Incomplete Markets: Some Reflections

AFIR ASTIN

September 7 2005

Phelim Boyle

University of Waterloo and Tirgarvil Capital

Outline

- Introduction and Background
- Finance and insurance: Divergence and convergence
- Complete markets and incomplete markets
- Pricing: Hedging and reserving
- Case Study One: Guaranteed Annuity Options
- Case Study Two: Guaranteed Minimum Withdrawal Benefits
- Analysis of the risks
- Managing the risks in practice
- Some implementation challenges
- Summary

Actuarial science: The early years

- Profession to serve a public purpose.
- Started with Equitable 1762
- Actuaries computed premiums
- Estimated reserves
- Assessment of solvency
- Concepts used
  1. Basic probability
  2. Compound interest
- Finance ideas used were state of the art at the time

How we drifted apart

- Big advances in finance
  1. Bachelier(1900)
  2. Markowitz(1952)
  3. Sharpe Linter CAPM(1960's)
  4. Black Scholes Merton(1973)
- In the beginning actuaries tended to ignore these developments
- However new products were introduced that needed these ideas
- Financial economics now generally accepted as useful by the profession
- Hans Bühlmann's *Actuaries of the third kind* 1987 Astin editorial
- Struggle still goes on in some professional actuarial bodies

Signs of Convergence

Conferences

- Symposia: "Geld,Banken und Versicherungen" in Karlsruhe
- Interplay Between Insurance, Finance and Statistics, Italy(1998)
- Stochastics for Risk, Insurance and Finance (The Danes are Coming!), UK(2002)
- Scientific Conference on Insurance and Finance, Germany (2002)
- International Symposium on Insurance and Finance Norway (2003)
- Interface Between Quantitative Finance and Insurance(UK)(2005)
- Insurance and Finance: Two solitudes no longer
- Actuarial science and finance: Friends reunited

Finance and Insurance

- Nature of markets
  - Finance: Efficient secondary market
  - Insurance: No secondary market
- Arbitrage
  - Finance: No arbitrage
  - Insurance: Hard to make arbitrage profits
- Completeness of market
  - Finance: Often assumed to be complete
  - Insurance: Incomplete not all state contingent claims available
- Life insurance and pensions very long term contracts

Finance and Insurance:

Hedging

With complete markets and no arbitrage financial securities can be replicated (hedged). For this to work

- Model is assumed to be known
- Model parameters are assumed to be known
- No credit risk

Reserving

The traditional insurance method. Set aside capital to cover risk at some confidence level. Insurance risks are generally diversifiable. Appeal to law of large numbers.

Borrowing from each other

Premium principles

Traditional actuarial method of assigning premiums to risks.

What does insurer do with the premium?

- Seller decides on the price according to some formula.
- Ignores buyer's preferences
- Ignores other sellers

Criticism not new: Dates back to Borch. Economic based approaches by Borch and Bühlmann and others.
Perhaps think of premium principle as first step. Then consider competitive forces.

Dealing with incompleteness

When market is incomplete there is no unique price. Some significant progress.

- Föllmer-Schweizer-Sondermann approach: Minimize squared hedging error.
- Super hedging: El Karoui and Quenez
- Quantile Hedging: Föllmer Leukart

These methods have been applied to the pricing of insurance contracts by Møller. Kolkiewicz and Tan have implemented a robust hedging approach for regime switching models.

Guaranteed Annuity Options

- Insurance companies (UK) issued very long term complex options
- Started in 1950's very popular in 1970's and early 1980's
- Options combined equity risk, interest rate risk mortality risk
- Caused the downfall of Equitable Life
- Lead to critical reviews of UK actuarial profession
  - The Penrose Report
  - The Morris Review
- Insurers sold options that they could not hedge
- Price at inception almost zero

Guaranteed Annuity Options

- Provides option at age 65
- Option to convert proceeds into annuity at a fixed conversion rate
- Rate for males 111 per annum for 1000 proceeds
- If the conversion rate is better than the market option is in the money
- Like a put option on interest rates but there is more $\cdots$
- Option becomes more valuable as annuities become expensive. This happens if interest rates fall and/or mortality improves.
- Option applies to entire proceeds

Typical contract

- Contract matures at time T when policyholder is 65
- Single Premium invested in equity portfolio S
- Maturity value $S(T)$
- Market value at time T of annuity to life age 65 is $a_{65}(T)$.
- g is the guaranteed conversion rate $=9$ in our case
- Maturity value of the option

$$S(T) \ \max\left[\ \left(\frac{a_{65}(T)}{9} - 1\right), \ 0\right] \tag{1}$$

Factors that affect the cost

- Option is in the money if

$$a_{65}(T) > 9.$$

  $a_{65}(T)$ increases as interest rates fall and as mortality improves.

- Magnitude of payment is also proportional to the proceeds $S(T)$. If premium invested in equities and stock market does well then liability increases.

