu(x, s)$.

Now define a counting process $N(x) = (N(x,t))_{0 \leq t \leq T}$ by

$$N(x,t) = \sum_{i=1}^n 1_{(T_i \leq t)},$$

which keeps track of the number of deaths in the portfolio of insured lives. We denote by $I\!H = (\mathcal{H}(t))_{0 \leq t \leq T}$ the natural filtration generated by $N(x)$. It follows that $N(x)$ is an $I\!H \vee I\!I$-Markov process, and the stochastic intensity process $\lambda(x) = (\lambda(x,t))_{0 \leq t \leq T}$ of $N(x)$ under P can be informally defined by

$$\lambda(x,t)dt \equiv E_P\left[dN(x,t) | \mathcal{H}(t-) \vee \mathcal{I}(t)\right] = (n - N(x,t-))\mu(x,t)dt, \quad (5.1)$$

which is proportional to the product of the number of survivors and the mortality intensity. It is well-known, that the process $M(x) = (M(x,t))_{0 \leq t \leq T}$ defined by

$$dM(x,t) = dN(x,t) - \lambda(x,t)dt, \quad 0 \leq t \leq T, \quad (5.2)$$

is an $(I\!H \vee I\!I, P)$-martingale.

6 The combined model

The filtration $I\!F = (\mathcal{F}(t))_{0 \leq t \leq T}$ introduced earlier is given by $\mathcal{F}(t) = \mathcal{G}(t) \vee \mathcal{H}(t) \vee \mathcal{I}(t)$. Thus, $I\!F$ is the filtration for the combined model of the financial market, the mortality intensity and the insurance portfolio. Moreover, we assume that the financial market is stochastically independent of the insurance portfolio and the mortality intensity, i.e. $\mathcal{G}(T)$ and $(\mathcal{H}(T), \mathcal{I}(T))$ are independent. In particular, this implies that the properties of the underlying processes are preserved. For example, $M(x)$ is also an $(I\!F, P)$-martingale, and the $(I\!F, P)$-intensity process is identical to the $(I\!H \vee I\!I, P)$-intensity process $\lambda(x)$. We note that the combined model is on the general index-form studied in Steffensen (2000). However, Steffensen (2000) contains no explicit remarks or calculations regarding a stochastic mortality intensity.

6.1 A class of equivalent martingale measures

If we consider the financial market only, i.e. if we restrict ourselves to the filtration $I\!G$, we found in Section 4 that (given some regularity conditions) there exists a unique

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equivalent martingale measure. This is not the case when analyzing the combined model of the financial market and the insurance portfolio, see e.g. Møller (1998, 2001c) for a discussion of this problem. In the present model, we can also perform a change of measure for the counting process $N(x)$ and for the underlying mortality intensity; we refer to Dahl (2004) for a more detailed treatment of these aspects. Consider a likelihood process on the form

$$d\Lambda(t) = \Lambda(t-)\Big(h^r(t)dW^r(t) + h^\mu(t)dW^\mu(t) + g(t)dM(x,t)\Big), \qquad (6.1)$$

with $\Lambda(0) = 1$. We assume that $E_P[\Lambda(T)] = 1$ and define an equivalent martingale measure Q via $\frac{dQ}{dP} = \Lambda(T)$. In the following, we describe the terms in (6.1) in more detail. The process h^r, which is defined in (4.6), is related to the change of measure for the underlying bond market. It is uniquely determined by requiring that the discounted bond price process is a Q-martingale.

The term involving h^μ leads to a change of measure for the Brownian motion which drives the mortality intensity process μ. Hence, $dW^{\mu,Q}(t) = dW^\mu(t) - h^\mu(t)dt$ defines a standard Brownian motion under Q. Here, we restrict ourselves to h^μ's of the form

$$h^\mu(t, \zeta(x,t)) = -\beta(x,t)\frac{\sqrt{\zeta(x,t)}}{\sigma(x,t)} + \frac{\beta^*(x,t)}{\sigma(x,t)\sqrt{\zeta(x,t)}} \qquad (6.2)$$

for some continuous functions β and β^*. In this case, the Q-dynamics of $\zeta(x,t)$ are given by

$$d\zeta(x,t) = \big(\gamma^Q(x,t) - \delta^Q(x,t)\zeta(x,t)\big)\,dt + \sigma(x,t)\sqrt{\zeta(x,t)}dW^{\mu,Q}(t),$$

where

$$\gamma^Q(x,t) = \gamma(x,t) + \beta^*(x,t), \qquad (6.3)$$
$$\delta^Q(x,t) = \delta(x,t) + \beta(x,t). \qquad (6.4)$$

Hence, ζ also follows a time-inhomogeneous CIR model under Q. A necessary condition for the equivalence between P and Q is that ζ is strictly positive under Q. Thus, we observe from (6.3) that we must require that $\beta^*(x,t) \geq (\sigma(x,t))^2/2 - \gamma(x,t)$. The Q-dynamics of $\mu(x)$ are now given by

$$d\mu(x,t) = (\gamma^{\mu,Q}(x,t) - \delta^{\mu,Q}(x,t)\mu(x,t))dt + \sigma^\mu(x,t)\sqrt{\mu(x,t)}dW^{\mu,Q}(t), \qquad (6.5)$$

where $\gamma^{\mu,Q}(x,t)$ and $\delta^{\mu,Q}(x,t)$ are given by (3.8) and (3.9) with $\gamma(x,t)$ and $\delta(x,t)$ replaced by $\gamma^Q(x,t)$ and $\delta^Q(x,t)$, respectively. If $h^\mu = 0$, i.e. if the dynamics of ζ (and thus μ) are identical under P and Q, we say the market is risk-neutral with respect to systematic mortality risk.

The last term in (6.1) involves a predictable process $g > -1$. This term affects the intensity for the counting process. More precisely, it can be shown, see e.g. Andersen, Borgan, Gill and Keiding (1993), that the intensity process under Q is given by $\lambda^Q(x,t) = (1 + g(t))\lambda(x,t)$. Using (5.1), we see that

$$\lambda^Q(x,t) = (n - N(x,t-))(1 + g(t))\mu(x,t),$$

such that $\mu^Q(x,t) = (1+g(t))\mu(x,t)$ can be viewed as the mortality intensity under Q. Hence the process $M^Q(x) = (M^Q(x,t))_{0 \le t \le T}$ defined by

$$dM^Q(x,t) = dN(x,t) - \lambda^Q(x,t)dt, \quad 0 \le t \le T, \tag{6.6}$$

is an $(\mathbb{F}, Q)$-martingale. If $g = 0$, the market is said to be risk-neutral with respect to unsystematic mortality risk. This choice of g can be motivated by the law of large numbers. In this paper, we restrict the analysis to the case, where g is a deterministic, continuously differentiable function. Combined with the definition of h^r in (4.6) and the restricted form of h^μ in (6.2), this implies that the independence between $\mathcal{G}(T)$ and $(\mathcal{H}(T), \mathcal{I}(T))$ is preserved under Q.

Now define the Q-survival probability and the associated Q-martingale by

$$\mathcal{S}^Q(x,t,T) = E_Q\left[e^{-\int_t^T \mu^Q(x,\tau)d\tau} \,\Big|\, \zeta(x,t) \right]$$

and

$$\mathcal{S}^{Q,M}(x,t,T) = E_Q\left[e^{-\int_0^T \mu^Q(x,\tau)d\tau} \,\Big|\, \zeta(x,t) \right] = e^{-\int_0^t \mu^Q(x,\tau)d\tau} \mathcal{S}^Q(x,t,T).$$

Calculations similar to those in Section 3.3 give the following Q-dynamics of $\mu^Q(x)$

$$d\mu^Q(x,t) = (\gamma^{\mu,Q,g}(x,t) - \delta^{\mu,Q,g}(x,t)\mu^Q(x,t))dt + \sigma^{\mu,Q,g}(x,t)\sqrt{\mu^Q(x,t)}dW^{\mu,Q},$$

where

$$\gamma^{\mu,Q,g}(x,t) = (1+g(t))\gamma^{\mu,Q}(x,t),$$

$$\delta^{\mu,Q,g}(x,t) = \delta^{\mu,Q}(x,t) - \frac{\frac{d}{dt}g(t)}{1+g(t)},$$

$$\sigma^{\mu,Q,g}(x,t) = \sqrt{1+g(t)}\sigma^\mu(x,t).$$

Since the drift and squared diffusion terms for $\mu^Q(x,t)$ are affine in $\mu^Q(x,t)$, we have the following proposition

Proposition 6.1 *(Affine mortality structure under Q)*
The Q-survival probability $\mathcal{S}^Q(x,t,T)$ is given by

$$\mathcal{S}^Q(x,t,T) = e^{A^{\mu,Q}(x,t,T) - B^{\mu,Q}(x,t,T)(1+g(t))\mu(x,t)},$$

where $A^{\mu,Q}$ and $B^{\mu,Q}$ are determined from (3.11) and (3.12) with $\gamma^\mu(x,t)$, $\delta^\mu(x,t)$ and $\sigma^\mu(x,t)$ replaced by $\gamma^{\mu,Q,g}(x,t)$, $\delta^{\mu,Q,g}(x,t)$ and $\sigma^{\mu,Q,g}(x,t)$, respectively. The dynamics of the Q-martingale associated with the Q-survival probability are given by

$$d\mathcal{S}^{Q,M}(x,t,T) = -(1+g(t))\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^{\mu,Q}(x,t,T)\mathcal{S}^{Q,M}(x,t,T)dW^{\mu,Q}(t). \tag{6.7}$$

Similarly to the forward mortality intensities, the Q-forward mortality intensities are given by

$$f^{\mu,Q}(x,t,T) = -\frac{\partial}{\partial T}\log \mathcal{S}^Q(x,t,T) = \mu^Q(x,t)\frac{\partial}{\partial T}B^{\mu,Q}(x,t,T) - \frac{\partial}{\partial T}A^{\mu,Q}(x,t,T). \tag{6.8}$$

6.2 The payment process

The total benefits less premiums on the insurance portfolio is described by a payment process A. Thus, $dA(t)$ are the net payments to the policy-holders during an infinitesimal interval $[t, t + dt)$. We take A of the form

$$dA(t) = -n\pi(0)dI_{\{t\geq 0\}} + (n - N(x, \overline{T}))\Delta A_0(\overline{T})dI_{\{t\geq \overline{T}\}}$$
$$+ a_0(t)(n - N(x,t))dt + a_1(t)dN(x,t), \quad (6.9)$$

for $0 \leq t \leq T$. The first term, $n\pi(0)$ is the single premium paid at time 0 by all policy-holders. The second term involves a fixed time $\overline{T} \leq T$, which represents the retirement time of the insured lives. This term states that each of the surviving policy-holders receive the fixed amount $\Delta A_0(\overline{T})$ upon retirement. The third term involves a piecewise continuous function

$$a_0(t) = -\pi^c(t)1_{\{0\leq t<\overline{T}\}} + a^p(t)1_{\{\overline{T}\leq t\leq T\}},$$

where $\pi^c(t)$ are continuous premiums paid by the policy-holders (as long as they are alive) and $a^p(t)$ corresponds to a life annuity benefit received by the policy-holders. Finally, the last term in (6.9) represents payments immediately upon a death, and we assume that a_1 is some piecewise continuous function.

