

AN ANALYSIS OF THE UTILITY FUNCTIONS OF MEMBERS OF RETIREMENT FUNDS

Author: Rob Thomson
Professional affiliation: Fellow of the Institute of Actuaries; Fellow of the Actuarial Society of South Africa; Visiting Professor, University of the Witwatersrand
Address: 27 Marico Road, Emmarentia, Johannesburg 2195, South Africa
Telephone number: 27 11 646-5332
Fax number: 27 11 339-6640
E-mail address: rthomson@icon.co.za

ABSTRACT

In this article the author analyses the observed values of utility functions of post-retirement income elicited from members of retirement funds. Difficulties in the process of elicitation are discussed. Parametric functions suitable for use in advising members on investment channel choice to maximise their expected utility are fitted to the observed values. Issues relating to the advice to be given by trustees to risk-seeking members are considered.

KEYWORDS

Expected utility theory; decision theory; investment channel choice; defined contribution retirement funds

1. INTRODUCTION

1.1 In a number of countries in recent years there has been a marked trend from defined-benefit to defined-contribution retirement funds (IACA, 1998). Pursuant to the resulting transfer of risks from the employers to the members of such funds, there has been increasing interest in allowing members the facility of choosing between a number of investment channels. The returns on a member's interest in such a fund are then determined by the member's choice of investment channels.

1.2 In a previous article (Thomson, 1998a) the author suggested a method of recommending an apportionment of a member's interest in a defined-contribution retirement fund that was based on the maximisation of the expected utility of the benefit that would be payable by the fund on the date on which the member intended to retire or withdraw from the fund. In a further article (Thomson, 1998b) he defended the use of expected utility theory for such applications. In both those articles it was envisaged that a computer system could be developed to elicit the utility function of a member of such a fund. The system could be programmed to model stochastically the returns from the investment channel choices available and thus to obtain the distribution of a pseudo-random sample of the member's benefits at retirement or withdrawal. By means of dynamic modelling, allowance could be made for the fact that the member would be able to make further choices in the future, depending on intermediate outcomes.

1.3 This system could be installed in the offices of employers participating in the fund. Members would have access to the system and after responding to questions posed by it on a monitor, would receive recommendations for the apportionment of their interests in the fund between the channels available.

1.4 In this article the author analyses the observed values of utility functions of post-retirement income elicited from members of retirement funds in order to assess:

- whether subjects generally answer the necessary questions consistently;
- if so, whether they generally exhibit risk-aversion, and
- whether the functional form proposed in Thomson (1998a) is appropriate for risk-averse subjects.

Difficulties in the process of elicitation are discussed. Parametric functions suitable for use in advising members on investment channel choice to maximise their expected utility are fitted to the observed values. Issues relating to the advice to be given by trustees to risk-seeking members are considered.

2. ELICITATION OF UTILITY FUNCTIONS

2.1 For the purposes of this article, the equally likely certainty equivalent (ELCE) method (Anderson, Dillon & Hardaker, 1977) is used for the elicitation of the utility functions of the subjects. The questionnaire used to elicit the utility functions is shown in the appendix.

2.2 The questionnaire was used by third-year actuarial science students at the University of the Witwatersrand in 1998 and 1999. Each student selected one subject. The only constraint placed on the selection was that the subjects should be members of retirement funds not yet on pension. The students administered the questionnaire and ensured that each subject understood the questions. 50 correctly completed questionnaires were returned. (A questionnaire was rejected if questions 2 to 13 were not properly answered or if it did not reflect non-satiation over the whole range: the latter condition requires that $\forall x : u'(x) > 0$.) These data are available from the author.

2.3 Whereas the proposed system would present decision-makers with choices involving specified monetary amounts (in current money terms), the questionnaires presented subjects with choices involving post-retirement income expressed as percentages of current income from employment. In other respects the format of the questions posed in the questionnaire corresponded to those which would be posed by the proposed system.

