ARBITRAGE IN ASSET MODELING FOR INTEGRATED RISK MANAGEMENT

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There has been much discussion of the importance of arbitrage-free as a criteria of asset modeling. This paper will explore some of the issues involved when the models are for use in Integrated Risk Management analysis. The complexities of the issue increase when the method of application is considered—a model may be arbitrage-free in infinite state space, but what remains when a finite sample space is used for analysis in computer simulation? This paper will explore the problems involved and suggest some remedies.

KEY WORDS: Arbitrage, discrete, interest rate, model, term structure of interest rates, integrated risk management, stochastic.

1.0 INTRODUCTION

Arbitrage-free is well accepted as an important feature for interest rate modeling (Heath-Jarrow-Morton; Ho-Lee). While providing a rational market feature (no riskless profits), arbitrage-free also allows application of the fundamental theorem of asset pricing. As such, arbitrage-free has rightly become a cornerstone of derivative pricing. This paper examines the importance of arbitrage-free, not to the application of pricing, but rather to the application of strategic planning by use of Integrated Risk Management (IRM) stochastic modeling.

With the recent growth of Integrated Risk Management (IRM) modeling (Sweeney-Correnti 1994 & 1995) attention has come to bear on what criteria should be required for asset modeling in this framework. Many have looked to the methods of interest rate modeling used in derivative pricing (D'Arcy-Gorvett-Herbers-Hettinger). Some have called for models that are arbitrage-free (Smith-Speed).

This paper will examine the importance of arbitrage-free as a modeling criteria for Integrate Risk Management stochastic modeling, discuss some difficulties for simulation application, and propose some remedies.

What is the purpose of Integrated Risk Management modeling? IRM, sometimes referred to as Dynamic Financial Analysis or Asset Liability Management, is an evolving financial practice of strategic risk analysis. The potential applications are broad, including product design, investment strategy, reinsurance structure, capital structure, merger/acquisition analysis, and rating agency/regulatory evaluation. To achieve these tasks with IRM the modeling should:

a) Determine realistic probabilistic outcomes for future financial results; and
b) Analyze strategies to optimize future financial risk and reward results.

IRM models involve explicit modeling of the uncertainties facing a financial organization, and integration of this modeling into the financial structure of the organization to produce models of the uncertainty of the financial results of the organization. This paper will not discuss the aspects of this practice, but rather focus on a component of these models, economic and capital market modeling.

For effective IRM, accuracy and realism are important criteria and as such the simulation must provide an accurate representation of the probabilities of potential economic and market outcomes. These features are discussed in papers describing economic and market modeling for IRM (Mulvey-Thorlacius; Dempster-Thorlacius). The Wilkie models for actuarial use are an excellent additional reference (Wilkie 1987 & 1995).

The focus of this paper is on the issue of arbitrage in economic and market simulation models for IRM. Arbitrage-free is usually defined as a variation on the theme that no investment strategy can be found which involves:

a) no initial cost;
b) nonnegative future return in all scenarios; and
c) a strictly positive return in some scenarios.

Throughout this paper, the term "arbitrage-free" is used to refer to a model that does not allow arbitrage investment strategies, or to this feature of the model.

Most of the difficulties in designing models without arbitrage arise in respect to modeling fixed income assets and their related interest rate curve. When modeling interest rate curves one is essentially modeling a wide array of related investment alternatives. If the term is limited to 30 years and monthly increments, 360 different initial investment choices are available. However, there should be a strong relationship between the asset represented by the 200th month forward rate and the 201st month forward rate.

For simplification and to reflect the relationship between rates at different terms, interest rates are often generated based on one to three independent uncertainty processes (random variables). Three degrees of uncertainty are driving the return on all 360 investment products—this essential means that all 360 products are combinations of three basic products (which can be selected from the 360 products).

If the prices (or interest rates) of those 360 products are not consistent then there will be different prices for the three underlying basic products and thus an arbitrage opportunity. From the numbers in this simple example, it should be clear that without careful coordination, it will be easy for the inconsistencies to appear. The HJM model (Heath-Jarrow-Morton) provides the standard explanation of the criteria that must be followed.