6.3 Market reserves

In the following we consider an arbitrary, but fixed, equivalent martingale measure Q from the class of measures introduced in Section 6.1 and define the process

$$V^{*,Q}(t) = E_Q\left[\int_{[0,T]} e^{-\int_0^\tau r(u)du}dA(\tau)\bigg| \mathcal{F}(t)\right], \quad (6.10)$$

which is the conditional expected value, calculated at time t, of discounted benefits less premiums, where all payments are discounted to time 0. Using that the processes A and r are adapted, and introducing the discounted payment process A^* defined by

$$dA^*(t) = e^{-\int_0^t r(u)du}dA(t),$$

we see that

$$V^{*,Q}(t) = \int_{[0,t]} e^{-\int_0^\tau r(u)du}dA(\tau) + e^{-\int_0^t r(u)du}E_Q\left[\int_{(t,T]} e^{-\int_t^\tau r(u)du}dA(\tau)\bigg| \mathcal{F}(t)\right]$$
$$= A^*(t) + e^{-\int_0^t r(u)du}\widetilde{V}^Q(t). \quad (6.11)$$

In the literature, the process $V^{*,Q}$ is called the intrinsic value process, see Föllmer and Sondermann (1986) and Møller (2001c). The process $\widetilde{V}^Q(t)$ introduced in (6.11) represents the conditional expected value at time t, of future payments. We shall refer to this quantity as the market reserve. We have the following result:

Proposition 6.2 *The market reserve $\widetilde{V}^Q$ is given by*

$$\widetilde{V}^Q(t) = (n - N(x,t))V^Q(t, r(t), \mu(x,t)), \quad (6.12)$$

where

$$V^Q(t, r(t), \mu(x,t)) = \int_t^T P(t,\tau)\mathcal{S}^Q(x,t,\tau)\left(a_0(\tau) + a_1(\tau)f^{\mu,Q}(x,t,\tau)\right)d\tau$$
$$+ P(t,\overline{T})\mathcal{S}^Q(x,t,\overline{T})\Delta A_0(\overline{T}). \tag{6.13}$$

This can be verified by using methods similar to the ones used in Møller (2001c) and Dahl (2004). A sketch of proof is given below.

Some comments on this results: The quantity $V^Q(t, r(t), \mu(x,t))$ is the market reserve at time t for one policy-holder who is alive, given the current level for the short rate and the mortality intensity. The market reserve has the same structure as standard reserves. However, the usual discount factor has been replaced by a zero coupon bond price $P(t,T)$ and the usual (deterministic) survival probability of the form $\exp(-\int_t^\tau \mu^\circ(x,u)du)$ has been replaced by the term $\mathcal{S}^Q(x,t,\tau)$. In addition, the Q-forward mortality intensity, $f^{\mu,Q}(x,t,\tau)$, now appears instead of the deterministic mortality intensity $\mu^\circ(x,\tau)$ in connection with the sum $a_1(\tau)$ payable upon a death.

Sketch of proof of Proposition 6.2: The proposition follows by exploiting the independence between the financial market and the insured lives. In addition, we use that for any predictable, sufficiently integrable process $\widetilde{g}$,

$$\int_0^t \widetilde{g}(s)(dN(x,s) - \lambda^Q(x,s)ds) \tag{6.14}$$

is an $(I\!\!F, Q)$-martingale. For example, this implies that

$$E_Q\left[\left.\int_t^T e^{-\int_t^\tau r(u)du}a_1(\tau)dN(x,\tau)\right| \mathcal{F}(t)\right]$$
$$= E_Q\left[\left.\int_t^T e^{-\int_t^\tau r(u)du}a_1(\tau)\lambda^Q(x,\tau)d\tau\right| \mathcal{F}(t)\right]$$
$$= \int_t^T P(t,\tau)a_1(\tau)E_Q\left[\left.(n - N(x,\tau))\mu^Q(x,\tau)\right| \mathcal{F}(t)\right]d\tau. \tag{6.15}$$

Here, the second equality follows by changing the order of integration and by using the independence between r and (N,μ). By iterated expectations, we get that

$$E_Q\left[(n - N(x,\tau))|\,\mathcal{F}(t)\right] = E_Q\left[E_Q\left[(n - N(x,\tau))|\,\mathcal{F}(t)\vee\mathcal{I}(T)\right]|\,\mathcal{F}(t)\right]$$
$$= E_Q\left[\left.(n - N(x,t))e^{-\int_t^\tau \mu^Q(x,u)du}\right|\mathcal{F}(t)\right]$$
$$= (n - N(x,t))\mathcal{S}^Q(x,t,\tau), \tag{6.16}$$

where the second equality follows by using that, given $\mathcal{I}(T)$, the lifetimes are i.i.d. under Q with mortality intensity $\mu^Q(x)$, and the third equality is the definition of

the Q-survival probability. Similarly, we have that

$$
\begin{aligned}
&E_Q \left[(n - N(x, \tau)) \mu^Q(x, \tau) \middle| \mathcal{F}(t) \right] \\
&= E_Q \left[E_Q \left[(n - N(x, \tau)) \mu^Q(x, \tau) \middle| \mathcal{F}(t) \vee \mathcal{I}(T) \right] \middle| \mathcal{F}(t) \right] \\
&= E_Q \left[(n - N(x, t)) \mu^Q(x, \tau) e^{-\int_t^\tau \mu^Q(x, u) du} \middle| \mathcal{F}(t) \right] \\
&= (n - N(x, t)) \frac{\partial}{\partial \tau} \mathcal{S}^Q(x, t, \tau) \\
&= (n - N(x, t)) \mathcal{S}^Q(x, t, \tau) f^{\mu, Q}(x, t, \tau).
\end{aligned} \tag{6.17}
$$

Here, the third equality follows by differentiating $\mathcal{S}^Q(x, t, \tau)$ under the integral. The result now follows by using (6.15)–(6.17).

$\square$

We emphasize that the market reserve depends on the choice of equivalent martingale measure Q.

7 Risk-minimizing strategies

The discounted insurance payment process A^* is subject to both financial and mortality risk. This implies that the insurance liabilities typically cannot be hedged and priced uniquely by trading on the financial market. Møller (1998) applied the criterion of risk-minimization suggested by Föllmer and Sondermann (1986) for the handling of this combined risk for unit-linked life insurance contracts. This analysis led to so-called risk-minimizing hedging strategies, that essentially minimized the variance of the insurance liabilities calculated with respect to some equivalent martingale measure. Here, we follow Møller (2001c), who extended the approach of Föllmer and Sondermann (1986) to the case of a payment process. Further applications of the criterion of risk-minimization to insurance contracts can be found in Møller (2001a, 2002).

7.1 A review of risk-minimization

Consider the financial market introduced in Section 4 consisting of a zero coupon bond expiring at T and a savings account. We denote by $X(t) = P^*(t, T)$ the discounted price process of the zero coupon bond. A strategy is a process $\varphi = (\xi, \eta)$, where ξ is the number of zero coupon bonds held and η is the discounted deposit on the savings account. The discounted value process $V(\varphi)$ associated with φ is defined by $V(t, \varphi) = \xi(t) X(t) + \eta(t)$, and the cost process $C(\varphi)$ is defined by

$$
C(t, \varphi) = V(t, \varphi) - \int_0^t \xi(u) dX(u) + A^*(t). \tag{7.1}
$$

The accumulated costs $C(t, \varphi)$ at time t are the discounted value $V(t, \varphi)$ of the portfolio reduced by discounted trading gains (the integral) and added discounted net payments to the policy-holders. A strategy is called risk-minimizing, if it minimizes

$$
R(t, \varphi) = E_Q \left[(C(T, \varphi) - C(t, \varphi))^2 \middle| \mathcal{F}(t) \right] \tag{7.2}
$$

for all t with respect to all strategies, and a strategy φ with $V(T, \varphi) = 0$ is called 0-admissible. The process $R(\varphi)$ is called the risk process. Föllmer and Sondermann (1986) realized that the risk-minimizing strategies are related to the so-called Galtchouk-Kunita-Watanabe decomposition,

$$V^{*,Q}(t) = E_Q\left[A^*(T)\middle|\mathcal{F}(t)\right] = V^{*,Q}(0) + \int_0^t \xi^Q(u)dX(u) + L^Q(t), \tag{7.3}$$

where ξ^Q is a predictable process and where L^Q is a zero-mean Q-martingale orthogonal to X. It now follows by Møller (2001c, Theorem 2.1) that there exists a unique 0-admissible risk-minimizing strategy $\varphi^* = (\xi^*, \eta^*)$ given by

$$\varphi^*(t) = (\xi^*(t), \eta^*(t)) = \left(\xi^Q(t),\, V^{*,Q}(t) - \xi^Q(t)X(t) - A^*(t)\right). \tag{7.4}$$

In particular, it follows that the cost process associated with the risk-minimizing strategy is given by

$$C(t, \varphi^*) = V^{*,Q}(0) + L^Q(t). \tag{7.5}$$

The risk process associated with the risk-minimizing strategy, the so-called intrinsic risk process, is given by

$$R(t, \varphi^*) = E_Q\left[(L^Q(T) - L^Q(t))^2\middle|\mathcal{F}(t)\right]. \tag{7.6}$$

It follows from (7.4) that $V(t, \varphi^*) = V^{*,Q}(t) - A^*(t)$, i.e. the discounted value process associated with the risk-minimizing strategy coincides with the intrinsic value process reduced by the discounted payments.