- Let us recall what happened

Long term interest rates fell

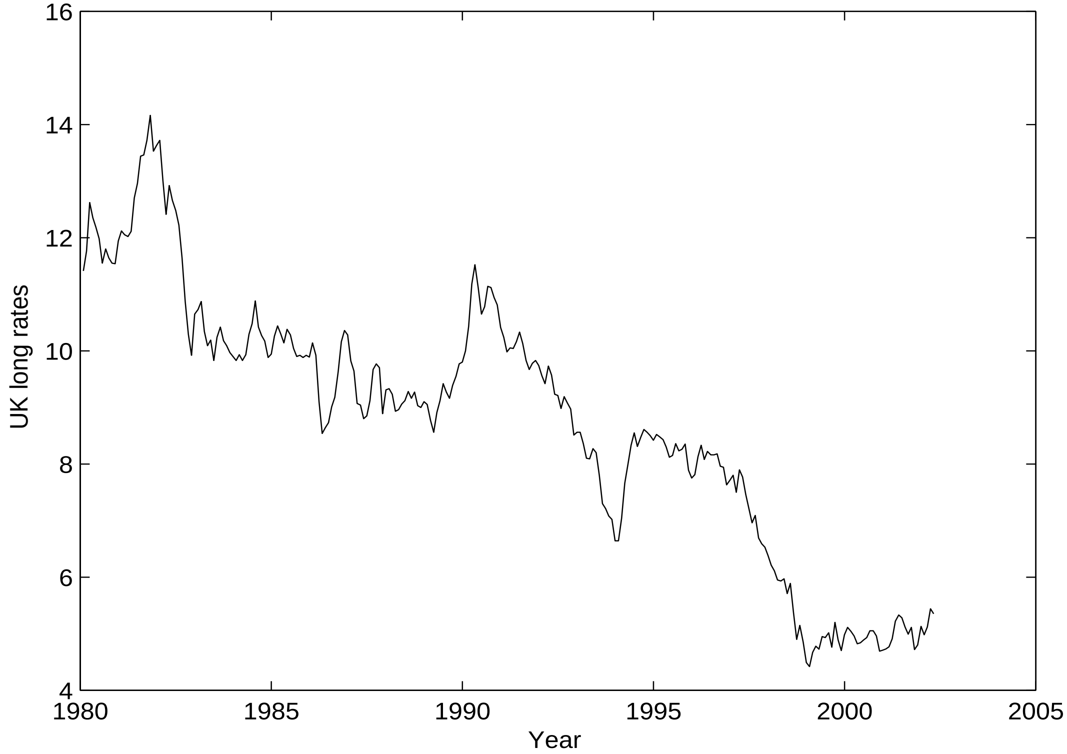


Figure 1: UK long term interest rates

Mortality Improved

During the period 1970-2000 there was a dramatic improvement in the mortality of UK males especially at the older ages. Three tables

- The a(55) table used to compute annuity values in the 1970's.
- PMA80(C10) table: UK experience for the period 1979-1982 projected to 2010
- PMA92(C20) based on UK experience for 1991-1994 and projected to 2020

Table	Expectation of life at age 65
a55	14.3
PMA80(C10)	16.9
PMA92(C20)	19.8

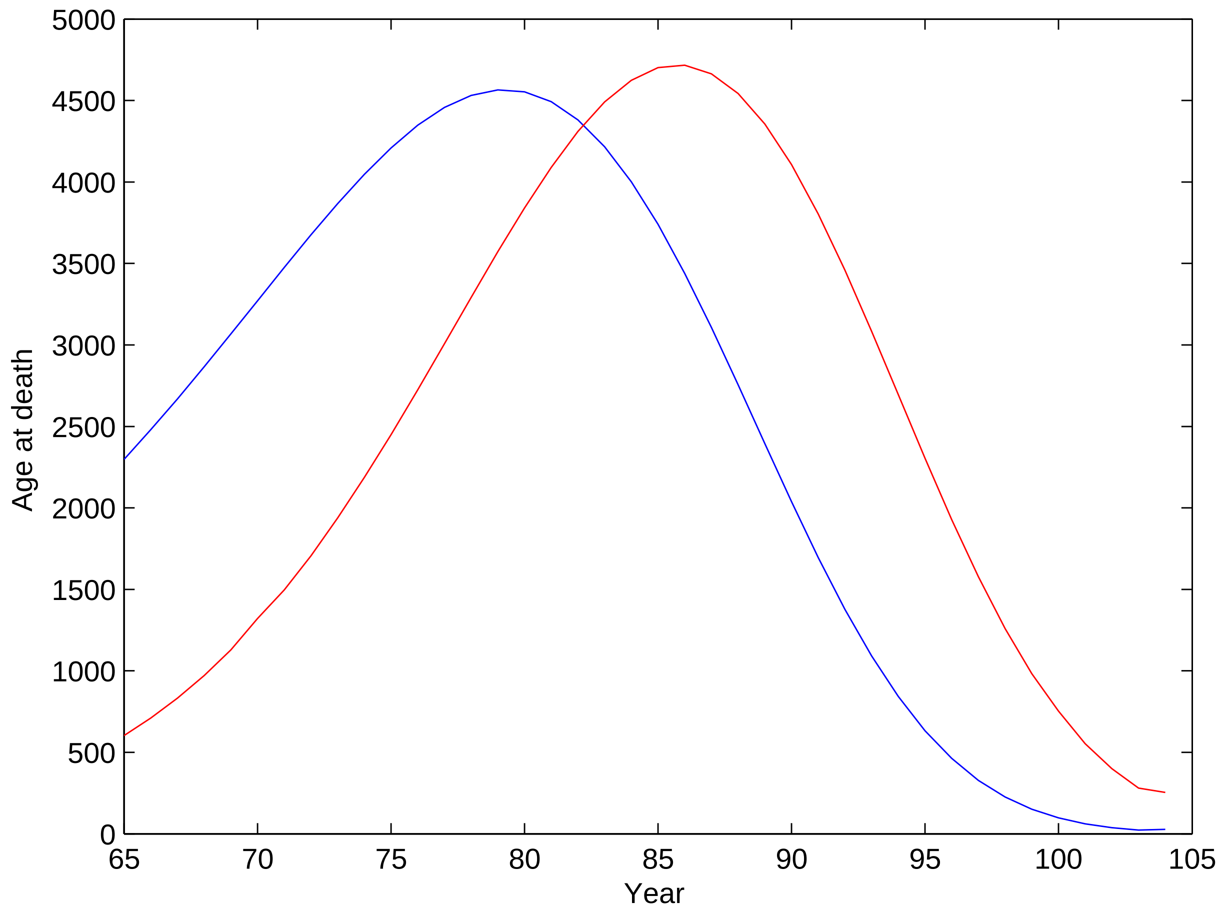


Figure 2: Distribution of age at death conditional on reaching age 65. Blue line corresponds to original assumption a55. Red line to PMA920(C20).

Interaction of interest and mortality

Because of mortality improvement the break even interest rate (strike price of option) increased significantly

- Break-even rate under the a(55) table is 5.6%.
- Under the PMA80(C10) table it is 7.0%
- under the PMA92(C20) table it is 8.2%