2.4 From the questionnaires the elicited values of the subjects' utility functions were obtained as shown in Table 1. That table also gives typical values of x . It may be noted from that table that two values of x are obtained for $u(x) = 0.5$. The purpose of this is to validate each subject's preferences. In terms of expected utility theory, if a subject is indifferent as between a certain outcome x_2 (or x'_2) and a 50-50 chance of two uncertain outcomes x_0 and x_4 , (or x_1 and x_3) then:

$$u(x_2) = \frac{1}{2} \{u(x_0) + u(x_4)\} = 0.5; \text{ and}$$

$$u(x'_2) = \frac{1}{2} \{u(x_1) + u(x_3)\} = 0.5.$$

If the subject's utility function is monotonically increasing (i.e. if the subject is non-satiated) then $u^{-1}(0.5)$ is unique. In fact subjects did not always give the same answers for x'_2 as for x_2 . In practice, decision-makers would be asked to reconsider such inconsistencies. But for the purposes of this article it was thought best to retain them in order to give an understanding of the extent to which such inconsistencies might occur in practice.

Table 1. Derivation of utility functions from questionnaires

Post-retirement income (% of current income from employment)	Typical value	Utility
x	x	$u(x)$
$x_0 = 20\%$	20%	0.00
$x_1 = \text{answer 6 or 7 or, if blank, answer 5(2)}$	23%	0.25
$x_2 = \text{answer 3 or 4 or, if blank, 50\%}$	30%	0.50
$x'_2 = \text{question 12 or 13 or, if blank, 11(2)}$	25%	0.50
$x_3 = \text{answer 9 or 10 or, if blank, answer 8(2)}$	40%	0.75
$x_4 = 100\%$	100%	1.00

2.5 The typical values listed in Table 1 are shown in Figure 1. Whereas x_2 is elicited as the certainty equivalent of a 50-50 chance of x_0 and x_4 , x'_2 is elicited as the certainty equivalent of a 50-50 chance of x_1 and x_3 .

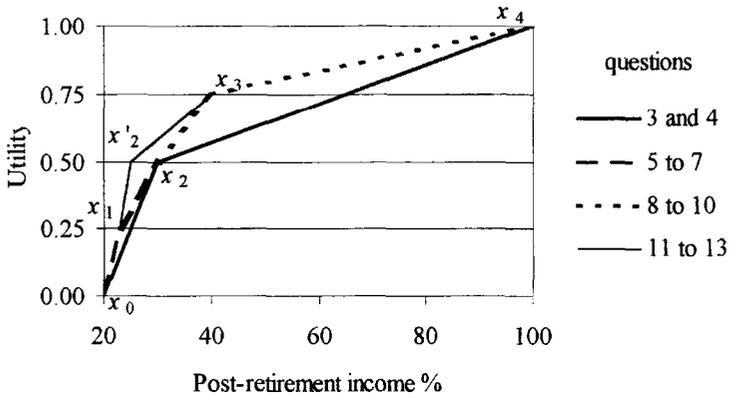


Figure 1. Typical utility function elicited by questionnaire

2.6 By considering the angle formed by the lines joining the point representing each certainty equivalent with those representing the alternatives from which it was elicited, it may be ascertained whether that point reflects risk-aversion or not. If the angle shows a downward deflection then the point reflects risk-aversion. If there is a straight line through the point then it reflects risk-neutrality. If there is an upward deflection then it represents risk-seeking.

2.7 Table 2 gives a cross-tabulation of risk-aversion at the various points, and Table 3 summarises this information. From Table 2 it may be seen that, of the 50 completed questionnaires, only 20 exhibited global risk aversion. However, from Table 3 it may be seen that, whereas only 27 are risk-averse at x_1 , the proportion of subjects showing risk-aversion tends to increase with increasing benefit levels. It has often been noted (e.g. Friedman & Savage, 1948) that utility functions may be concave over certain ranges and convex over others. Of the 23 subjects who are risk-seeking at x_1 , 16 are risk-averse or risk-neutral at x_3 . This phenomenon is referred to as "aspiration".

Table 2. Analysis of risk-aversion

	risk-averse/neutral at x_2 :		risk-seeking at x_2 :	
	risk-averse/ neutral at x_1	risk-seeking at x_1	risk-averse/ neutral at x_1	risk-seeking at x_1
risk-averse/neutral at x'_2 :				
risk-averse/neutral at x_3	20	4	0	3
risk-seeking at x_3	0	1	0	1
risk-seeking at x'_2 :				
risk-averse/neutral at x_3	5	8	2	1
risk-seeking at x_3	0	2	0	3

Table 3. Summary of risk-aversion

Point	Risk-averse/neutral	Risk-seeking
x_1	27	23
x'_2	29	21
x_2	40	10
x_3	43	7

2.8 As noted in ¶2.4, subjects did not always give the same answers for x'_2 as for x_2 . The sample frequency of $x'_2 - x_2$ is shown in Figure 2. The mean is 1.0 and the standard deviation is 10.3. The large standard deviation suggests that a substantial proportion of the subjects do not follow expected utility theory or are not entirely consistent in their numerical logic. However, for those subjects that showed global risk aversion, the mean is -4.6 and the standard deviation is 5.4. The sample frequency for those members is shown in Figure 3.