Note that the risk-minimizing strategy depends on the choice of martingale measure Q. In the literature, the minimal martingale measure has been applied for determining risk-minimizing strategies, since this essentially corresponds to the criterion of local risk-minimization, which is a criterion in terms of P, see Schweizer (2001a).

7.2 Risk-minimizing strategies for the insurance payment process

As noted in Section 6.3, the intrinsic value process $V^{*,Q}$ associated with the payment process A is given by

$$V^{*,Q}(t) = A^*(t) + (n - N(x,t))B(t)^{-1}V^Q(t, r(t), \mu(x,t)), \tag{7.7}$$

where $V^Q(t, r(t), \mu(x,t))$ is defined by (6.13). The Galtchouk-Kunita-Watanabe decomposition of $V^{*,Q}$ is determined by the following lemma:

Lemma 7.1 *The Galtchouk-Kunita-Watanabe decomposition of $V^{*,Q}$ is given by*

$$V^{*,Q}(t) = V^{*,Q}(0) + \int_0^t \xi^Q(\tau)dP^*(\tau, T) + L^Q(t), \tag{7.8}$$

where

$$V^{*,Q}(0) = -n\pi(0) + nV^Q(0, r(0), \mu(x,0)), \tag{7.9}$$

$$L^Q(t) = \int_0^t \nu^Q(\tau)dM^Q(x,\tau) + \int_0^t \kappa^Q(\tau)d\mathcal{S}^{Q,M}(x,\tau,T), \tag{7.10}$$

and

$$\xi^Q(t) = (n - N(x,t-)) \left(\int_t^T \frac{B^r(t,\tau)P^*(t,\tau)}{B^r(t,T)P^*(t,T)} \mathcal{S}^Q(x,t,\tau) \left(a_0(\tau) + a_1(\tau) f^{\mu,Q}(x,t,\tau) \right) d\tau \right.$$

$$\left. + \frac{B^r(t,\overline{T})P^*(t,\overline{T})}{B^r(t,T)P^*(t,T)} \mathcal{S}^Q(x,t,\overline{T}) \Delta A_0(\overline{T}) \right), \tag{7.11}$$

$$\nu^Q(t) = B(t)^{-1} \left(a_1(t) - V^Q(t, r(t), \mu(x,t)) \right), \tag{7.12}$$

$$\kappa^Q(t) = (n - N(x,t-)) \left(\int_t^T P^*(t,\tau) \frac{B^{\mu,Q}(x,t,\tau)\mathcal{S}^Q(x,t,\tau)}{B^{\mu,Q}(x,t,T)\mathcal{S}^{Q,M}(x,t,T)} \right.$$

$$\times \left(a_0(\tau) + a_1(\tau) \left(f^{\mu,Q}(x,t,\tau) - \frac{\frac{\partial}{\partial\tau}B^{\mu,Q}(x,t,\tau)}{B^{\mu,Q}(x,t,\tau)} \right) \right) d\tau$$

$$\left. + P^*(t,\overline{T}) \frac{B^{\mu,Q}(x,t,\overline{T})\mathcal{S}^Q(x,t,\overline{T})}{B^{\mu,Q}(x,t,T)\mathcal{S}^{Q,M}(x,t,T)} \Delta A_0(\overline{T}) \right). \tag{7.13}$$

Proof of Lemma 7.1: See Appendix A.1.

In the decomposition obtained in Lemma 7.1, the integrals with respect to the compensated counting process $M^Q(x)$ and the Q-martingale $\mathcal{S}^{Q,M}(x,\cdot,T)$ associated with the Q-survival probability comprise the non-hedgeable part of the payment process. The factor $\nu^Q(t)$ appearing in the integral with respect to $M^Q(x)$ in (7.10) represents the discounted extra cost for the insurer associated with a death within the portfolio of insured lives. It consists of the discounted value of the amount $a_1(t)$ to be paid out immediately upon death, reduced by the discounted market reserve of one policy-holder $B(t)^{-1}V^Q(t, r(t), \mu(x,t))$. In traditional life insurance, $\nu^Q(t)$ is known as the (discounted) sum at risk associated with a death in the insured portfolio at time t, see e.g. Norberg (2001); in Møller (1998), a similar result is obtained with deterministic mortality intensities.

Changes in the mortality intensity lead to new Q-survival probabilities, and this affects the expected present value under Q of future payments. This sensitivity is described by the process $\kappa^Q(t)$ appearing in (7.10), which can be interpreted as the change in the discounted value of expected future payments associated with a change in the Q-martingale associated with the Q-survival probability. It follows

from (7.6) that the intrinsic risk process is given by

$$
R(t, \varphi)
$$

$$
= E_Q \left[\left(\int_t^T \nu^Q(u) dM^Q(x, u) + \kappa^Q(u) d\mathcal{S}^{Q,M}(x, u, T) \right)^2 \middle| \mathcal{F}(t) \right]
$$

$$
= E_Q \left[\int_t^T \left(\nu^Q(u) \right)^2 d\langle M^Q \rangle(x, u) + \left(\kappa^Q(u) \right)^2 d\langle \mathcal{S}^{Q,M}(x, u, T) \rangle \middle| \mathcal{F}(t) \right]
$$

$$
= E_Q \Big[\int_t^T \left(\nu^Q(u) \right)^2 (n - N(x, u-)) (1 + g(u)) \mu(x, u) du
$$

$$
+ \left(\kappa^Q(u)(1 + g(u)) \sigma^\mu(x, u) \sqrt{\mu(x, u)} B^{\mu,Q}(x, u, T) \mathcal{S}^{Q,M}(x, u, T) \right)^2 du \Big| \mathcal{F}(t) \Big].
$$

Here, we have used the square bracket processes, that $M^Q(x)$ is an adapted process with finite variation, and that $\mathcal{S}^{Q,M}(x, \cdot, T)$ is continuous (hence predictable), such that the martingales are orthogonal.

Using the general results on risk-minimization and Lemma 7.1, we get the following result.

Theorem 7.2 *The unique 0-admissible risk-minimizing strategy for the payment process (6.9) is*

$$
(\xi^*(t), \eta^*(t)) = \left(\xi^Q(t), (n - N(x, t)) B(t)^{-1} V^Q(t, r(t), \mu(x, t)) - \xi^Q(t) P^*(t, T) \right),
$$

where ξ^Q is given by (7.11).

This result is similar to the risk-minimizing hedging strategy obtained in Møller (2001c, Theorem 3.4). However, our results differ from the ones obtained there in that the market reserves depend on the current value of the mortality intensity. The fact that the strategies are similar is reasonable, since we are essentially adding a stochastic mortality to the model of Møller, and this does not change the market in which the hedger is allowed to trade. As in Møller (2001c), the discounted value process associated with the risk-minimizing strategy φ^* is

$$
V(t, \varphi^*) = (n - N(x, t)) B(t)^{-1} V^Q(t, r(t), \mu(x, t)), \tag{7.14}
$$

where $V^Q(t, r(t), \mu(x, t))$ is given by (6.13). This shows that the portfolio is currently adjusted, such that the value at any time t is exactly the market reserve. Inserting (7.9) and (7.10) in (7.5) gives

$$
C(t, \varphi^*) = nV^Q(0, r(0), \mu(x, 0)) - n\pi(0)
$$

$$
+ \int_0^t \nu^Q(\tau) dM^Q(x, \tau) + \int_0^t \kappa^Q(\tau) d\mathcal{S}^{Q,M}(x, \tau, T).
$$

Hence the hedger's loss is driven by $M^Q(x)$ and $\mathcal{S}^{Q,M}(x, \cdot, T)$. The first three terms are similar to the ones obtained by Møller (2001c). The last term, which accounts for costs associated with changes in the mortality intensity, did not appear in his model, since he worked with deterministic mortality intensities.

Example 7.3 Consider the case where $\overline{T} = T$, and where all n insured purchase a pure endowment of $\Delta A_0(T)$ paid by a single premium at time 0. In this case, the Galtchouk-Kunita-Watanabe decomposition (7.8) of $V^{*,Q}$ is determined via

$$\xi^Q(t) = (n - N(x,t-))\mathcal{S}^Q(x,t,T)\Delta A_0(T),$$
$$\nu^Q(t) = -P^*(t,T)\mathcal{S}^Q(x,t,T)\Delta A_0(T),$$
$$\kappa^Q(t) = (n - N(x,t-))P^*(t,T)e^{\int_0^t \mu^Q(x,u)du}\Delta A_0(T),$$

since $V^Q(t,r(t),\mu(x,t)) = P(t,T)\mathcal{S}^Q(x,t,T)\Delta A_0(T)$, and

$$\frac{\mathcal{S}^Q(x,t,T)}{\mathcal{S}^{Q,M}(x,t,T)} = e^{\int_0^t \mu^Q(x,u)du}.$$

This gives the 0-admissible risk-minimizing strategy

$$\xi^*(t) = (n - N(x,t-))\mathcal{S}^Q(x,t,T)\Delta A_0(T),$$
$$\eta^*(t) = (N(x,t-) - N(x,t))\,P^*(t,T)\mathcal{S}^Q(x,t,T)\Delta A_0(T).$$

The risk-minimizing strategy has the following interpretation: The number of bonds held at time t is equal to the Q-expected number of bonds needed in order to cover the benefits at time T, conditional on the information available at time $t-$. The investments in the savings account only differ from 0 if a death occurs at time t, and in this case it consists of a withdrawal (loan) equal to the market reserve for one insured individual who is alive.

$\square$

8 Mean-variance indifference pricing

Methods developed for incomplete markets have been applied for the handling of the combined risk inherent in a life insurance contract in Møller (2001b, 2002, 2003a, 2003b) with focus on the mean-variance indifference pricing principles of Schweizer (2001b). In this section, these results are reviewed and indifference prices and optimal hedging strategies are derived.