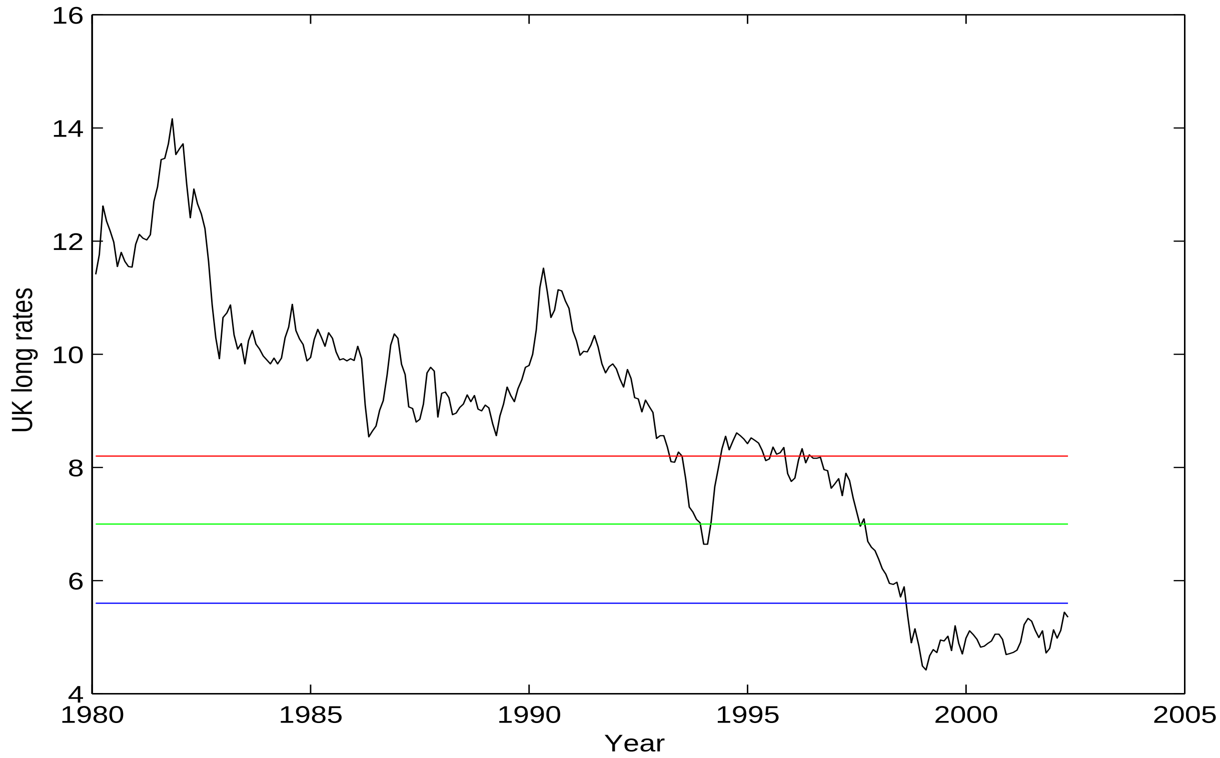


Figure 3: Graph of long term UK interest rates and break even interest rates for various mortality assumptions. Blue line 5.6% corresponds to original assumption a55. Green line 7.0% to PMA80(C10)and red 8.2% to PMA920(C20).

Summary

During 1980-2000 UK equities averaged 18% compound. Cost of guarantee increased because of

- Fall in interest rates
- Improvement in mortality
- Strong stock market performance

All 3 factors impacted the value of the option.

A Thought Experiment

Suppose we could travel back to 1980 armed with leading textbooks and research papers in modern finance and insurance. What advice would we give to the UK actuaries.

Just say no

Do not write these options

The interest rate risk

Look first the interest rate risk. Now have several sophisticated multi factor stochastic interest rate models. Which one to use? Difficult to hedge interest rate risks over a long period. We need to

- Get the right model
- Estimate parameters of the model
- Have liquid hedging instruments
- Worry about credit risk

For example suppose you were using a Vasicek or CIR model in 1980 to price long term interest rate options. What parameters would you use?

The interest rate risk

Emergence of long dated swaptions provides a natural vehicle for hedging the risk of falling interest rates. Pelsser(2003)

Swaptions help make the interest rate market more complete

However it is the joint occurrence of falling interest rates and mortality improvement that increased the likelihood of the option being in the money.

The mortality risk

- Conventional wisdom was that mortality risk is diversifiable.
- Force of mortality is μ_{x+t}. Hazard rate
- Assumed force of mortality is a deterministic function of time.
- Some recent work assumes a stochastic intensity
- Sometimes modelled with interest rate diffusions (tractability)
- Lee Carter approach: Projecting mortality and providing distribution of future mortality
- Systematic risk cannot be diversified away. Can it be hedged?

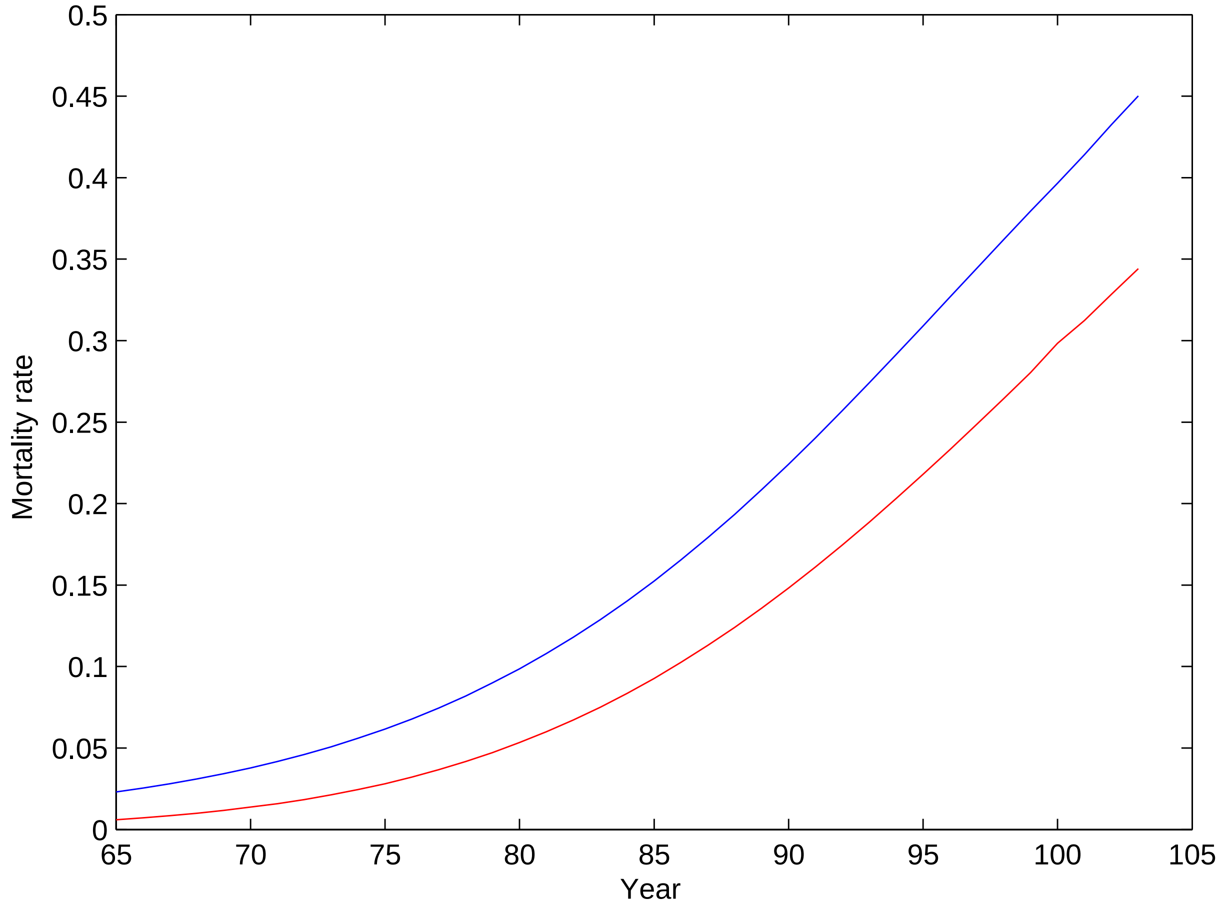


Figure 4: Age specific mortality rates for original and improved mortality assumptions. Blue line corresponds to original assumption a55. Red line to PMA920(C20).