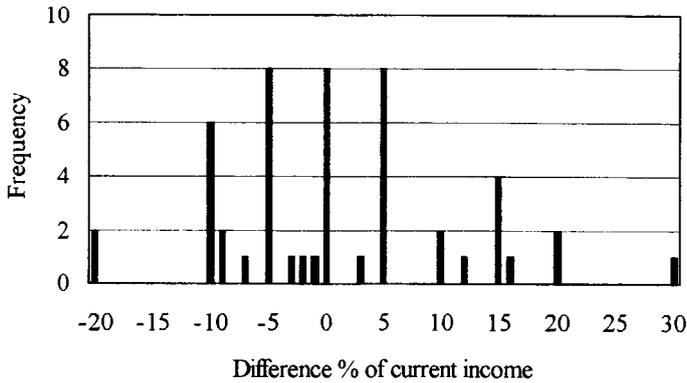


Figure 2. Frequency of difference: $x'_2 - x_2$ for all subjects

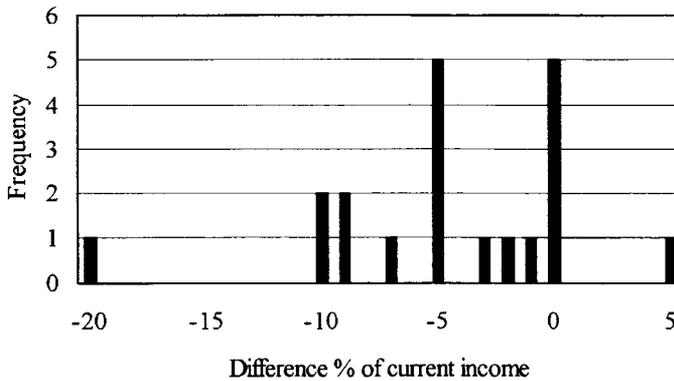


Figure 3. Frequency of difference: $x'_2 - x_2$ for risk-averse subjects

3. FITTING OF PARAMETRIC FORMS

3.1 As explained in Thomson (1998a), in order to use a member's utility function for the purpose intended, it is necessary to fit a parametric function to the values elicited.

3.2 As discussed in Thomson (1998b) it is questionable whether the trustees of a retirement fund would be willing to recommend investment channel choices based on the maximisation of the expected utility of a member's retirement benefit if the member's utility function exhibited local risk-seeking. The trustees might justifiably regard such a utility function as a speculative basis for decision-making. In Thomson (1998a) a method of fitting a parametric form to the elicited utility function is discussed. The form proposed is:

$$u(x) = a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) + a_4,$$

where the functions $f_i(x)$ are suitable functions and a_i represent parameters. In that paper, the functions used for illustrative purposes were as follows:

$$\begin{aligned} f_1(x) &= x; \\ f_2(x) &= \ln x; \\ f_3(x) &= -\frac{1}{x}. \end{aligned}$$

However, it was acknowledged that functions of the form

$$f(x) = -\frac{1}{x^n}, \text{ where } n > 1,$$

might be needed in order adequately to reflect the risk aversion of decision-makers. The values of a_i were constrained to $a_i > 0$ for all i in order to avoid risk-seeking and to ensure that $u(x)$ would be well-behaved.

3.3 As explained in Thomson (1998a), because of the fact that the error terms

$$\varepsilon_i = \overset{o}{u}(x_i) - \{a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) + a_4\}$$

are not independent, it is necessary to sum the squares of the error terms

$$\eta_i = 2\varepsilon_i - \varepsilon_2 \text{ for } i = 1, 2, 3.$$

For each subject s , the parameters a_i may accordingly be calculated by minimising

$$S_s = \sum_{i=1}^3 \eta_i^2.$$

3.4 In order to accommodate risk-neutrality (and to avoid imposing positive risk aversion on risk-seeking subjects) it was predetermined that

$$f_1(x) = x.$$

It was assumed that

$$f_2(x) = -\frac{1}{x^{n_2}}; \text{ and}$$

$$f_3(x) = -\frac{1}{x^{n_3}}$$

where:

$$0 < n_2 < n_3.$$

(The requirement that n_2 and n_3 be positive is not a precondition, but it was found that only positive values were required in practice.)