8.1 A review of mean-variance indifference pricing

Denote by K^* the discounted wealth of the insurer at time T and consider the mean-variance utility-functions

$$u_i(K^*) = E_P[K^*] - a_i(\text{Var}_P[K^*])^{\beta_i}, \tag{8.1}$$

$i = 1,2$, where $a_i > 0$ are so-called risk-loading parameters and where we take $\beta_1 = 1$ and $\beta_2 = 1/2$. It can be shown that the equations $u_i(K^*) = u_i(0)$ indeed lead to the classical actuarial variance (i=1) and standard deviation principle (i=2), respectively, see e.g. Møller (2001b).

Schweizer (2001b) proposes to apply the mean-variance utility functions (8.1) in an indifference argument which takes into consideration the possibilities for trading

in the financial markets. Denote by Θ the space of admissible strategies and let $G_T(\Theta)$ be the space of discounted trading gains, i.e. random variables of the form $\int_0^T \xi(u)dX(u)$, where X is the price process associated with the discounted traded asset. Denote by c the insurer's initial capital at time 0. The u_i-indifference price v_i associated with the liability H is defined via

$$\sup_{\vartheta \in \Theta} \left(c + v_i + \int_0^T \vartheta(u)dX(u) - H^* \right) = \sup_{\widetilde{\vartheta} \in \Theta} \left(c + \int_0^T \widetilde{\vartheta}(u)dX(u) \right), \qquad (8.2)$$

where H^* is the discounted liability. The strategy ϑ^* which maximizes the left side of (8.2) will be called the optimal strategy for H. In order to formulate the main result, some more notation is needed. We denote by $\widetilde{P}$ the variance optimal martingale measure and let $\widetilde{\Lambda}(T) = \frac{d\widetilde{P}}{dP}$. In addition, we let $\pi(\cdot)$ be the projection in $L^2(P)$ on the space $G_T(\Theta)^\perp$ and write $1 - \pi(1) = \int_0^T \widetilde{\beta}(u)dX(u)$. It follows via the projection theorem that any discounted liability H^* allows for a unique decomposition on the form

$$H^* = c^H + \int_0^T \vartheta^H(u)dX(u) + N^H, \qquad (8.3)$$

where $\int_0^T \vartheta^H dX$ is an element of $G_T(\Theta)$ and where N^H is in the space $(I\!R + G_T(\Theta))^\perp$. From Schweizer (2001b) and Møller (2001b) we have that the indifference prices for H are:

$$v_1(H) = E_{\widetilde{P}}[H^*] + a_1 \text{Var}_P[N^H], \qquad (8.4)$$

$$v_2(H) = E_{\widetilde{P}}[H^*] + a_2 \sqrt{1 - \text{Var}_P[\widetilde{\Lambda}(T)]/a_2^2} \sqrt{\text{Var}_P[N^H]}, \qquad (8.5)$$

where (8.5) is only defined if $a_2^2 \geq \text{Var}_P[\widetilde{\Lambda}(T)]$. The optimal strategies associated with these two principles are:

$$\vartheta_1^*(t) = \vartheta^H(t) + \frac{1 + \text{Var}_P[\widetilde{\Lambda}(T)]}{2a_1}\widetilde{\beta}(t), \qquad (8.6)$$

$$\vartheta_2^*(t) = \vartheta^H(t) + \frac{1 + \text{Var}_P[\widetilde{\Lambda}(T)]}{a_2\sqrt{1 - \text{Var}_P[\widetilde{\Lambda}(T)]/a_2^2}}\sqrt{\text{Var}_P[N^H]}\widetilde{\beta}(t), \qquad (8.7)$$

where (8.7) is only well-defined if $a_2^2 > \text{Var}_P[\widetilde{\Lambda}(T)]$. For more details, we refer to Møller (2001b, 2003a, 2003b).

8.2   The variance optimal martingale measure

In order to determine the variance optimal martingale measure $\widetilde{P}$ we first turn our attention to the minimal martingale measure, which loosely speaking is "the equivalent martingale measure which disturbs the structure of the model as little as possible", see Schweizer (1995). The minimal martingale measure is obtained by

letting $h^\mu = 0$ and $g = 0$. Hence, we have from Section 4 that the Radon-Nikodym derivative $\widehat{\Lambda}(T)$ for the minimal martingale measure is given by

$$\widehat{\Lambda}(T) = \exp\left(\int_0^T h^r(u)dW^r(u) - \frac{1}{2}\int_0^T (h^r(u))^2 du\right),$$

where h^r is defined by (4.6).

In general, the variance optimal martingale measure $\widetilde{P}$ and the minimal martingale measure $\hat{P}$ differ. However, we find below that they coincide in our model. Since $h^r(t)$ is $\mathcal{G}(t)$-measurable, the density $\widehat{\Lambda}(T)$ is $\mathcal{G}(T)$-measurable, and therefore it can be represented by a constant D and a stochastic integral with respect to $P^*(\cdot, T)$, see e.g. Pham, Rheinländer and Schweizer (1998, Section 4.3). Thus, we have the following representation of $\widehat{\Lambda}(T)$

$$\widehat{\Lambda}(T) = D + \int_0^T \widetilde{\zeta}(u)dP^*(u, T). \tag{8.8}$$

Schweizer (1996, Lemma 1) gives that $\widehat{\Lambda}(T)$ is the density for the variance optimal martingale measure as well, i.e.

$$\frac{d\widetilde{P}}{dP} = \widehat{\Lambda}(T),$$

such that $\hat{P} = \widetilde{P}$. Hence, under the equivalent martingale measure $\widetilde{P}$, the dynamics of the mortality intensity and the intensity of the counting process $N(x)$ are unaltered. For later use, we introduce the $\widetilde{P}$-martingale

$$\widetilde{\Lambda}(t) := E_{\widetilde{P}}[\widehat{\Lambda}(T)|\mathcal{F}(t)] = E_{\widetilde{P}}[\widehat{\Lambda}(T)|\mathcal{G}(t)].$$

Note that $\widehat{\Lambda}(T) = \widetilde{\Lambda}(T)$. If h^r is constant, calculations similar to those in Møller (2003b) for the Black-Scholes case give that

$$\frac{\widehat{\Lambda}(t)}{\widetilde{\Lambda}(t)} = e^{-(h^r)^2(T-t)}.$$

8.3 Mean-variance indifference pricing for pure endowments

Let $\overline{T} = T$ and consider a portfolio of n individuals of the same age x each purchasing a pure endowment of $\Delta A_0(T)$ paid by a single premium at time 0. Thus, the discounted liability is given by

$$H^* = (n - N(x, T))B(T)^{-1}\Delta A_0(T).$$

In this case we have the following proposition

Proposition 8.1 *The indifference prices are given by inserting the following expressions for $E_{\widetilde{P}}[H^*]$ and $\mathrm{Var}_P(N^H)$ in (8.4) and (8.5):*

$$E_{\widetilde{P}}[H^*] = nP(0, T)\mathcal{S}(x, 0, T)\Delta A_0(T), \tag{8.9}$$

and

$$\mathrm{Var}_P[N^H] = n \int_0^T \Upsilon_1(t)\Upsilon_2(t)dt + n^2 \int_0^T \Upsilon_1(t)\Upsilon_3(t)dt, \qquad (8.10)$$

where

$$\Upsilon_1(t) = E_P\left[\frac{\widehat{\Lambda}(t)}{\widetilde{\Lambda}(t)}(P^*(t,T)\Delta A_0(T))^2\right],$$

$$\Upsilon_2(t) = E_P\Big[\left(\mathcal{S}(x,t,T)\right)^2 e^{-\int_0^t \mu(x,u)du}$$

$$\times \mu(x,t)\left(1 + (\sigma^\mu(x,t)B^\mu(x,t,T))^2(1 - e^{-\int_0^t \mu(x,u)du})\right)\Big],$$

$$\Upsilon_3(t) = E_P\left[\left(\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}(x,t,T)e^{-\int_0^t \mu(x,u)du}\right)^2\right].$$

Proof of Proposition 8.1: The independence between r and (N,μ) under $\widetilde{P}$ immediately gives (8.9). The expression for the variance of N^H in (8.10) follows from calculations similar to those in Møller (2001b). For completeness the calculations are carried out in Appendix A.2.

$\square$

We see from (8.10) that the variance of N^H can be split into two terms. The first term, which is proportional to the number of insured, stems from both the systematic and unsystematic mortality risk. Møller (2001b) also obtained a term proportional to the number of insured in the case with deterministic mortality intensity and hence only unsystematic mortality risk. The second term which is proportional to the squared number of survivors stems solely from the systematic mortality risk. Hence, the uncertainty associated with the future mortality intensity becomes increasingly important, when determining indifference prices for a portfolio of pure endowments, as the size of the portfolio increases. There are two reasons for this. Firstly, changes in the mortality intensity, as opposed to the randomness associated with the deaths within the portfolio, are non-diversifiable; in particular they affect all insured individuals in the same way. Secondly, this risk is not hedgeable in the market.

Proposition 8.2 *The optimal strategies are given by inserting (8.10) and the following expression for ϑ^H in (8.6) and (8.7):*

$$\vartheta^H(t) = \xi^{\widetilde{P}}(t) - \widetilde{\zeta}(t)\int_0^{t-} \frac{1}{\widetilde{\Lambda}(u)}\left(\nu^{\widetilde{P}}(u)dM(x,u) + \kappa^{\widetilde{P}}(u)d\mathcal{S}^M(x,u,T)\right), \qquad (8.11)$$

where

$$\xi^{\widetilde{P}}(t) = (n - N(x,t-))\mathcal{S}(x,t,T)\Delta A_0(T),$$

$$\nu^{\widetilde{P}}(t) = -P^*(t,T)\mathcal{S}(x,t,T)\Delta A_0(T), \qquad (8.12)$$

$$\kappa^{\widetilde{P}}(t) = (n - N(x,t-))P^*(t,T)e^{\int_0^t \mu(x,u)du}\Delta A_0(T). \qquad (8.13)$$

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Proof of Proposition 8.2: Expression (8.11) follows from Schweizer (2001a, Theorem 4.6) (Theorem A.1), which relates the decomposition in (8.3) to the Galtchouk-Kunita-Watanabe decomposition of the $\widetilde{P}$-martingale $V^{*,\widetilde{P}}(t) = E_{\widetilde{P}}[H^*|\mathcal{F}(t)]$ given in Example 7.3.