As a side note, if a three factor model is used for risk analysis then three bonds are all that are required to match any element of interest rate risk—such a strong result does not seem sensible for situations where a high degree of interest rate risk control is needed (in such cases a model with more factors would be more appropriate).

The problem comes in trying to create robust model characteristics that reflect those observed in the real world while at the same time confining the computational demands of the model. Structures that ensure arbitrage-free interest rates tend to be too simple (and thus produce unrealistic scenarios) or require a large amount of computation power. Fortunately, increasing computer capacity is making it much more plausible to work with the later although this is still no trivial matter. The question is whether it can be appropriate to sacrifice arbitrage-free in order to create a functionally simple model that produces realistic scenarios. The answer would seem to depend on the application. Section 4.0 briefly describes a simple technique to remove arbitrage.

Asset models come in many structural forms. Some structures are discrete and some are continuous, some have infinite state space and some have a finite realization. Practice in IRM calls for models with relatively complex features to capture the financial dynamics involved and for realistic probability evaluations. These features are difficult to handle analytically in a continuous time, infinite state space structure so that numerical methods are commonly used.

Whether starting with a finite and discrete model or a continuous and infinite state space model, it is common practice in IRM is to realize a finite number of scenarios on a discrete time series. Section 2.0 will explore why such a discretization introduces arbitrage and section 3.0 will describe one approach to reducing the arbitrage.

The main argument against using a model with arbitrage is that certain investment strategies will appear to perform unreasonably well, because they are exploiting an arbitrage opportunity. This problem is heightened if optimization methods are employed, as they will be drawn to the arbitrage opportunities. One counter-argument is that for "reasonable" investment strategies the degree to which the arbitrage is exploited is not significant. For example, a model may allow a 1 basis point gain for buying a 7 year bond and selling equal amounts of a 6 year bond, but only in scenarios where rates are greater than 8%. Exploiting this will require extensive short selling to produce a significant advantage, and this short selling will stand out in the strategy description as "unreasonable".

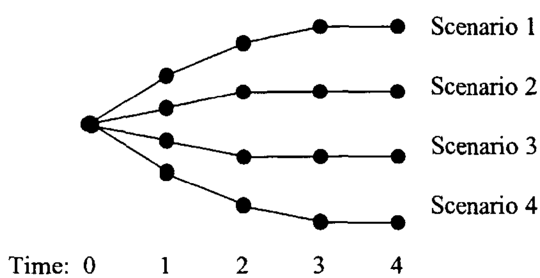
It is often said that models with arbitrage are not realistic. This argument is questionable as the real world does indeed contain arbitrage opportunities and companies attempt to make money from them every day. The amounts of money earned are limited by trading costs, credit line limits and the fact that trading to take advantage of an arbitrage opportunity will move the prices involved in an adverse way. These issues suggest ways to measure arbitrage and ways to design an IRM model to avoid problems from arbitrage.

Section 5.0 will explore some techniques to measure the influence of arbitrage. Also discussed in this section are ways in which the application of the asset model in the overall IRM structure can be performed so as to reduce the potential for bias in the analysis due to arbitrage.

2.0 FINITE REALIZATION AND ARBITRAGE

Analytic solutions can create powerful and elegant solutions for simple models, but these techniques are very difficult to apply to more realistic and thus more complex models. To approach these problems, numerical methods including computer simulation are often used. Computer simulation is a key feature in most IRM modeling.

Figure 1: The string scenario structure:



Due to the limits of time and digital architecture, computer simulation imposes the limit of a finite and discrete realization of the model scenarios. A typical scenario structure used in IRM modeling is a string structure that involves n time series scenarios, which have a single common starting stage (see figure 1). Having taken this approach, the most carefully constructed model will now have arbitrage (except for some trivial cases where all assets have the same return in each period—but this has no value for probability assessment). In the string structure, knowledge of the state after time 0 can determine which scenario is involved and thus allow strategies that key off a single scenario. Because it is a single scenario, the future is perfectly known providing an easy means for arbitrage-free returns.