3.5 Of the 20 subjects that showed global risk aversion, there were 17 for whom $|x'_2 - x_2| < 10\%$. In order to find appropriate values of n_2 and n_3 , the sum of the squares of the error terms η_i was calculated over the latter subjects. The values adopted were those which minimised

$$SS = \sum_s S_s.$$

For this purpose, the value of x_2 was taken as $\frac{1}{2}(x'_2 + x_2)$.

3.6 The resulting values were:

$$n_2 = 0.6; \text{ and}$$

$$n_3 = 2.1.$$

For these values there was one subject for whom

$$a_1 = a_2 = 0,$$

indicating that, but for the constraints, the value of n_3 would have been higher. In addition, one subject showed a particularly high value of S_s . These subjects, referred to as subjects A and B respectively, are considered below.

3.7 For subject A the value of n_3 was increased to the point at which the value of S_s was minimised. The value of n_3 at this point was 2.4, and S_s was reduced from 0,174 to 0,168. In view of the insignificance of the improvement, it appears that it is not necessary to increase the value of n_3 .

3.8 For subject *B* the observed values and the fitted utility function (for which $a_3 = 0$) are shown in Figure 4. Also shown is the alternative curve for $a_1 = a_3 = 0$.

3.9 It may appear paradoxical that the fitted curve lies entirely below the observed values. One might have expected that, in order better to fit the observed values, greater curvature would have been required, as shown by the alternative function. However, it should be recognised that, as explained in ¶3.3, it is not the usual error terms that are used in minimising the sum of the squares, but the independent portions of those errors. Thus, in a sense, the method attempts to fit the shape of the observed values rather than the values themselves. The reason that it fails is that the shape of the observed values is very complex. First, there is an infinitely positive slope from x_0 to x_1 . This means that the error term ε_1 is 0,25 regardless of the parameters. For the fitted curve, $\varepsilon_2 = 0,100$, whereas for the alternative shown, $\varepsilon_2 = -0,007$. This means that the values of η_1 are 0,400 and 0,507 respectively, thus making the fitted curve preferable. Although the reverse applies at x_2 and x_3 , it is not sufficient to offset the effect at x_1 . The curvature of the fitted function is more important than its fit to individual observations. Figure 4 shows that, as one would expect, its curvature is intermediate between the curvature excluding the end-points and the curvature including those points. In practice, as mentioned in ¶2.4, decision-makers would be asked to reconsider such inconsistencies. This case does, however, highlight the necessity of achieving consistency rather than merely averaging inconsistent answers.

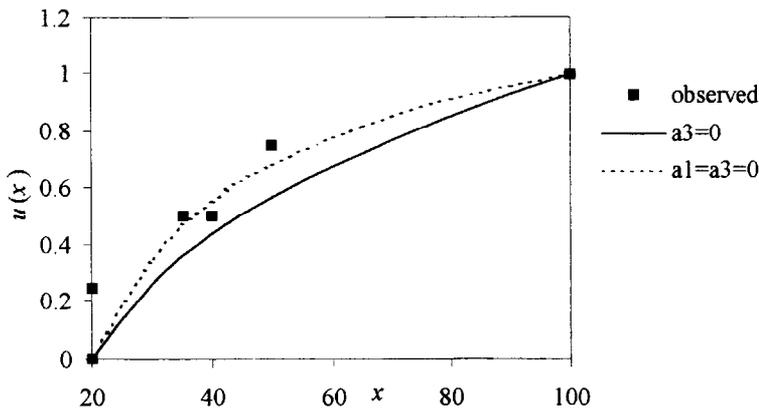


Figure 4. Utility function of subject *B*

3.10 It is of interest to consider whether subjects' income from employment affected their risk-aversion. Table 4 analyses the 50 completed questionnaires by gross annual income from employment and whether or not the subjects exhibited global risk-aversion or neutrality. (At the time of writing $R1 = 1,26\text{NOK}$.)