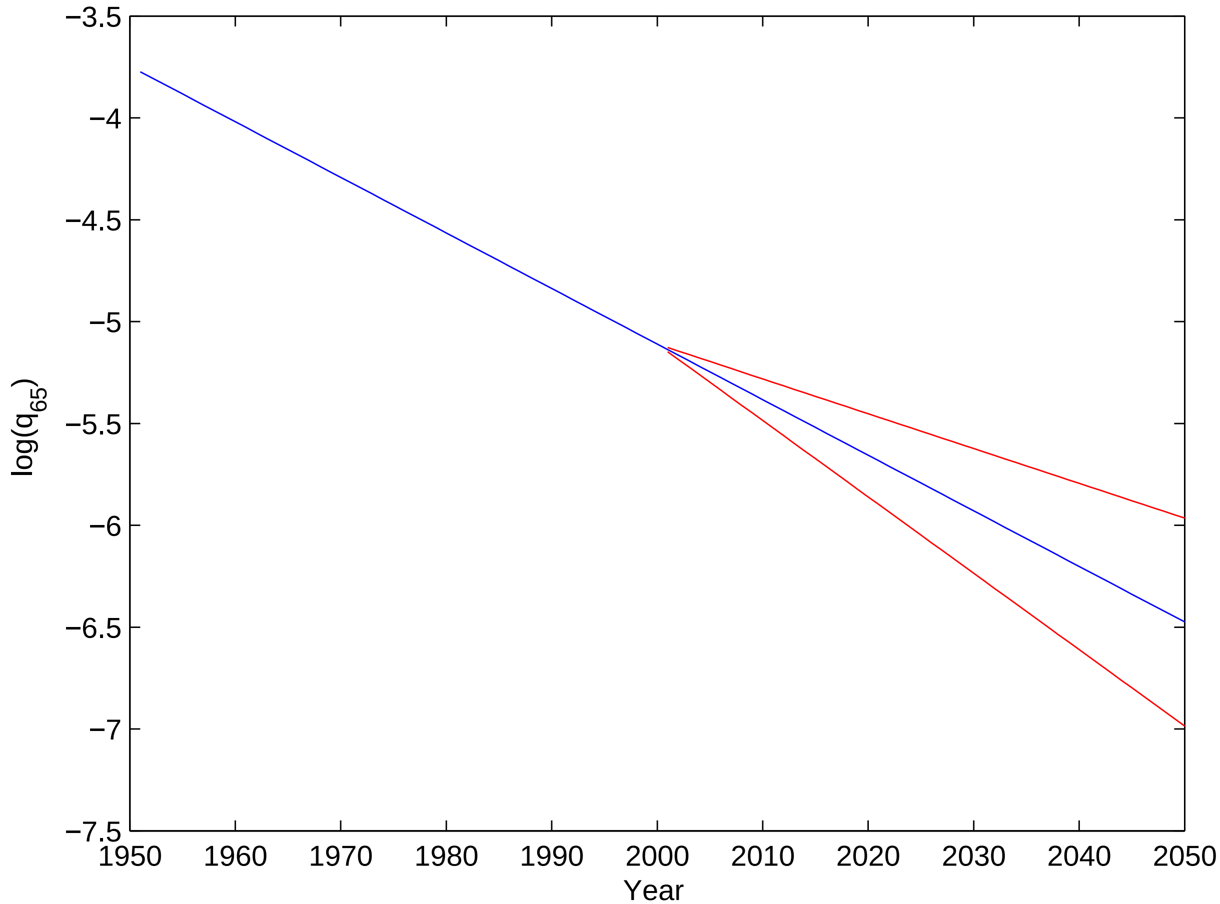


Figure 5: Graph of $log(q_{65})$ and projection with 95% confidence band. Distribution based on Lee Carter type forecast

Transferring the risk

Mortality linked securities

- BNP Paribas European Investment Bank Survivor Bond
  1. 25 year bond
  2. Coupon payments based on survivors of base group aged 65 in 2003
  3. Capacity limited

Reinsurance

Traditional method of risk sharing with insurers. Reinsurer appetite for longevity risk is very limited.

Natural Hedges?

The equity risk

Recall that size of payoff is proportional to $S(T)$. We can derive a formula for the quanto option and hedge according to this formula. However difficult to know

- Process for equity prices for next 30 years
- Interest rate process for next 30 years
- Joint process. What is the correlation?

Summary

- Option depends on three risks
- We have made progress in modelling each risk in the last 20 years .
- Challenges remain
- Ask investment bank to write a product to meet the liability
- This would be very expensive
- But it would force insurer to rethink matters

Guaranteed Minimum Withdrawal Benefit(GMWB)

- Variable Annuities
- Basic features
- Pricing in a complete market
- Actuarial variables
- Modeling in an incomplete market
- Dynamic behavior
- Concluding remarks

Background

- Big market for retirement products(lots of baby boomers)
- Investors like upside appreciation: concerned about downside risk
- Proliferation of products that combine both features
- Equity Indexed Annuities and Variable Annuities(VA)
- Variable annuities include different embedded options
- Recent innovation Guaranteed Minimum Withdrawal Benefit(GMWB)
- Very popular in the market
- Milevsky and Salisbury (2004)

GMWB

- GMWB provides a guaranteed level of income
- Suppose investor puts $100,000 in a VA: invested in equities
- She can withdraw a fixed percentage (7% is typical) every year until the initial premium is withdrawn
- Can withdraw $7,000 p.a. year for 14.28 years.
- Can withdraw funds irrespective of how the investment account performs
- Example. Market does well at first and then collapses.