$\square$

8.4 Mean-variance hedging

We now briefly mention the principle of mean-variance hedging used for hedging and pricing in incomplete financial markets. This short review follows a similar review in Møller (2001b). With mean-variance hedging, the aim is to determine the self-financing strategy $\hat{\varphi} = (\hat{\vartheta}, \hat{\eta})$ which minimizes

$$E_P\left[(H^* - V(T, \varphi))^2\right].$$

The main idea is thus to approximate the discounted claim H^* as closely as possible in $L^2(P)$ by the discounted terminal value of a self-financing portfolio φ. Since we consider self-financing portfolios only, the optimal portfolio is uniquely determined by the pair $(V(0, \hat{\varphi}), \hat{\vartheta})$, where $V(0, \hat{\varphi})$ is known as the approximation price for H and $\hat{\vartheta}$ is the mean-variance optimal hedging strategy. Schweizer (2001a, Theorem 4.6) gives that $V(0, \hat{\varphi}) = E_{\widetilde{P}}[H^*]$ and $\hat{\vartheta} = \vartheta^H$. Thus, we recognize the approximation price and the mean-variance hedging strategy as the first part of the mean-variance indifference prices and optimal hedging strategies, respectively. Note that even though the minimization criterion is in terms of P, the solution is given (partly) in terms of $\widetilde{P}$.

9 Numerical examples

In this section, we present some numerical examples with calculations of the market reserves of Proposition 6.2. Furthermore, we investigate two different parameterizations within the class of time-inhomogeneous CIR models and compare these to the 2003 mortality intensities and a deterministic projection for the mortality intensities.

Calculation method
A useful way of evaluating the expression (6.13) is to define auxiliary functions

$$\widehat{V}^Q(t, t') = \int_t^T e^{-\int_t^\tau (f^r(t',u) + f^{\mu,Q}(x,t',u))du} \left(a_0(\tau) + a_1(\tau)f^{\mu,Q}(x,t',\tau)\right) d\tau$$
$$+ e^{-\int_t^T (f^r(t',u) + f^{\mu,Q}(x,t',u))du} \Delta A_0(\overline{T}), \tag{9.1}$$

where the zero coupon bond price and the Q-survival probabilities are expressed in terms of the relevant forward rates and Q-forward mortality intensities. Note moreover, that we have introduced the additional parameter t'. We note that $V^Q(t, r(t), \mu(x,t)) = \widehat{V}^Q(t,t)$, whereas these two quantities differ if $t \neq t'$. It follows immediately, that on the set $(0, \overline{T}) \cup (\overline{T}, T)$, $\widehat{V}^Q(t, t')$ satisfies for fixed t' the

differential equation

$$\frac{\partial}{\partial t}\widehat{V}^Q(t,t') = (f^r(t',t) + f^{\mu,Q}(x,t',t))\widehat{V}^Q(t,t') - a_0(t) - a_1(t)f^{\mu,Q}(x,t',t), \quad (9.2)$$

subject to the terminal condition $\widehat{V}^Q(T,t') = 0$ and with

$$\widehat{V}^Q(\overline{T}-,t') = \Delta A_0(\overline{T}) + \widehat{V}^Q(\overline{T},t').$$

Alternatively, the expression (6.13) can be determined by solving the following partial differential equation on $(0,\overline{T}) \cup (\overline{T},T) \times I\!R \times I\!R_+$

$$0 = \frac{\partial}{\partial t}V^Q(t,r,\mu) + (\gamma^{\mu,Q}(x,t) - \delta^{\mu,Q}(x,t)\mu)\frac{\partial}{\partial \mu}V^Q(t,r,\mu)$$

$$+ \left(\gamma^{r,\alpha,Q} - \delta^{r,\alpha,Q}r\right)\frac{\partial}{\partial r}V^Q(t,r,\mu) + \frac{1}{2}(\gamma^{r,\sigma} + \delta^{r,\sigma}r)\frac{\partial^2}{\partial r^2}V^Q(t,r,\mu) - rV^Q(t,r,\mu)$$

$$+ a_0(t) + (1 + g(t))\mu(a_1(t) - V^Q(t,r,\mu)) + \frac{1}{2}(\sigma^\mu(x,t))^2\mu\frac{\partial^2}{\partial \mu^2}V^Q(t,r,\mu),$$

with terminal condition $V^Q(T,r,\mu) = 0$ and with

$$V^Q(\overline{T}-,r,\mu) = \Delta A_0(\overline{T}) + V^Q(\overline{T},r,\mu).$$

The partial differential equation follows either as a byproduct from the proof of Lemma 7.1 in Appendix A.1 or as a special case of the generalized Thiele's differential equation in Steffensen (2000). A similar partial differential equation can be found in Dahl (2004).
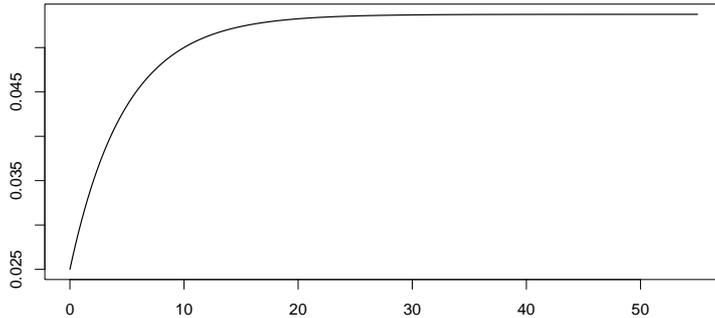

Figure 5: *Forward rate curve for the Vasiček model.*

Parameters for financial market

We now present the parameters which will be used in the numerical examples. The financial market will be described via a standard Vasiček model with parameters $\gamma^{r,\alpha} = 0.008$, $\delta^{r,\alpha} = 0.2$, $\gamma^{r,\sigma} = 0.0001$, $\delta^{r,\sigma} = 0$, $\widetilde{c} = -0.003$ and $r_0 = 0.025$. Given these parameters, the mean reversion level for the short rate is $\gamma^{r,\alpha}/\delta^{r,\alpha} = 0.04$ under P and $(\gamma^{r,\alpha} - \widetilde{c})/\delta^{r,\alpha} = 0.055$ under Q. The short rate volatility is given by $\sqrt{\gamma^{r,\sigma}} = 0.01$ and the speed of mean reversion is $\delta^{r,\alpha} = 0.2$. The parameter

$-\widetilde{c}/\delta^{r,\alpha} = 0.015$ can be interpreted as the typical difference between the long and short term zero coupon yield, see Poulsen (2003) for more details. The initial short rate is given by $r_0 = 0.025$, which corresponds to the present short rate level. The forward rate curve $f^r(0,\tau)$ can be found in Figure 5.

	Males			Females		
Year	α	β	c	α	β	c
1980	0.000233	0.0000658	1.0959	0.000220	0.0000197	1.1063
2003	0.000134	0.0000353	1.1020	0.000080	0.0000163	1.1074

Table 1: *Estimated Gompertz-Makeham parameters for 1980 and 2003.*

Parameters for insurance portfolio
We have fitted the parameters for the underlying Gompertz-Makeham distributions at various time. In Table 1 below, we present the numbers for 1980 and 2003 which

	$\delta(x,t)$	$\gamma(x,t)$	$\sigma(x,t)$
Case I	$\widetilde{\delta}$	$\widetilde{\delta}e^{-\widetilde{\gamma}t}$	$\widetilde{\sigma}$
Case II	$\widetilde{\gamma}$	$\frac{1}{2}\widetilde{\sigma}^2$	$\widetilde{\sigma}$

Table 2: *Parametrization for the underlying process ζ.*

have been obtained by standard methods. We now list some parameters for the underlying mortality improvement process ζ defined by (3.6), which is supposed to capture the variation present in Figure 4 from Section 2. We consider two different

	$\widetilde{\delta}$	$\widetilde{\gamma}$	$\widetilde{\sigma}$	5%	25%	50%	75%	95%
Case I	0.2	0.008	0.02	0.838	0.867	0.887	0.907	0.937
	1	0.008	0.02	0.837	0.850	0.859	0.868	0.881
	0.2	0.008	0.03	0.814	0.856	0.886	0.917	0.962
	1	0.008	0.03	0.827	0.846	0.859	0.872	0.892
Case II		0.008	0.02	0.726	0.801	0.854	0.909	0.990