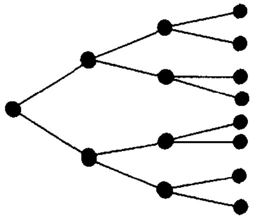
This can be seen by considering strategies which are contingent on the state of the model at the second time point. For example, assuming there is only one scenario with a time two 1 year interest rate of exactly 4.32% then a rule such as "if the 1 year interest rate is exactly 4.32% at time two then buy a 1 year bond and sell the 2 year bond and close out the position one period later" would be specific to the scenario and take advantage of the perfect information this reveals about subsequent period returns. Because the time series have only a single path, knowledge of the second period values determines all subsequent values and thus a strategy can be designed to create arbitrage by using this information (except for the uninteresting case where in each subsequent period all assets have the same return).

LEMMA 1: Given an arbitrage-free, discrete time, finite scenario interest rate model which has n distinct branches (no two are identical at any node) at time 0 and none thereafter (creating a string structure) then after time 1 all bonds must have the same return and follow the expectations hypothesis.

PROOF: To prove this assertion we need that each time 1 node is somehow different than other nodes which is given from the distinct branch criteria. This difference allows specification of a contingent strategy based on this difference i.e. do nothing unless in state i at time1. If an arbitrage profit can be earned contingent on this then overall the structure is not arbitrage-free (as all strings have non-zero probability).

Having reached state i at time 1 there is only one future path for the world and it is known. If the return on bond A is greater that that of bond B in time period p then arbitrage can be achieved by selling B and buying A at the beginning of p and reversing the deal at the end of the period. To avoid this arbitrage the return on all bonds must be the same within a specific string, at a specific time period (except the first period). This is the expectations hypothesis fully realized in each string at time period 1.□

Figure 2: The tree scenario structure:



Time: 0 1 2 3

The problem of arbitrage created by a string scenario structure can be overcome in a number of ways. One technique involves multiple branching of the scenarios at each time point, creating a tree structure (see Figure 2). The tree structure breaks the ability of the state of a scenario to be used to specifically determine which scenario will be followed in the future.

To avoid arbitrage opportunities at each time point, branching is required at each node. It turns out that two branches are often insufficient and the number of branches at each node must be at least equal to the number of independent investment alternatives (see lemma 2 below). Consider the case of IRM model involving 20 independent investment choices over a five year time frame on a monthly basis—the required number of simulations is 20 to the power of (5 x 12) = a number with 78 digits. This is not practical. We are faced with the tradeoff between too simple a model (i.e. only a few independent investment alternatives) or too many branches at each node (leading to massive scenario numbers for significant time periods that are required in integrated risk management modeling).

It should also be noted that to ensure absence of arbitrage in a tree structure the relative realization of the different branches must be carefully coordinated. It is not sufficient just to have enough branches. Some models are designed for just this purpose (see for example, Ho-Lee), but they do not contain the number of independent factors required of IRM modeling.

LEMMA 2: If we have k independent asset classes and are using a tree structure then a minimum of k branches are required at every node. Further, if there is a node with $q<k$ branches then the last k-q assets must have returns that can be replicated by a portfolio of the first q assets or an arbitrage will occur. [By independent we will require that there is no linear function which can be used to describe the 1 period returns of one of the independent assets as a function of the other independent assets over all branches from any specific node.]

PROOF: Assume at state i, time period t there are only q branches and $q<k$. Let R be the matrix of returns of the k independent assets over the q branching states at this node. Let Q be the matrix of the first q rows of R. By independence of these rows, the inverse matrix of Q, Q^{-1} exists. Let b be the pth row of R, where $p>q$ and $p<=k$, and let $c^T=b\ Q^{-1}$. Then, $c^TQ = b\ Q^{-1}Q = b$. But this means that the pth row of R is a linear combination of the first q rows of R, contradicting the assumption of independence.