Table 4. Analysis of global risk-aversion/neutrality by gross annual income from employment

Gross annual income from employment	Subjects globally risk-averse/neutral	Subjects locally risk-seeking	Total
A up to R20 000	0	0	0
B R20 001 to R50 000	2	0	2
C R50 001 to R100 000	4	7	11
D over R100 000	14	22	36
not stated	0	1	1
Total	20	30	50

3.11 Using the chi-square test of homogeneity at the 5% level, the hypothesis that global risk-aversion (including risk-neutrality) is homogeneous with respect to gross income from employment cannot be rejected.

3.12 It is evident that the students tended to select subjects in the higher income-bands. This bias may be explained partly by the wealthier backgrounds of most actuarial students and partly by their perception that it would be easier to interview subjects from similar backgrounds.

3.13 The averages of the observed values of the 17 subjects that showed global risk aversion and for whom $|x'_2 - x_2| < 10\%$ are shown in Figure 5, together with an unconstrained fit and a constrained fit of the utility function to those values, against the left-hand axis. Also shown is the absolute risk-aversion

$$R(x) = -\frac{u''(x)}{u'(x)}$$

for the constrained fit, against the right-hand axis.

3.14 It may be noted from Figure 5 that, with the unconstrained utility function, although it fits better than the constrained function, satiation is reached at a benefit of 100%. The constraint applies to a_1 and a_3 , so that increasing n_3 would not improve the fit of the constrained function.

3.15 A corresponding graph for all 50 completed questionnaires is shown in Figure 6, from which the aspiration phenomenon may be seen in the unconstrained fit. Once again, however, that function reaches satiation at 100%.

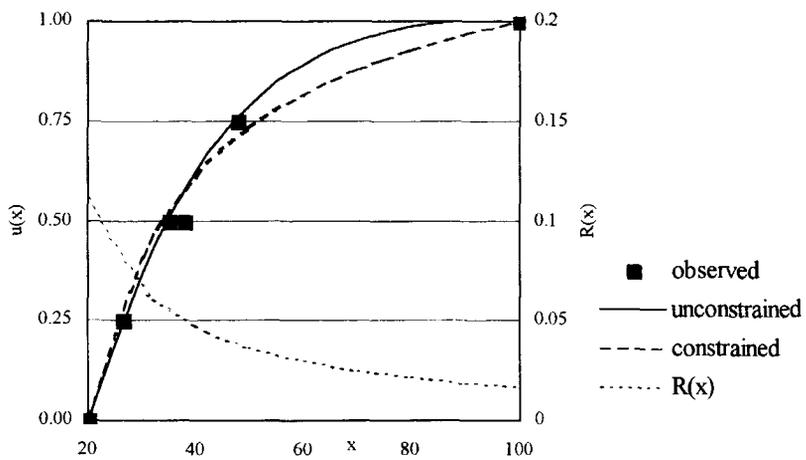


Figure 5. Average utility for risk-averse subjects

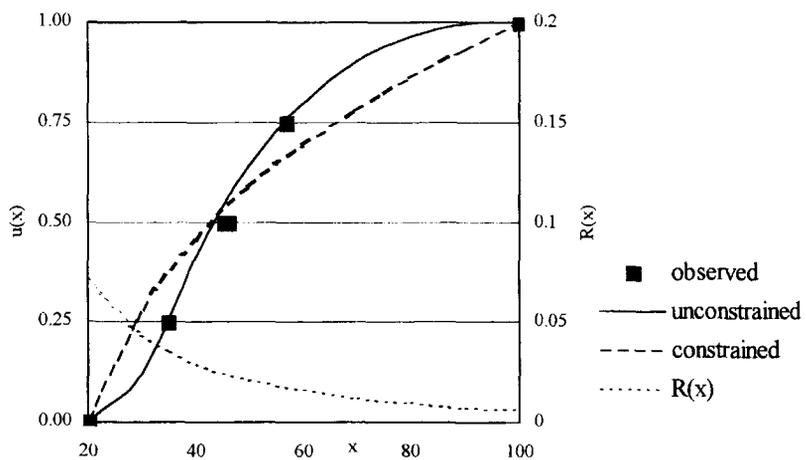


Figure 6. Average utility for all subjects

4. CONCLUSION

4.1 It appears from the above analysis that it would be appropriate to use a utility function of the form:

$$u(x) = a_1x - \frac{a_2}{x^{0.7}} - \frac{a_3}{x^{2.1}} + a_4,$$

where

$$\forall i : a_i \geq 0.$$

4.2 Before fitting such curves, however, it is necessary to ensure that the decision-makers are consistent in their answers to the required questions. This can be done by means of questions such as questions 11 to 13 of the appendix. If possible, without exerting influence on the decision-makers preferences, consistency should be achieved by discussion with the decision-maker. Otherwise it should be acknowledged that, for that decision-maker, the method cannot be applied.