Year	Rate on fund	Fund before withdrawal	Fund after withdrawal	Amount withdrawn	Balance remaining
1	10%	110,000	103,000	7,000	93,000
2	10%	113,300	106,300	7,000	86,000
3	-60%	42,520	35,520	7,000	79,000
4	-60%	14,208	7, 208	7,000	72,000
5	- 2.8857%	7,000	Zero	7,000	65,000
6	r%	0	0	7,000	58,000
⋮	⋮	⋮	⋮	⋮	⋮
14	r%	0	0	7,000	2000

The GMWB

Fee charged for the GMWB is expressed as a percentage (say fifty basis points) of either

- The investment account or
- The outstanding guaranteed withdrawal benefit.

Assumptions

- Perfect frictionless complete market
- No arbitrage
- Assume max amount withdrawn each year
- Fixed term contract over $[0, T]$
- Investment fund dynamics

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

where B_t is a Brownian motion under P
μ is the drift
σ is the volatility

Ubiquitous lognormal assumption

Investor's account

- Let V_t be the value of the investors account at time t.
- V has an absorbing barrier at zero. Suppose first time it hits zero is τ.
- Dynamics of V for $0 < t < \tau$ are

$$dV_t = [(\mu - q)V_t - g]dt + \sigma V_t dB_t$$

  where q is the fee and g is the withdrawal rate.

- If the initial investment amount is I_0 then

$$g = \frac{I_0}{T}$$

Pricing the contract in a complete market

Two ways to decompose the GMWB

Call Option decomposition

Investor pays I_0 at time zero. Benefit is a guaranteed stream of g per annum plus a European call option on the maturity amount V_T. Strike price is zero. Find fee q so that

$$I_0 = \int_0^T ge^{-ru}du \ + \ C_0$$

where r is the (constant) risk free rate and C_0 is the time zero value of the call.

Pricing the contract in a complete market

Put Option decomposition

- Investment often in mutual fund.
- Insurer guarantees to pay remaining withdrawal benefits if $V(t)$ reaches zero in $[0, T]$.
- Guarantee provided by the insurance is a put option.
- If $V(t)$ stays positive in $[0, T]$ no payment under the put.
- Put is exercised when the account balance first becomes zero.

Put Option decomposition

- When this happens the insurer pays the remaining stream of withdrawal benefits of g per annum. Value at time τ is

$$\int_{\tau}^{T} g \; e^{-ru} du$$

- Put option has a random exercise time τ. Put is funded by the fee payable until time τ.

- Guarantee backed solely by the claims paying ability of insurance co.

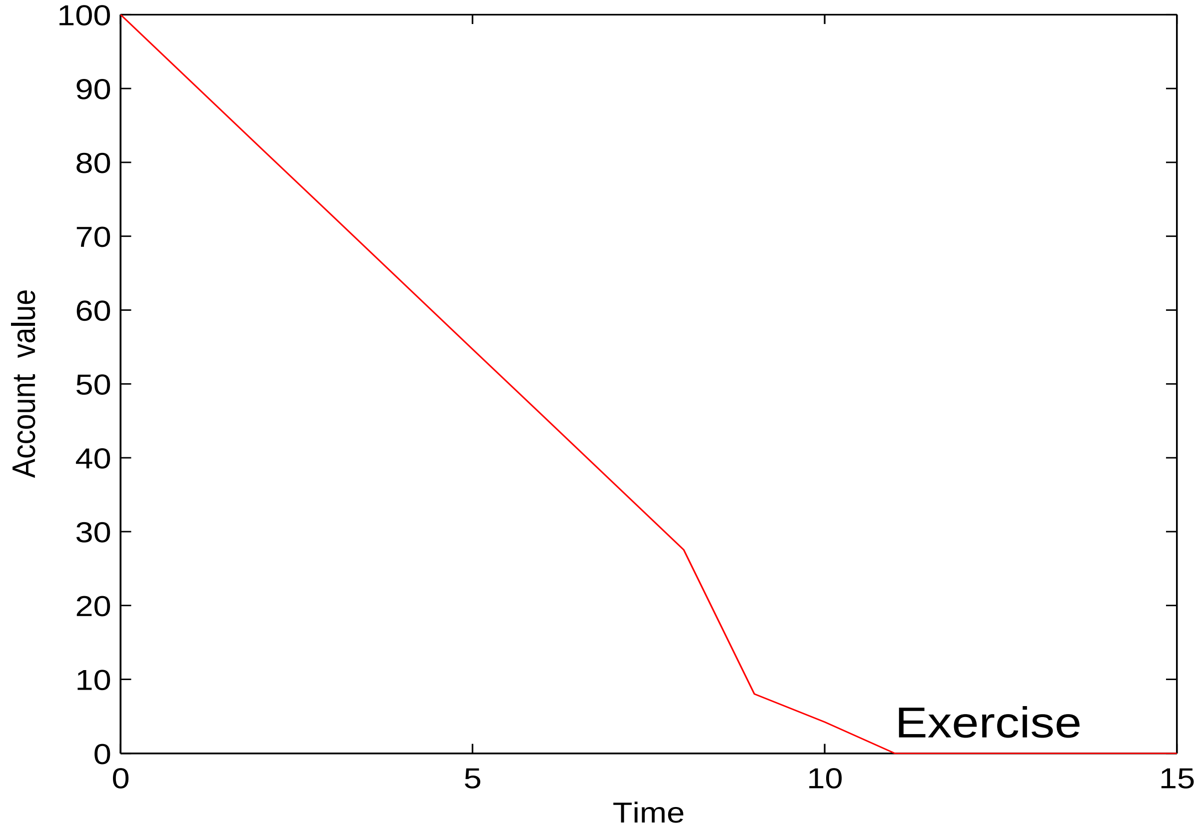


Figure 6: Path of V reaching zero at year 11. Put option exercised when V hits zero. Value of put then is $\int_{11}^{15} ge^{-ru}du$

Idealized Contract: Numerical Example

Benchmark contract, fifteen year term. No lapses no deaths. All policyholders start to withdraw funds at max rate from the outset. Input parameters are

Parameter	Symbol	Benchmark value
Initial investment	I_0	100
Contract term	T	15 years
Withdrawal rate	g	6.6667
Volatility	σ	0.20
Riskfree rate	r	0.05

Now compute put prices and value of contributions for different fee levels.