Let us further assume that the indices of b do not sum to 1, but to a value d. If $d > 1$ then consider the portfolio of buying $1 of asset p and selling $1 of the c weighted combination of the first q assets. This portfolio will have zero cost and always return $(d-1)>0. Similarly, if d< 1, sell $1 of asset p and buy $1 of the c weighted combination of the first q assets. This portfolio will have zero cost and always return

(1-d)>0. Thus unless d=1, arbitrage will exist. But d=1 means that b describes portfolios of the first q assets which replicate the returns of the remaining assets. □

An alternate technique to scenario trees involves a process of testing the extent to which arbitrage is being exploited, and then requiring more and more simulations until the amount is acceptable. This concept could be applied in the manner of

a) First find the best strategy given X scenarios (originally set X to 100).
b) Test the effectiveness relative to alternatives on a second set of X scenarios.
c) If the best strategy still looks best on the second set then end, else return to a) and double the size of X.

Essentially this method is looking for convergence. If the underlying model is arbitrage-free then convergence will obliterate any arbitrage being exploited. The problem with such an approach is that a very large number of scenarios may be required to achieve convergence within an acceptable tolerance, or worse some structures will never achieve convergence. Even if convergence is achieved, there would be a statistical chance that the result would still be influenced by an arbitrage element.

One way to increase the rate of convergence would be to try to adjust the simulated scenarios to reflect the properties of the overall distribution. We will discuss such an approach in section 3.0.

3.0 ADJUSTMENT OF A FINITE SET OF SCENARIOS

Consider the case where a finite group of time series interest rate scenarios have been generated, with finite and discrete time and term intervals. Is there a method whereby the scenarios can be adjusted in such a way as to make them arbitrage-free? From our earlier discussion it is clear that this is not possible for general contingent investment strategies without reducing the scenarios extensively (one could imagine adjusting the scenarios past the first time node so that within the scenario all investments provide the same return, one can also imagine creating an effective tree structure). What if we attempted to eliminate arbitrage from some narrower set of investment alternatives? What if we limited the strategies to those where no contingency was allowed? In other words, the amount of buying and selling would be the same for all scenarios. What if we limited to strategies which could be purchased at time 0, with complete portfolio liquidation at some late period Ts common to all scenarios?

In order to explore these questions, let us introduce some notation. For ease, all discretization will be on a unit basis (i.e. time is measured in units, and there are m string scenarios).

$fo(t,T,i)$ = original forward rate in scenario i, available at time t where the bond principal will be lent at time $T-1$ and repaid at time T (note: for the case

$t = 0$, reference to the scenario index, i, will be omitted as all scenarios share a common state space for $t = 0$).

$fn(t,T,i)$ = adjusted forward rate in scenario i, available at time t where the bond principal will be lent at time $T-1$ and repaid at time T

$$Fo(t,T,i) = 1 + fo(t,T,i) \tag{3.1}$$

$$Fn(t,T,i) = 1 + fn(t,T,i) \tag{3.2}$$

One approach to ensure arbitrage free is apply the fundamental theorem of asset pricing (Harrison-Kreps), which states that no arbitrage exists if and only if there exists an alternative probability measure (with the same probability zero sets) under which all assets have the same expected return in all time periods. Essentially, if we can find an alternative set of probabilities under which the expected return on all bonds are the same then we will have no arbitrage under the original probability structure.

These alternative probabilities could be achieved with two steps:

Step 1 : using optimization or other means, adjust the string probabilities so that the expected return on several key bonds have the same returns

Step 2 : adjust the yields so that the expected returns are all identical (or close to the same) under the new probability measure.

Having completed these steps, the adjusted yields could be used with the original probability measure as arbitrage-free under the limited set of strategies.

Step 1 can be approached by constructing an optimization problem, whose variables represent the probability adjustment, constrained to ensure the total probability adds to one and individual probabilities remain strictly positive, and whose objective function quantifies the difference in expected returns for specified bonds. Although including more bonds in this objective function would increase the effectiveness, a smaller number of bonds (reflecting the number of model factors) would likely be practically just as effective—the structure of this problem is complex enough that current optimization technologies would be unlikely to find a true global optimal solution.