Table 3: *Quantiles for the mortality improvement process ζ for time horizon 20 years. (Numbers are based on 100000 simulations with 100 steps per year in an Euler scheme. The results are indistinguishable if we altenatively use a Milstein scheme).*

parameterizations, Case I and Case II, see Table 2. Case I: We take $\delta(x,t) = \widetilde{\delta}$ constant and assume that $\gamma(x,t) = \widetilde{\delta}e^{-\widetilde{\gamma}t}$, where $\log(1+\widetilde{\gamma})$ represents the expected yearly relative decline in the mortality intensity. Thus, $e^{-\widetilde{\gamma}t}$ is the (time-dependent) level to which the process ζ adapts and $\widetilde{\delta}$ controls how fast it adapts to this level. Finally, we propose to let $\sigma(x,t) = \widetilde{\sigma}$, which describes the "noise". Case II: Here, we

let $\delta(x,t) = \widetilde{\gamma}$, $\gamma(x,t) = \frac{1}{2}\widetilde{\sigma}^2$ and $\sigma(x,t) = \widetilde{\sigma}$. This means that we expect a relative yearly decline in ζ of approximately $\widetilde{\gamma}$. Note that Case II has one parameter less than case I. The choice $\gamma(x,t) = \frac{1}{2}\widetilde{\sigma}^2$ in Case II ensures that ζ remains strictly positive. Quantiles for the mortality improvement process ζ for the two parameterizations can be found in Table 3. In Case II, the mean reversion level is $\gamma(t,x)/\delta(t,x) = \frac{1}{2}\widetilde{\sigma}^2/\widetilde{\gamma}$. With $\widetilde{\gamma} = 0.008$ and $\widetilde{\sigma} = 0.02$, this leads to the mean reversion level of 0.025, which is negligible. A comparison of the mortality for 2003 for males and the corresponding
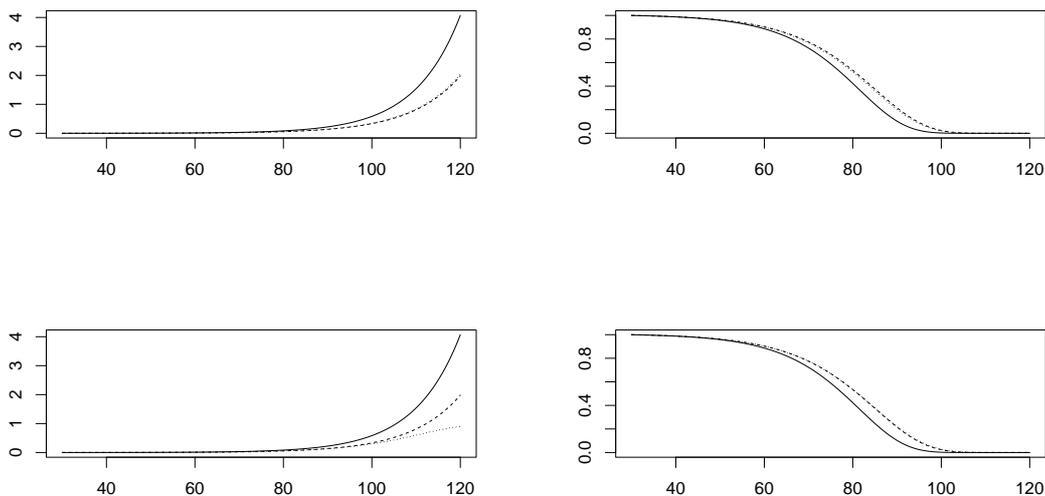

Figure 6: *Top pictures are Case I and bottom pictures are Case II. To the left: Mortality intensity curve for 30 year old males for 2003 (solid line), exponentially corrected with factor $\exp(-\widetilde{\gamma}t)$ (dashed line) and forward mortality intensities (dotted line). To the right: The corresponding survival probabilities.*

forward mortality intensities in Case I with parameters $(\widetilde{\delta}, \widetilde{\gamma}, \widetilde{\sigma}) = (0.2, 0.008, 0.02)$ can be found at the top of Figure 6. The figure shows a rather limited difference between the forward mortality intensities and the exponentially corrected intensities (they essentially coincide), whereas there is a big difference between these two curves and the 2003 estimate for the mortality intensities. For Case II, there is a more substantial difference between the forward mortality intensities and the exponentially corrected mortality intensities at very high ages.

Expected lifetimes
Figure 7(a) shows the histogram for the expected lifetime of a policyholder aged 30 for case I with parameters $(0.2, 0.008, 0.03)$. As a comparison, the expected lifetime for a male policyholder aged 30 is 75.8, 79.0 and 78.6 if we use the 2003 estimate, the exponentially corrected mortality and the forward mortality intensities, respectively. The variability in the figure reflects the uncertainty related to changes in the future mortality intensities. The histogram shows that there is a relatively small uncertainty associated with the expected lifetime in Case I. This is explained by the fact that the model for the mortality improvement process is mean-reverting with a relatively small volatility. If we instead consider case II, the expected lifetime
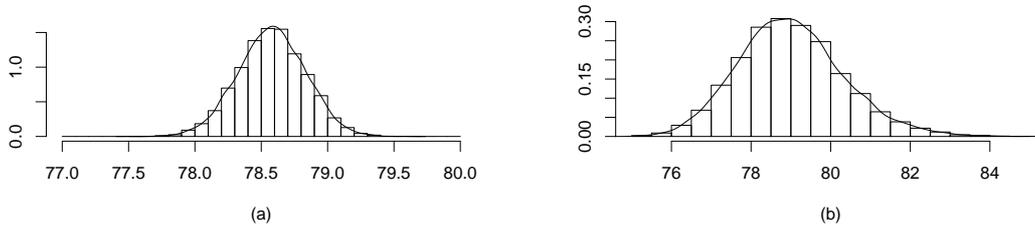
(a)

(b)

Figure 7: *Histograms for the expected lifetime for a policy-holder aged 30 for Case I with parameters* $(0.2, 0.008, 0.03)$ *(figure a) and Case II (figure b). (Histograms are based on 10000 simulations with 100 steps per year in an Euler scheme.)*

changes to 79.2, and we now get substantially bigger variation into the expected lifetimes, see the histogram for the expected lifetime in Figure 7(b).

Market reserves

In Figure 8, we have plotted the functions $\widehat{V}^Q(t, t'|x)$ for fixed $t = t' = 0$ as a function of age x in the case where Q is the minimal martingale measure. (Here, we have added an x to the function $\widehat{V}^Q$ in order to underline its dependence on the initial age x.) We have considered Case I with parameters $(0.2, 0.008, 0.03)$ and studied a life annuity starting at age 65. Moreover, we have compared this with the reserves obtained by using the 2003 estimate without any correction for future mortality improvements, and the mortality intensities obtained by reducing the mortality intensities exponentially. For each initial age x, we have calculated the relevant forward mortalities and solved the differential equation for $\widehat{V}^Q(t, t')$. We
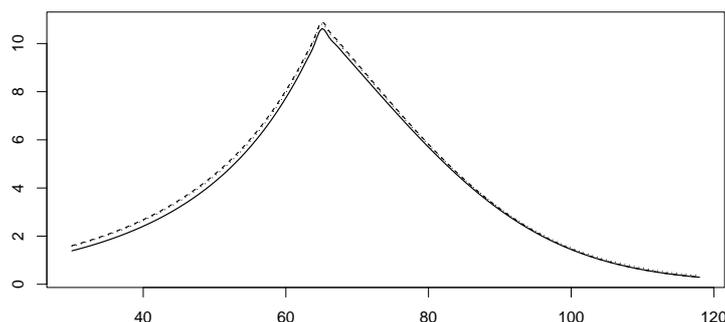

Figure 8: *Reserves for a life annuity starting at age 65. Reserve based on 2003-estimate for males (solid line), exponentially corrected mortality intensities (dashed line) and forward mortalities (dotted line).*

see only very little difference between the reserves obtained by using the forward mortality intensities and the exponentially corrected mortality intensities.

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Risk-minimizing strategies and mean-variance indifference pricing

The risk-minimizing strategies and mean-variance indifference hedging strategies obtained in Section 7 and 8 can also be determined numerically. Møller (2001b) contains a section with numerical examples for a similar contract (without systematic mortality risk), where the strategies have been determined for a couple of simulations. In addition, the methods listed there may be used for determining the mean-variance indifference prices of Proposition 8.1.

Acknowledgment

References

Andersen, P. K., Borgan, Ø., Gill, R. D. and Keiding, N. (1993). *Statistical Models Based on Counting Processes*, Springer-Verlag, New York.

Andreev, K. F. (2002). *Evolution of the Danish Population from 1835 to 2000*, Monograph no. 9/Aging Research Center, University of Southern Denmark, Odense. Download: `http://www.demogr.mpg.de/books/odense/9/`.

Becherer, D. (2003). Rational Hedging and Valuation of Integrated Risks under Constant Absolute Risk Aversion, *Insurance: Mathematics and Economics* **33**, 1–28.

Biffis, E. and Millossovich, P. (2004). The Fair Value of Guaranteed Annuity Options, Preprint, Istituto di Metodi Quantitativi, Bocconi University. Download: `http://papers.ssrn.com/sol3/papers.cfm?abstract_id=647402`.

Björk, T. (2004). *Arbitrage Theory in Continuous Time*, 2nd edn, Oxford University Press.

Cairns, A., Blake, D. and Dowd, K. (2004). Pricing Death: Frameworks for the Valuation and Securitization of Mortality Risk, Preprint, Actuarial Mathematics and Statistics, School of Mathematical and Computer Sciences, Heriot-Watt University.

Cheridito, P., Filipović, D. and Kimmel, R. L. (2003). Market Price of Risk Specifications for Affine Models: Theory and Evidence, Working Paper, Princeton University.

Cox, J., Ingersoll, J. and Ross, S. (1985). A Theory of the Term-Structure of Interest Rates, *Econometrica* **53**, 385–408.

Dahl, M. (2004). Stochastic Mortality in Life Insurance: Market Reserves and Mortality-Linked Insurance Contracts, *Insurance: Mathematics and Economics* **35**, 113–136.

Fledelius, P. and Nielsen, J. P. (2002). Prediction of Mortalities. A Comparative Danish Study, Paper presented at XXVII'th International Congress of Actuaries, Cancun, Mexico. Download: `http://www.actuaries.org/EVENTS/Congresses/Cancun/ica2002.htm`.

Föllmer, H. and Sondermann, D. (1986). Hedging of Non-Redundant Contingent Claims, *in* W. Hildenbrand and A. Mas-Colell (eds), *Contributions to Mathematical Economics*, North-Holland, 205–223.

Hull, J. and White, A. (1990). Pricing Interest-Rate Derivative Securities, *Review of Financial Studies* **3**, 573–592.

Jamshidian, F. (1995). A Simple Class of Square-Root Interest-Rate Models, *Applied Mathematical Finance* **2**, 61–72.

Maghsoodi, Y. (1996). Solution of the Extended CIR Term Structure and Bond Option Valuation, *Mathematical Finance* **6**, 89–109.

Marocco, P. and Pitacco, E. (1998). Longevity Risk and Life Annuity Reinsurance, Transactions of the 26th International Congree of Actuaries, Birmingham, England **6**, 453–479.

Milevsky, M. A. and Promislow, S. D. (2001). Mortality Derivatives and the Option to Annuitise, *Insurance: Mathematics and Economics* **29**, 299–318.