Put values and value of contributions for benchmark GMWB

Value of q basis points	Present value of Contributions	Put option
0	0	3.98
10	0.96	4.07
25	2.36	4.20
48	4.40	4.40
75	6.79	4.67
100	8.87	4.91
200	16.33	5.98
300	22.63	7.17

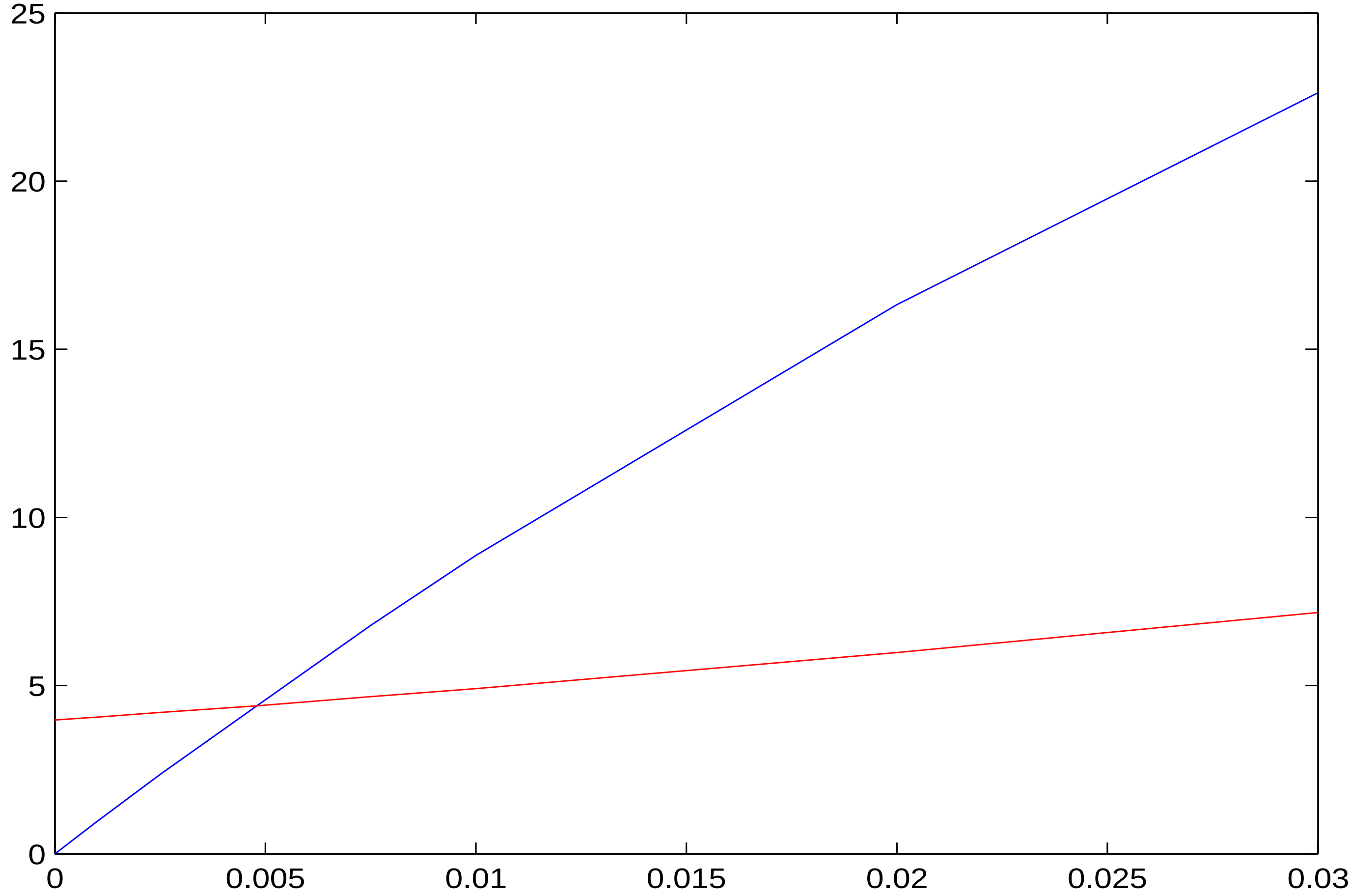


Figure 7: Plot of present value of contributions against value of put option. Lines cross when q= .0048 when put value =4.40

No Arbitrage Value

For this example the no arbitrage value of q is

$$q = 0.004751$$

Values when q = 0.004751

Value of Entity	Value (sd)
Contributions	4.40
put option	4.40

Comments on Hedging in the US

Hedging is now more attractive because of new accounting regulations. Under SFAS 133 guarantee is classified as a derivative and hedging portfolio can be marked to market. Hedging can be done in-house or using external agencies.

Relaxing the assumptions

In practice market is incomplete. Need to consider

- Lapses
- Rate of utilization
- Mortality

First two interact with financial variables.

Jon Boscia, Chairman and CEO of Lincoln Financial Group

Quote from Feb 2004 webcast

GMWB accounted for 40% of total sales. Our GMWB rider gained a strong foothold in key distribution channels as a result of strong sales," Boscia said. He continued, "Only 7% electing the rider are taking withdrawal at the maximum rate allowed with the guarantee. This level of utilization is very good because it is an indication that we are reaching the target group we aimed for when designing this rider - the comfort buyer still in accumulation stages of life.