4.3 Most of the subjects exhibited risk-seeking tendencies over at least part of the range considered. Whether it is appropriate to make recommendations to such subjects is a matter for discussion. The following alternatives could be considered:

- (1) relax the constraint that $\forall i : a_i \geq 0$;
- (2) devise a different formula to reflect risk-seeking while retaining non-satiation;
- (3) apply the constraint; or
- (4) advise the decision-maker that the method cannot be used in his or her case.

4.4 The problem with alternative (1) is that the resulting function will not necessarily reflect non-satiation over the whole range. Alternative (2) is outside the scope of this paper. The author thinks that trustees will not be willing to permit the recommendation of investment channel choices based on risk-seeking utility functions. In any case, the use of risk-neutral functions, which are accommodated by the proposed method, will generally result in the maximum possible exposure to those channels with relatively high risks and expected returns. For this reason it may not be inappropriate to adopt alternative (3), and alternative (4) may not be particularly helpful. If alternative (3) is adopted it is also debatable whether and to what extent the reason for and the effects of the constraints should be explained. Similarly, if alternative (4) is adopted, it is debatable whether and to what extent the reason for the refusal to advise should be explained. In either case it may be difficult to explain the matter in a way that is both satisfying to more sophisticated decision-makers and intelligible to others.

4.5 As explained above, the sample selected for this analysis is not a random sample, and there is some evidence of bias. By the same token, the users of the proposed system are also likely to be a biased sample of members. Conclusions drawn from this sample should be treated with circumspection. However, it can form the basis of the next step, which will be to introduce the system on a trial basis to selected retirement funds. Those trials can then be used to see how members of retirement funds use the system in making decisions. Recommendations acted on can then form the basis of further analysis. In particular, it would be of interest to establish

whether indicators of the shape of members' utility functions can be obtained from questions that may be easier for them to answer.

4.6 It is argued by Asher (1999) that the imposition of investment channel choice on the members of a retirement fund places an unnecessary burden on them, which should be borne by the trustees. There is also considerable antagonism from South African trades unions against allowing investment channel choice to members of retirement funds. The omission from this article of discussion of this issue is not intended to imply that it can be ignored.

4.7 In recent years, in addition to the concepts of "descriptive" and "normative" analyses (which seek respectively to explain and, as in this article, to recommend decisions) the concept of "prescriptive" analyses has been propounded (e.g. Bell et al, 1988, French & Xie, 1994). Although in the present author's opinion "prescriptive" is an unfortunate epithet, the concept is useful. It envisages using decision theory as a process to assist the decision-maker to understand the analysis and thereby to make a more informed decision. In discussions with Professor S. French, the present author was strongly urged to ensure that, if a normative system was made available to the members of a retirement fund to make recommendations regarding investment channel choice, trained personnel should be available to assist in this process and to help members to understand the model being used and the basis on which the recommendation was made. The extent to which this will be possible may be limited by considerations of cost-effectiveness. But further research into methods of facilitating the decision-making process, and into the application of recently developed theory to investment channel choice, may be worthwhile.

ACKNOWLEDGEMENTS

The author acknowledges with thanks the help of Professors S. French and A. Asher with regard to certain aspects of this article, as well as the work of the students that returned completed questionnaires and the participants in seminars at the Universities of the Witwatersrand and Cape Town.

REFERENCES

- ANDERSON, J.R., DILLON, J.L. & HARDAKER, B. (1977). *Agricultural Decision Analysis*, Iowa State University Press, Ames; cited in Farquhar (1984).
- ASHER, A. (1999). Loading retirement fund members with investment channel choice, *Journal of Pensions Management*, 5 (1), 76-84.
- BELL, D.E., RAIFFA, H. & TVERSKY, A. (EDS.) (1988). *Decision Making: Descriptive, Normative and Prescriptive Interactions*. Cambridge; cited in French & Xie (1994).
- FARQUHAR, P.H. (1984). Utility assessment methods, *Management Science* 30, 1283-94.
- FRENCH, S. & XIE, Z. (1994). A perspective on recent developments in utility theory. In Rios (1994).