Møller, T. (1998). Risk-Minimizing Hedging Strategies for Unit-Linked Life Insurance Contracts, *ASTIN Bulletin* **28**, 17–47.

Møller, T. (2000). Quadratic Hedging Approaches and Indifference Pricing in Insurance, Ph.D. thesis, University of Copenhagen.

Møller, T. (2001a). Hedging Equity-Linked Life Insurance Contracts, *North American Actuarial Journal* **5**(2), 79–95.

Møller, T. (2001b). On Transformations of Actuarial Valuation Principles, *Insurance: Mathematics and Economics* **28**, 281–303.

Møller, T. (2001c). Risk-Minimizing Hedging Strategies for Insurance Payment Processes, *Finance and Stochastics* **5**(4), 419–446.

Møller, T. (2002). On Valuation and Risk Management at the Interface of Insurance and Finance, *British Actuarial Journal* **8**(4), 787–828.

Møller, T. (2003a). Indifference Pricing of Insurance Contracts in a Product Space Model, *Finance and Stochastics* **7**(2), 197–217.

Møller, T. (2003b). Indifference Pricing of Insurance Contracts in a Product Space Model: Applications, *Insurance: Mathematics and Economics* **32**(2), 295–315.

Norberg, R. (2000). *Basic Life Insurance Mathematics*, Lecture notes, Laboratory of Actuarial Mathematics, University of Copenhagen.

Norberg, R. (2001). On Bonus and Bonus Prognoses in Life Insurance, *Scandinavian Actuarial Journal*, 126–147.

Pham, H., Rheinländer, T. and Schweizer, M. (1998). Mean-Variance Hedging for Continuous Processes: New Proofs and Examples, *Finance and Stochastics* **2**, 173–198.

Poulsen, R. (2003). Mortgage Choice: Who should have their ARMs Capped?, Preprint, Institute for Mathematical Sciences, University of Copenhagen. Download: `http://www.math.ku.dk/~rolf/papers.html`.

Rogers, L. C. G. (1995). Which Models for Term-Structures of Interest Rates should one use, *in* D. Duffie and S. Shreve (eds), *Mathematical Finance, IMA Volume 65*, Springer, New York.

Schweizer, M. (1995). On the Minimal Martingale Measure and the Föllmer-Schweizer Decomposition, *Stochastic Analysis and Applications* **13**, 573–599.

Schweizer, M. (1996). Approximation Pricing and the Variance-Optimal Martingale Measure, *Annals of Probability* **24**(1), 206–236.

Schweizer, M. (2001a). A Guided Tour through Quadratic Hedging Approaches, *in* E. Jouini, J. Cvitanić and M. Musiela (eds), *Option Pricing, Interest Rates and Risk Management*, Cambridge University Press, 538–574.

Schweizer, M. (2001b). From Actuarial to Financial Valuation Principles, *Insurance: Mathematics and Economics* **28**, 31–47.

Steffensen, M. (2000). A No Arbitrage Approach to Thiele's Differential Equation, *Insurance: Mathematics and Economics* **27**, 201–214.

Vasiček, O. (1977). An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* **5**, 177–188.

A   Appendix

A.1   Proof of Lemma 7.1

We verify the Galtchouk-Kunita-Watanabe decomposition under the additional assumption that $V^Q(t, r, \mu) \in C^{1,2,2}$, i.e. $V^Q(t, r, \mu)$ is continuously differentiable with respect to t and twice continuously differentiable with respect to r and μ. Recall from (7.7) that the Q-martingale $V^{*,Q}$ can be written as

$$V^{*,Q}(t) = A^*(t) + (n - N(x,t))B(t)^{-1}V^Q(t, r(t), \mu(x,t)).$$

Differentiating under the integral gives

$$\frac{\partial}{\partial r}V^Q(t, r, \mu) = -\int_t^T B^r(t, \tau)P(t, \tau)\mathcal{S}^Q(x, t, \tau)\left(a_0(\tau) + a_1(\tau)f^{\mu,Q}(x, t, \tau)\right)d\tau$$
$$- B^r(t, \overline{T})P(t, \overline{T})\mathcal{S}^Q(x, t, \overline{T})\Delta A_0(\overline{T}), \tag{A.1}$$

and

$$\frac{\partial}{\partial \mu}V^Q(t, r, \mu) = -(1 + g(t))\Bigg(\int_t^T P(t, \tau)B^{\mu,Q}(x, t, \tau)\mathcal{S}^Q(x, t, \tau)$$
$$\times \left(a_0(\tau) + a_1(\tau)\left(f^{\mu,Q}(x, t, \tau) - \frac{\frac{\partial}{\partial \tau}B^{\mu,Q}(x, t, \tau)}{B^{\mu,Q}(x, t, \tau)}\right)\right)d\tau$$
$$+ P(t, \overline{T})B^{\mu,Q}(x, t, \overline{T})\mathcal{S}^Q(x, t, \overline{T})\Delta A_0(\overline{T})\Bigg), \tag{A.2}$$

where we have used

$$\frac{\partial}{\partial \mu}f^{\mu,Q}(x, t, \tau) = (1 + g(t))\frac{\partial}{\partial \tau}B^{\mu,Q}(x, t, \tau).$$

Integration by parts used on $(n - N(x,t))B(t)^{-1}V^Q(t, r(t), \mu(x,t))$ yields

$$
\begin{aligned}
V^{*,Q}(t) = {}& A^*(t) + nV^Q(0, r(0), \mu(x,0)) \\
&+ \int_0^t (n - N(x,u))V^Q(u, r(u), \mu(x,u))dB(u)^{-1} \\
&+ \int_0^t B(u)^{-1}(n - N(x,u-))dV^Q(u, r(u), \mu(x,u)) \\
&- \int_0^t B(u)^{-1}V^Q(u, r(u), \mu(x,u))dN(x,u). \tag{A.3}
\end{aligned}
$$

In order to calculate the fourth term in (A.3), we need to find $dV^Q(u, r(u), \mu(x,u))$. Recall from (4.7) and (6.5) that the dynamics of r and $\mu(x)$ under Q are given by

$$
\begin{aligned}
dr(t) &= \alpha^{r,Q}(r(t))dt + \sigma^r(t, r(t))dW^{r,Q}(t), \\
d\mu(x,t) &= \alpha^{\mu,Q}(t, \mu(x,t))dt + \sigma^\mu(t, \mu(x,t))\sqrt{\mu(x,t)}dW^{\mu,Q}(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha^{r,Q}(r(t)) &= \gamma^{r,\alpha,Q} - \delta^{r,\alpha,Q}r(t), \\
\alpha^{\mu,Q}(t, \mu(x,t)) &= \gamma^{\mu,Q}(x,t) - \delta^{\mu,Q}(x,t)\mu(x,t).
\end{aligned}
$$

In the rest of the proof we use the shorthand notation $V^Q(u) = V^Q(u, r(u), \mu(x,u))$. Furthermore we only include explicitly the time argument in the coefficient functions. The additional assumption $V^Q(t) \in C^{1,2,2}$ allows us to apply Itô's formula. We obtain

$$
\begin{aligned}
dV^Q(u) = {}& \left(\frac{\partial}{\partial u}V^Q(u) + \alpha^{\mu,Q}(u)\frac{\partial}{\partial \mu}V^Q(u) + \frac{1}{2}(\sigma^\mu(u))^2\mu(x,u)\frac{\partial^2}{\partial \mu^2}V^Q(u) \right. \\
&\left. + \alpha^{r,Q}(u)\frac{\partial}{\partial r}V^Q(u) + \frac{1}{2}(\sigma^r(u))^2\frac{\partial^2}{\partial r^2}V^Q(u) \right) du + \sigma^r(u)\frac{\partial}{\partial r}V^Q(u)dW^{r,Q}(u) \\
&+ \sigma^\mu(u)\sqrt{\mu(x,u)}\frac{\partial}{\partial \mu}V^Q(u)dW^{\mu,Q}(u) \\
= {}& \left(\frac{\partial}{\partial u}V^Q(u) + \alpha^{\mu,Q}(u)\frac{\partial}{\partial \mu}V^Q(u) + \frac{1}{2}(\sigma^\mu(u))^2\mu(x,u)\frac{\partial^2}{\partial \mu^2}V^Q(u) \right. \\
&\left. + \alpha^{r,Q}(u)\frac{\partial}{\partial r}V^Q(u) + \frac{1}{2}(\sigma^r(u))^2\frac{\partial^2}{\partial r^2}V^Q(u) \right) du \\
&- \frac{\frac{\partial}{\partial r}V^Q(u)}{B^r(u,T)P^*(u,T)}dP^*(u,T) \\
&- \frac{\frac{\partial}{\partial \mu}V^Q(u)}{(1 + g(u))B^{\mu,Q}(x,u,T)\mathcal{S}^{Q,M}(x,u,T)}d\mathcal{S}^{Q,M}(x,u,T).
\end{aligned}
$$

In the first equality we have used the dynamics of r and $\mu(x)$ and that the Brownian motions $W^{r,Q}$ and $W^{\mu,Q}$ are independent, such that we do not get any mixed second order terms. In the second equality we use (A.1) and (A.2) together with

the dynamics of $\mathcal{S}^{Q,M}(x,\cdot,T)$ and $P^*(\cdot,T)$ given in (6.7) and (4.10), respectively. Rewriting A^* in terms of the Q-martingale $M^Q(x)$ we get

$$
\begin{aligned}
A^*(t) = {} & -n\pi(0) \\
& + \int_0^t B(\tau)^{-1}\left(a_0(\tau)(n - N(x,\tau)) + a_1(\tau)(n - N(x,\tau-))\mu^Q(x,\tau)\right)d\tau \\
& + \int_0^t B(\overline{T})^{-1}(n - N(x,\overline{T}))\Delta A_0(\overline{T})dI_{\{\tau \geq \overline{T}\}} + \int_0^t B(\tau)^{-1}a_1(\tau)dM^Q(x,\tau).
\end{aligned}
$$

Collecting the terms from (A.3) involving integrals with respect to the martingales $P^*(\cdot,T)$, $\mathcal{S}^{Q,M}(x,\cdot,T)$ and $M^Q(x)$, respectively, we get the last three terms in (7.8). Since these three terms and $V^{*,Q}$ are Q-martingales, the remaining terms constitute a Q-martingale as well. Since this process is continuous (hence predictable) and of finite variation, it is constant. Inserting $t = 0$ we immediately get that $V^{*,Q}(0) = -n\pi(0) + nV^Q(0, r(0), \mu(x,0))$. Thus, we have proved the decomposition in (7.8).