Comments

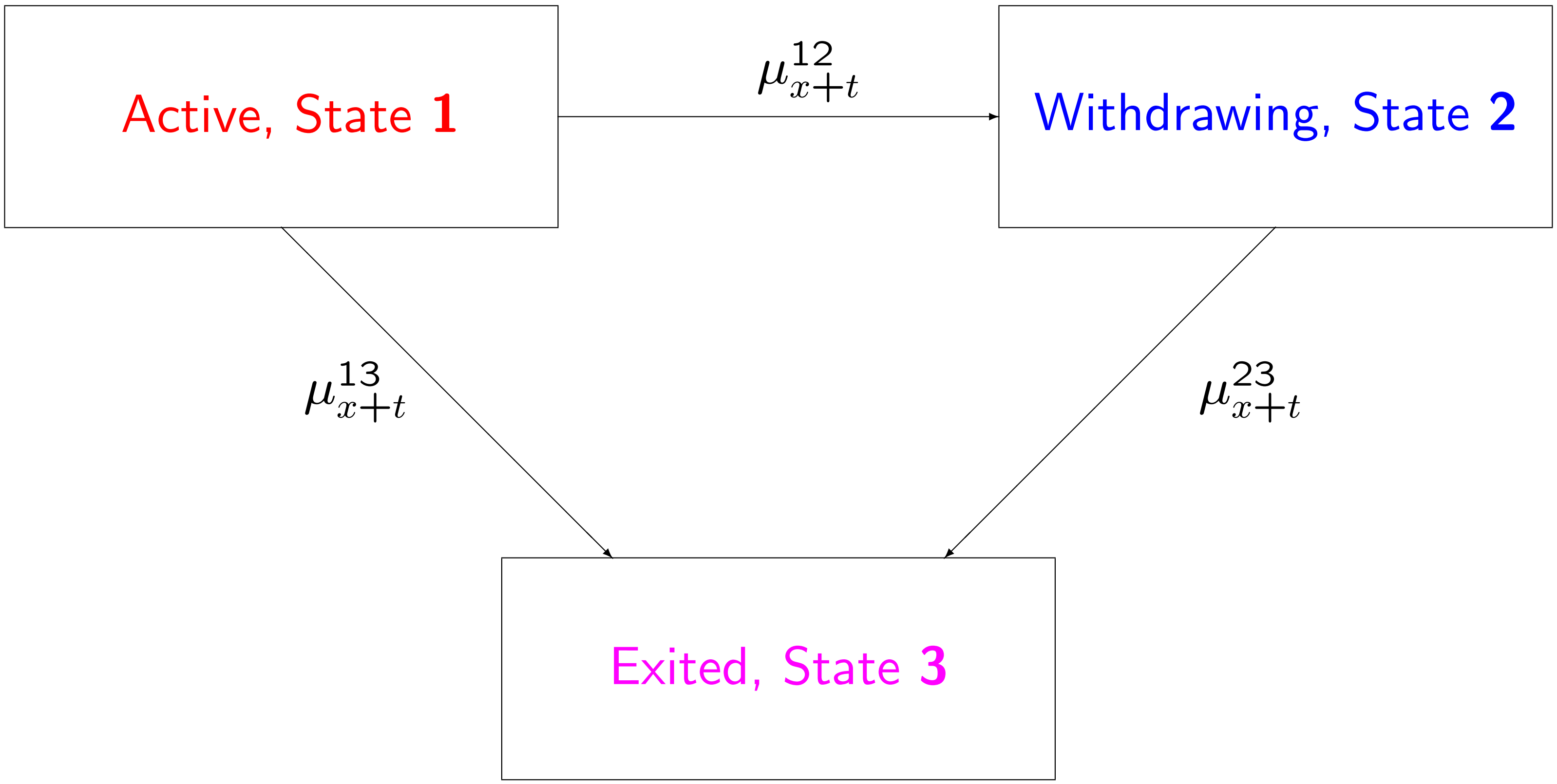
- Surrender behaviour influenced by the value of guarantee.
- When V is low a policyholder has an incentive to keep up the policy
- Should be factored into the hedge. Not easy.
  1. Few theories on exactly how policyholders will behave
  2. No published data on actual lapse rates for this product
  3. Product is a new one.

Multiple State Models

Use Multiple State framework with three states

- In force actives not yet using the withdrawal feature
- Policyholders who are withdrawing under the GMWB
- Policyholders who have exited (died or lapsed)

Let μ_{x+t}^{ij} be transition intensity from state i to state j at time t. First assume intensities are deterministic functions of time. Later they will depend on economic covariates.


Active, State 1
μ_{x+t}^{12}
Withdrawing, State 2
μ_{x+t}^{13}
μ_{x+t}^{23}
Exited, State 3

Transition Intensities

$$\begin{bmatrix} 0 & \mu_{x+t}^{12} & \mu_{x+t}^{13} \\ 0 & 0 & \mu_{x+t}^{23} \\ 0 & 0 & 0 \end{bmatrix}$$

We assume μ_{x+t}^{12} is constant (20% baseline assumption)
Surrenders tend to peak when surrender charges drop off .

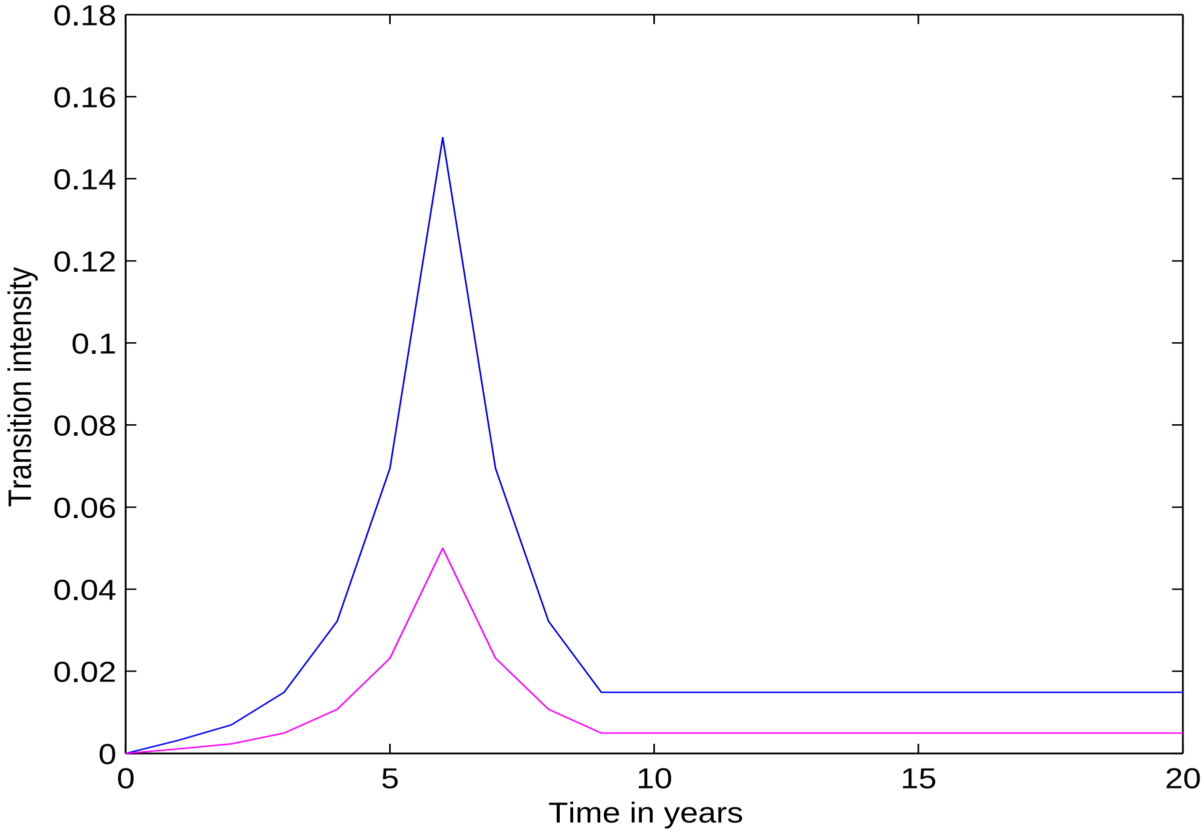


Figure 8: Graph of μ_{x+t}^{13} and μ_{x+t}^{23}

Transition probability matrix

Assume that transition intensities are piecewise linear(constant) over range $[t, t+1]$. We can compute the transition probability matrix on this basis. Thus for $t = 0$ this matrix is