- FRIEDMAN, M. & SAVAGE, L.J. (1948). The utility analysis of choices involving risk. *Journal of Political Economy*, **56**, 279-304.
- IACA (1998). National reports. *Transactions: 16th Conference, International Association of Consulting Actuaries*, Cape Town.
- RIOS, S. (ED.) (1994). *Decision Theory and Decision Analysis: Trends and Challenges*. Kluwer, Dordrecht.
- THOMSON, R.J. (1998a). Investment channel choice in defined contribution retirement funds: the use of utility functions. *8th International AFIR Colloquium*, 489-508.
- THOMSON, R.J. (1998b). A defence of expected utility theory for certain actuarial applications. *Transactions of the Actuarial Society of South Africa* **12(1)**, 174-234.

**QUESTIONNAIRE USED FOR THE
ELICITATION OF UTILITY FUNCTIONS**

1. What is your gross annual income from employment?

up to R20 000	
R20 001 to R50 000	
R50 001 to R100 000	
over R100 000	

2. If you could choose between:

- (1) a 50-50 chance that your income after retirement (gross of tax) in today's money terms will be:
 - (a) 20%; or
 - (b) 100%
 of your current annual income from employment; or
- (2) having an income after retirement equal to 50% of your current annual income from employment with certainty,

which of (1) and (2) would you choose?

Enter "1" or "2", or if you are indifferent, leave blank:

If your answer to question 2 was "1" go to question 3; if it was "2" go to question 4. Then proceed to question 5. If you left it blank go direct to question 5.

3. To what value would the percentage in 2(2) have to be increased in order to change your answer to question 2?

4. To what value would the percentage in 2(2) have to be decreased in order to change your answer to question 2?

5. If you could choose between:

- (1) a 50-50 chance that your income after retirement (gross of tax) in today's money terms will be:
 - (a) 20%; or
 - (b)%¹

¹ Insert here your answer to question 3 or 4 or, if you left your answer to question 2 blank, insert 50%.

of your current annual income from employment; or

- (2) having an income after retirement equal to%² of your current annual income from employment with certainty,

which of (1) and (2) would you choose?

Enter "1" or "2", or if you are indifferent, leave blank:

If your answer to question 5 was "1" go to question 6; if it was "2" go to question 7. Then proceed to question 8. If you left it blank go direct to question 8.

6. To what value would the percentage in 5(2) have to be increased in order to change your answer to question 5?

7. To what value would the percentage in 5(2) have to be decreased in order to change your answer to question 5?

8. If you could choose between:

- (1) a 50-50 chance that your income after retirement (gross of tax) in today's money terms will be:

(a)%³; or

(b) 100%

of your current annual income from employment; or

- (2) having an income after retirement equal to%⁴ of your current annual income from employment with certainty,

which of (1) and (2) would you choose?

Enter "1" or "2", or if you are indifferent, leave blank:

If your answer to question 8 was "1" go to question 9; if it was "2" go to question 10. Then proceed to question 11. If you left it blank go direct to question 11.

9. To what value would the percentage in 8(2) have to be increased in order to change your answer to question 8?

² Insert here a value equal to $\frac{2}{3} \times (a) + \frac{1}{3} \times (b)$.

³ Insert here the percentage in question 5(1)(b).

⁴ Insert here a value equal to $\frac{2}{3} \times (a) + \frac{1}{3} \times (b)$.

10. To what value would the percentage in 8(2) have to be decreased in order to change your answer to question 8?

11. If you could choose between:

(1) a 50-50 chance that your income after retirement (gross of tax) in today's money terms will be:

(a)%⁵; or

(b)%⁶

of your current annual income from employment; or

(2) having an income after retirement equal to%⁷ of your current annual income from employment with certainty,

which of (1) and (2) would you choose?

Enter "1" or "2", or if you are indifferent, leave blank:

If your answer to question 11 was "1" go to question 12, if it was "2" go to question 13.

12. To what value would the percentage in 11(2) have to be increased in order to change your answer to question 11?

13. To what value would the percentage in 11(2) have to be decreased in order to change your answer to question 11?

⁵ Insert here your answer to question 6 or 7 or, if you left your answer to question 5 blank, the percentage in question 5(2).

⁶ Insert here your answer to question 9 or 10 or, if you left your answer to question 8 blank, the percentage in question 8(2).

⁷ Insert here your answer to question 3 or 4 or, if you left your answer to question 2 blank, "50%".