A more crude approach would be to simply accept the initial probabilities and move on to step 2.

We now assume Step 1 is complete and the new probabilities are denoted p_i. With this information we will develop an approach to adjusting the yields to complete Step 2.

Let $Ro(j,k)$ denote the expected return of a strategy to buy term k bonds at time 0 and sell them at time j

Consider the 1 period strategies of buy term T bonds and sell after 1 period.

$$Ro(1,T) = \sum_{i=1}^{m} \frac{p_i \prod_{s=1}^{T} Fo(0,s,i)}{\prod_{s=2}^{T} Fo(1,s,i)} \tag{3.3}$$

Similarly for a n period strategy

$$Ro(n,T) = \sum_{i=1}^{m} \frac{p_i \prod_{s=1}^{T} Fo(0,s,i)}{\prod_{s=n+1}^{T} Fo(n,s,i)} \tag{3.4}$$

When $n = T$ we have the simplified case

$$Ro(T,T) = \sum_{i=1}^{m} p_i \prod_{s=1}^{T} Fo(0,s,i) \tag{3.5}$$

Which is completely determined by the initial yields.
We want initial rates to be unchanged, in order to reflect initial conditions.

$$fo(0,T,i) = fn(0,T,i) \tag{3.6}$$

In order for "all strategies of trade at time 0, and liquidation at time Ts (for any Ts) to have the same expected return" to be satisfied we have the constraint

$$Rn(n,T) = Ro(n,n) \text{ for every } T \geq n \tag{3.7}$$

We also want fn to be >0, and assume fo are all strictly positive.

Consider an adjustment

$$Fn(t,T,i) = d(t,T)Fo(t,T,i) \tag{3.8}$$

From (3.6) $d(0,T) = 1.0$

And thus $Rn(n,n) = Ro(n,n)$ (3.9)

Consider $T = n + 1$
From (3.7) we require $Rn(n, n+1) = Ro(n, n)$

$$\sum_{i=1}^{m} p_i \prod_{s=1}^{n+1} \frac{Fn(0,s,i)}{Fn(n,n+1,i)} = \sum_{i=1}^{m} p_i \prod_{s=1}^{n} Fo(0,s,i) \tag{3.10}$$

substituting

$$\sum_{i=1}^{m} p_i \prod_{s=1}^{n+1} \frac{Fo(0,s,i)}{d(n,n+1)Fo(n,n+1,i)} = \sum_{i=1}^{m} p_i \prod_{s=1}^{n} Fo(0,s,i) \tag{3.11}$$

rearranging

$$d(n,n+1) = \frac{\sum_{i=1}^{m} p_i \prod_{s=1}^{n+1} \frac{Fo(0,s,i)}{Fo(n,n+1,i)}}{\sum_{i=1}^{m} p_i \prod_{s=1}^{n} Fo(0,s,i)} \tag{3.12}$$

But time 0 terms are the same for all strings so using $\sum p_i = 1.0$

$$d(n,n+1) = Fo(0,n+1,)\sum_{i=1}^{m} p_i \,/\, Fo(n,n+1,i) \tag{3.13}$$

Now consider the case $T = n + q$

Similarly find

$$d(n,n+q) = \prod_{s=n+1}^{n+q} Fo(0,s,) \frac{\sum_{i=1}^{m} \left(p_i \,/ \prod_{s=n+1}^{n+q} Fo(n,s,i) \right)}{\prod_{k=1}^{q-1} \left(d(n,n+k) + 1 \right)} \tag{3.14}$$

Using this procedure we have found $d(t,T)$ which satisfy (3.7) for every n and every $T \geq n$.