A.2   Calculation of $\mathbf{Var}_P[N^H]$

The following theorem due to Schweizer (2001a, Theorem 4.6) relates the decomposition in (8.3) to the Galtchouk-Kunita-Watanabe decomposition of the $\widetilde{P}$-martingale $V^{*,\widetilde{P}}(t) = E_{\widetilde{P}}[H^*|\mathcal{F}(t)]$; see also Møller (2000).

Theorem A.1 *Assume that $H^* \in L^2(\mathcal{F}(T), P)$ and consider the Galtchouk-Kunita-Watanabe decomposition of $V^{*,\widetilde{P}}(t)$ given by*

$$
V^{*,\widetilde{P}}(t) = E_{\widetilde{P}}[H^*] + \int_0^t \xi^{\widetilde{P}}(u)dP^*(u,T) + L^{\widetilde{P}}(t), \quad 0 \leq t \leq \overline{T}. \tag{A.4}
$$

We can now express c^H, ϑ^H and N^H from (8.3) in terms of decomposition (A.4) by

$$
c^H = E_{\widetilde{P}}[H^*],
$$

$$
\vartheta^H(t) = \xi^{\widetilde{P}}(t) - \widetilde{\zeta}(t)\int_0^{t-} \frac{1}{\widetilde{\Lambda}(u)}dL^{\widetilde{P}}(u),
$$

$$
N^H = \widetilde{\Lambda}(\overline{T})\int_0^{\overline{T}} \frac{1}{\widetilde{\Lambda}(u)}dL^{\widetilde{P}}(u).
$$

Since

$$
L^{\widetilde{P}}(t) = \int_0^t \nu^{\widetilde{P}}(u)dM(x,u) + \int_0^t \kappa^{\widetilde{P}}(u)d\mathcal{S}(x,u,T),
$$

where $\nu^{\widetilde{P}}$ and $\kappa^{\widetilde{P}}$ are given by (8.12) and (8.13), respectively, Theorem A.1 gives the following expression for N^H:

$$
N^H = \widetilde{\Lambda}(T)\int_0^T \frac{1}{\widetilde{\Lambda}(t)}dL^{\widetilde{P}}(t) = \widetilde{\Lambda}(T)\int_0^T \frac{1}{\widetilde{\Lambda}(t)}\left(\nu^{\widetilde{P}}(t)dM(x,t) + \kappa^{\widetilde{P}}(t)d\mathcal{S}^M(x,t,T)\right).
$$

Since $E_P[N^H] = 0$, we first note that

$$\text{Var}_P[N^H] = E_P[(N^H)^2] = E_{\widetilde{P}}\left[\widetilde{\Lambda}(T)\left(\widetilde{L}(T) + \widetilde{R}(T)\right)^2\right]$$

$$= E_{\widetilde{P}}\left[\widetilde{\Lambda}(T)(\widetilde{L}(T))^2 + 2\widetilde{\Lambda}(T)\widetilde{L}(T)\widetilde{R}(T) + \widetilde{\Lambda}(T)(\widetilde{R}(T))^2\right], \qquad \text{(A.5)}$$

where we have defined $\widetilde{L}(t) = \int_0^t \frac{\nu^{\widetilde{P}}(u)}{\widetilde{\Lambda}(u)}dM(x,u)$ and $\widetilde{R}(t) = \int_0^t \frac{\kappa^{\widetilde{P}}(u)}{\widetilde{\Lambda}(u)}d\mathcal{S}^M(x,u,T)$. The three terms appearing in (A.5) can be rewritten using Itô's formula. For the first term we get

$$\widetilde{\Lambda}(T)(\widetilde{L}(T))^2 = \int_0^T (\widetilde{L}(t-))^2 d\widetilde{\Lambda}(t) + 2\int_0^T \widetilde{\Lambda}(t)\widetilde{L}(t-)d\widetilde{L}(t)$$
$$+ \int_0^T \widetilde{\Lambda}(t)\left(\frac{\nu^{\widetilde{P}}(t)}{\widetilde{\Lambda}(t)}\right)^2 dN(x,t),$$

and for the last term we find that

$$\widetilde{\Lambda}(T)(\widetilde{R}(T))^2 = \int_0^T \widetilde{R}(t)^2 d\widetilde{\Lambda}(t) + 2\int_0^T \widetilde{\Lambda}(t)\widetilde{R}(t)d\widetilde{R}(t) + \int_0^T \widetilde{\Lambda}(t)d\langle\widetilde{R}\rangle(t)$$

$$= \int_0^T \widetilde{R}(t)^2 d\widetilde{\Lambda}(t) + 2\int_0^T \widetilde{\Lambda}(t)\widetilde{R}(t)d\widetilde{R}(t)$$

$$+ \int_0^T \widetilde{\Lambda}(t)\left(\frac{\kappa^{\widetilde{P}}(t)}{\widetilde{\Lambda}(t)}\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}^M(x,t,T)\right)^2 dt.$$

The mixed term becomes

$$\widetilde{\Lambda}(T)\widetilde{R}(T)\widetilde{L}(T) = \int_0^T \widetilde{\Lambda}(t)\widetilde{R}(t)d\widetilde{L}(t) + \int_0^T \widetilde{L}(t)\widetilde{R}(t)d\widetilde{\Lambda}(t) + \int_0^T \widetilde{\Lambda}(t)\widetilde{L}(t)d\widetilde{R}(t)$$

$$+ \int_0^T \widetilde{L}(t)d[\widetilde{R}, \widetilde{\Lambda}](t).$$

Assuming all the local martingales are martingales, and using that the Brownian motions driving r and μ are independent, we get

$$\text{Var}_P\left[N^H\right] = E_{\widetilde{P}}\left[\int_0^T \frac{(\nu^{\widetilde{P}}(t))^2}{\widetilde{\Lambda}(t)}dN(x,t)\right] \qquad \text{(A.6)}$$

$$+ E_{\widetilde{P}}\left[\int_0^T \frac{\left(\kappa^{\widetilde{P}}(t)\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}^M(x,t,T)\right)^2}{\widetilde{\Lambda}(t)}dt\right].$$

We now investigate the two terms in (A.6) separately. The first term can be rewritten as

$$
E_{\widetilde{P}}\left[\int_0^T \frac{(\nu^{\widetilde{P}}(t))^2}{\widetilde{\Lambda}(t)} dN(x,t)\right]
$$

$$
= \int_0^T E_{\widetilde{P}}\left[\frac{(P^*(t,T)\Delta A_0(T))^2}{\widetilde{\Lambda}(t)}\right] E_{\widetilde{P}}\left[(\mathcal{S}(x,t,T))^2 (n - N(x,t-))\mu(x,t)\right] dt
$$

$$
= \int_0^T E_P\left[\frac{\widehat{\Lambda}(t)}{\widetilde{\Lambda}(t)}(P^*(t,T)\Delta A_0(T))^2\right] E_P\left[(\mathcal{S}(x,t,T))^2 (n - N(x,t-))\mu(x,t)\right] dt,
$$

where we have used the expression for $\nu^{\widetilde{P}}(t)$ from (8.12) and the independence between r and (N,μ). The second term is given by

$$
E_{\widetilde{P}}\left[\int_0^T \frac{\left(\kappa^{\widetilde{P}}(t)\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}^M(x,t,T)\right)^2}{\widetilde{\Lambda}(t)} dt\right]
$$

$$
= E_{\widetilde{P}}\left[\int_0^T \frac{\left((n - N(x,t-))P^*(t,T)\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}(x,t,T)\Delta A_0(T)\right)^2}{\widetilde{\Lambda}(t)} dt\right]
$$

$$
= \int_0^T E_P\left[\frac{\Lambda(t)}{\widetilde{\Lambda}(t)}(P^*(t,T)\Delta A_0(T))^2\right]
$$

$$
\times E_P\left[\left((n - N(x,t-))\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}(x,t,T)\right)^2\right] dt,
$$

where we have used the expression for $\kappa^{\widetilde{P}}(t)$ from (8.13) and once again the independence between r and (N,μ). Using that conditioned on $\mathcal{I}(t)$ the number of survivors at time t is binomially distributed with parameters $(n, e^{-\int_0^t \mu(x,u)du})$ under P, we get

$$
E_P\left[\left((n - N(x,t-))\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}(x,t,T)\right)^2\right]
$$

$$
= E_P E_P\left[\left((n - N(x,t-))\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}(x,t,T)\right)^2 \Big| \mathcal{I}(t)\right]
$$

$$
= E_P\left[\left(\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}(x,t,T)\right)^2 E_P\left[(n - N(x,t-))^2 \big| \mathcal{I}(t)\right]\right]
$$

$$
= E_P\left[\left(\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}(x,t,T)\right)^2\right.
$$

$$
\left.\times \left(ne^{-\int_0^t \mu(x,u)du}\left(1 - e^{-\int_0^t \mu(x,u)du}\right) + n^2\left(e^{-\int_0^t \mu(x,u)du}\right)^2\right)\right],
$$

and

$$E_P\left[(\mathcal{S}(x,t,T))^2\,(n-N(x,t-))\mu(x,t)\right]$$
$$= E_P E_P\left[(\mathcal{S}(x,t,T))^2\,(n-N(x,t-))\mu(x,t)\,\big|\,\mathcal{I}(t)\right]$$
$$= E_P\left[(\mathcal{S}(x,t,T))^2\,\mu(x,t)E_P\left[n-N(x,t-)\,|\,\mathcal{I}(t)\right]\right]$$
$$= n E_P\left[(\mathcal{S}(x,t,T))^2\,\mu(x,t)e^{-\int_0^t \mu(x,u)du}\right].$$

Collecting the terms proportional to n and n^2, respectively, we arrive at (8.10).