$$\begin{pmatrix} e^{-(\mu_x^{12}+\mu_x^{13})} & \frac{\mu_x^{12}}{\mu_x^{12}+\mu_x^{13}}\left(1-e^{-(\mu_x^{12}+\mu_x^{13})}\right) & \frac{\mu_x^{13}}{\mu_x^{12}+\mu_x^{13}}\left(1-e^{-(\mu_x^{12}+\mu_x^{13})}\right) \\ 0 & e^{-\mu_x^{23}} & \left(1-e^{-\mu_x^{23}}\right) \\ 0 & 0 & 1 \end{pmatrix}$$

Transition probability matrix

We now value the benchmark contract incorporating deterministic intensities. Contract details and assumptions

Parameter	Symbol	Benchmark value
Initial investment	I_0	100
Contract term	T	15 years
Withdrawal rate	g	6.6667
Volatility	σ	0.20
Riskfree rate	r	0.05

Assume that once a policyholder starts to withdraw they last 15 years unless terminated(lapse or death.)

Value of put option and contributions

Assume fee of 47.51 basis points.

Assumption	Value of contributions 47.51 basis points	Value of Put Option
Deterministic intensities	5.54	2.93
Zero lapses. All start to withdraw at outset.	4.40	4.40

Note value of contributions increases and value of put goes down.

Fee that equates cost and benefits

This fee is 24 basis points. With this fee the value of the contributions is 2.78 and the the value of the put is also 2.78.

Dynamic behavior

- We expect policyholders behaviour to respond to economic conditions.
- If option is in the money more likely policyholder will start to withdraw rather than surrender.

Let $\bar{\mu}^{12}_{x+t}$, $\bar{\mu}^{13}_{x+t}$ denote deterministic intensities. Then we assume

$$\mu^{12}_{x+t} = \bar{\mu}^{12}_{x+t}\; e^{\;\lambda\; max\left(0,\; 1-\frac{V(t)}{G(t)}\right)}$$

and

$$\mu^{13}_{x+t} = \bar{\mu}^{13}_{x+t}\; e^{\;-\lambda\; max\left(0,\; 1-\frac{V(t)}{G(t)}\right)}$$

where $V(t)$ is account at time t and $G(t)$ is outstanding balance on the total withdrawal amount at time t.

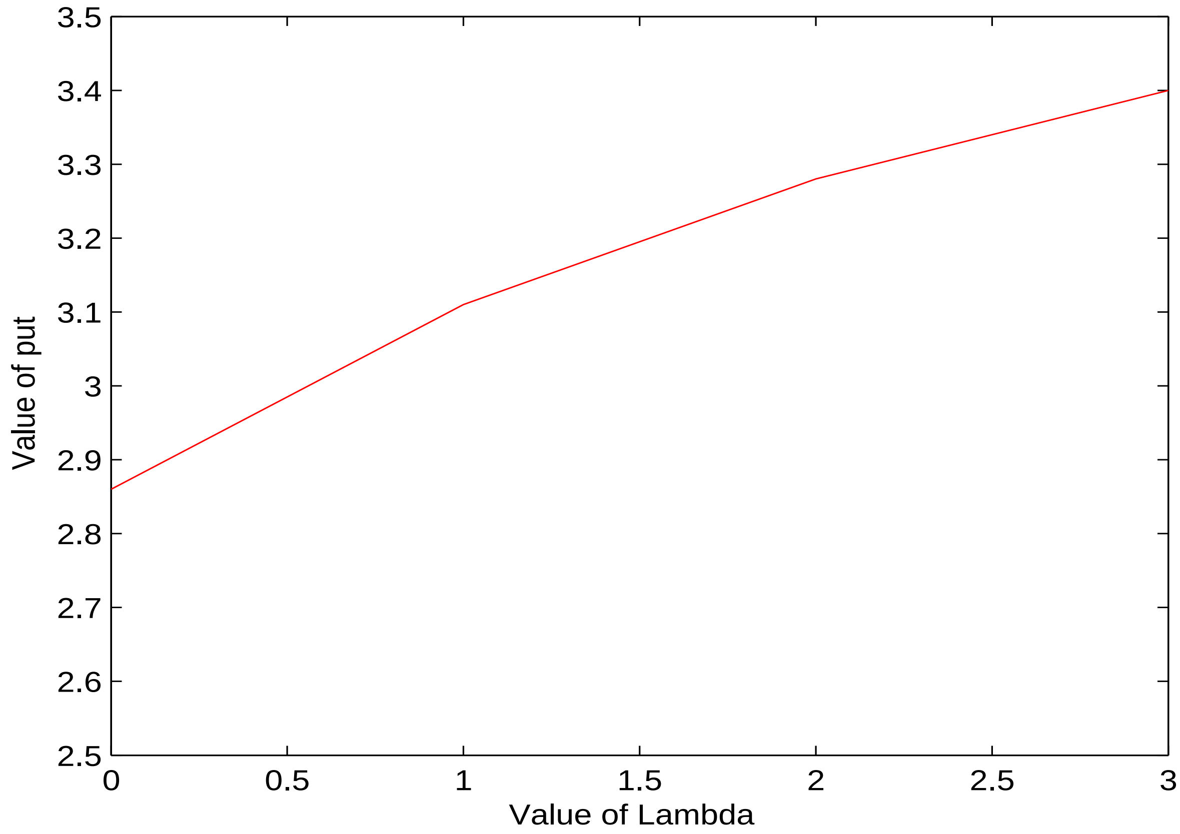


Figure 9: Put values for different values of λ. In all cases we assume a 30 basis points contribution fee

Comments

- Value of GMWB depends on consumer behavior
- Hard to model
- Option is hard to hedge
- Market is incomplete

Summary

- Insurance and finance
- Considerable progress made

  1. Progress in developing new hedging techniques

  2. New markets to reduce incompleteness

- Practical problems remain in dealing with real contracts
- Importance of model specification and parameter estimation
- Robust methods desirable
- Actuarial reserving can hedge against model error

Unified theory of insurance and finance still a challenge in the field