A problem with this procedure is that it may cause some fn to be negative. However, this can be avoided by using the alternative adjustment

$$fn(t,T,i) = d(t,T) fo(t,T,i) \tag{3.15}$$

Unfortunately the resulting equations are of the form

$$\prod_{s=n+1}^{n+q} Fo(0,s,)\sum_{i=1}^{m}\left(p_i / \prod_{s=n+1}^{n+q}(1+fo(n,s,i)d(n,s))\right)=1 \qquad (3.16)$$

Which does not produce as simple an equation for d. To be certain that no negative values of fn are produced, it would also be necessary to show that $d(t,T)>0$.

These techniques can be applied whether or not the model is designed to be arbitrage-free in infinite state space. When the model has been designed to be arbitrage-free in the limiting case, this technique should serve to increase the rate of convergence.

In any case, these adjustments do not satisfy the equivalence of return under a general stopping rule or for strategies which allow intermediate trading.

4.0 ADDING UNCERTAINTY PROCESSES TO CREATE ARBITRAGE-FREE

Suppose we have created an interest rate model that has attractive statistical characteristics, and we are only interested in discrete time, and we are only interested in a finite number of yield terms. Is there a method for adjusting this model so that it will be arbitrage-free?

One way to achieve this would be to add additional independent uncertainty processes to each yield term, thereby breaking the dependency and thus the arbitrage. The standard deviation of these processes could be made small so that the original statistics are not significantly disturbed. Lognormal distributions could be used to ensure the interest rate levels do not become negative. This should be done to yields on a forward rate basis to ensure no rate is negative.

It is relevant to note that such construction can be performed with exceedingly small standard deviations for the additional independent random elements, leaving little perceptible difference in the model but achieving arbitrage-free and thus calling into question whether arbitrage-free is a valuable addition. Alternatively, a moderate additional volatility can be used, reflecting the general bumpy character of interest rate changes.

Details of such a construction will be presented in a forthcoming paper (co-authored with David Heath), which describes such a methodology and the density of arbitrage-free models within the set of discrete terms structure models.

5.0 MEASURING AND OVERCOMING ARBITRAGE

From the perspective of Integrated Risk Management Modeling, the concern with arbitrage is that it is misleading us into believing a portfolio is more attractive than it otherwise would be due to an arbitrage advantage that is being realized. It would be useful to be able to measure the extent to which arbitrage has biased the result (or that unreasonable gains for very small risks are possible). If arbitrage exists, but if the amount of increased return that can be realized when the amount of short selling of any asset is limited to 100% is very modest, then it is reasonable to assert that the "flaw" is not meaningfully biasing the analysis.

There are several ways in which the amount of arbitrage in a model can be measured. These measures can be computed with respect to the original model or a finite sampling. These measures are based on market limitations that are faced by those trying to exploit arbitrage in the market. These methods will be described in a forthcoming paper but an overview description will be presented here.

The first measure identifies the amount of uniform trading cost (i.e. a transaction cost added to every trade at the same rate) which will eliminate arbitrage in the model. The lower the minimum required trading cost to eliminate arbitrage, the lower the level of arbitrage.

A second measure stems from the need to sell short to exploit arbitrage opportunities. Counterparties place margin and other requirements on short selling limiting the arbitrager. Additionally, the greater the amount of short selling required, the more potential for small errors to lead to disastrous results. Towards this end, the amount of arbitrage can be measured by searching for the strategy that has the greatest riskless profit under a fixed short selling ceiling. The higher the level of arbitrage profit for this strategy, the higher the level of arbitrage.

Application of these measures to a discrete simulation set can be achieved with the use of linear programming as an optimization tool, making the time of computation manageable if not negligible. If all investment strategies are considered, high levels of arbitrage measures will likely result for the reasons developed in section 2.0.

This leads to some techniques that can be used to minimize the influence of arbitrage:

1. Include trading costs.
2. Use a limited range of investment strategies by means of decision rules.
3. Constrain short selling.
4. Use representative assets and limit the number (for example, only include a small group of bond assets instead of all possible maturities)

Decision rules are a means of describing strategies. When applied to the investment strategy problem, they would limit the contingencies under which a different investment portfolio is held. In its simplest example, a fixed mix structure, the investment strategy must be the same in all situations and not contingent in any way on the state. More complex examples would allow the strategy to be contingent on a small number of state space indicators and limiting the extent to which this contingency can be expressed. In this way, the investment strategy cannot be explicitly contingent on the scenario.

Use of these techniques, combined with qualitative analysis (does it make sense? does it suggest short selling?) of the solution results allows for effective IRM in many situations where the underlying models are not strictly arbitrage-free.

6.0 CONCLUSION

Arbitrage-free is an important investment modeling principle and is fundamental to modern pricing theory. The rich structure required of integrated risk management modeling makes adherence to this principle difficult to achieve in IRM practice. To do so requires sacrificing model richness or accepting larger computation times. The importance of arbitrage in pricing stems from the fundamental theorem of asset pricing, which is not an underlying principle of IRM.

As analyzed in section 2.0, if simulation methods are used, the size of the minimum structure required to avoid arbitrage in finite realizations is significant and requires well-crafted tree structures. The literature on tree structure models focuses primarily on interest only models developed for pricing applications that do not begin to approach the dimensionality needed for meaningful IRM modeling. Section 3.0 describes some methods for adjusting scenarios. These methods could be applied on a node by node basis within a high dimensional generated tree structure to ensure the result is arbitrage free. Such an extension would be straightforward and could also be expanded to consider assets other than fixed income.

Section 4.0 indicates that if a model is developed to reflect realistic market behavior in a computationally efficient way at the sacrifice of strict arbitrage, then this model can be easily modified to remove the arbitrage in the infinite state case. This analysis suggests a very small leap between arbitrage and non-arbitrage.

Finally, in section 5.0, some methods for assessing the level of arbitrage in a model were described as well as some techniques for IRM application to limit the effect of arbitrage.

To conclude, it is desirable for asset models to be arbitrage-free. For some applications, such as many pricing situations, it is fundamental. However, IRM modeling applications are about assessing realistic probabilities of future results. The emphasis on model realism and determination of an explanation for why a strategy is expected to be successful should be viewed as at least as, if not more, important in IRM application than the requirement that the modeling be arbitrage-free.

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7.0 REFERENCES

D'Arcy, S., R. Gorvett, J. Herbers, and T. Hettinger, "A Basic Model for Dynamic Financial Analysis", Casualty Actuarial Society Special Interest Seminar, July 1997

Dempster, M.A.H., and A. E. Thorlacius, "Stochastic Simulation of International Economic Variables and Asset Returns: The Falcon Asset Model", 8th AFIR International Colloquium, 1998

Harrison, M., and D. Kreps, "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory*, 20, 381-408, 1979

Heath, D., R. Jarrow, and A. Morton "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuations," *Econometrica* 60(1):77-105, 1992

Ho, T. S. Y., and S.-B. Lee "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance* 41: 1011-1029, 1986

Mulvey, J., and A. E. Thorlacius, "The Towers Perrin Global Capital Market Scenario Generation System: CAP:Link," in *World Wide Asset and Liability Modeling*, (eds. W. Ziemba and J. Mulvey), Cambridge University Press, 1998. Also 6th AFIR International Colloquium, 1996

Smith, A., and C. Speed, "Gauge Transforms in Stochastic Investment Modelling", 8th AFIR International Colloquium, 1998

Sweeney, J., and S. Correnti "Asset-Liability Management and Asset Allocation for Property and Casualty Companies - The Final Frontier", 4th AFIR International Colloquium, 1994

Sweeney, J., and S. Correnti "The Dynamics of Interest Rate Movements and Inflation on Assets, Liabilities, and Surplus For Property and Casualty Insurers", 5th AFIR International Colloquium, 1995

Wilkie, A. D., "Stochastic Investment Models: Theory and Applications," *Insurance: Mathematics and Economics*, 6, 65-83, 1987.

Wilkie, A. D., "More on a Stochastic Asset Model for Actuarial Use," Institute of Actuaries and Faculty of Actuaries, 1995.