Contingent claims analysis in life and pension insurance

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Abstract

The application of mathematical finance to unit-linked life insurance is unified with the theory of distribution of surplus in life and pension insurance. The unification is based on a contractualization of the distribution of surplus. We suggest a distinction between the retrospective surplus and the prospective surplus and study these versions of the surplus in details. The retrospective surplus and the prospective surplus are proposed as indices in a type of index linked insurance which we appropriately call surplus-linked insurance.

Key words: Securitization, prospective/retrospective surplus, dividends.

1 Introduction

The term life and pension insurance is here used for the general type of life insurance where premiums and benefits are calculated on a certain basis at the time of issue and then revised, as time goes by, according to the performance of the insurance company. The revision of premiums and benefits can take various forms depending on the type of the contract. Examples are various types of participating life insurance (in some countries called with-profit life insurance) and various types of pension funding.

The revision of premiums and benefits is based on payment of *dividends*, which in general may be positive or negative, from the insurance company to the insurance contract holder. It is important to distinguish between two aspects of the revision, the *dividend plan* and the *bonus plan*. The dividend plan is the plan for recording of dividends. However, often the dividends are not paid out immediately as cash but are converted into a stream of future payments. The bonus plan is the plan for how the dividends are reflected in payments. In this paper, we shall focus on the dividend and the dividend plan.

In [6] a framework of securitization is developed where reserves are no longer defined as expected present values but as market prices of streams of payments (which, however, happen to be expressible as expected present values under adjusted measures). An insurance contract is defined as a stream of payments linked to dynamical indices, opening for a wide range of insurance contracts including various unit-linked contracts. The framework of securitization is one way of dealing with the dependence between the risk in the insurance contract and the risk in the financial market. It is built on the consideration of the stream of payments contracted in an insurance contract as a dynamically traded object on the financial market. The insurance company is then considered as a participant in this market and has to adapt prices and strategies to the conditions in the market.

In the present paper we construct a general life and pension insurance contract within the framework developed in [6]. Working with general index-linked payments in participating life insurance and pension funding we go beyond the traditional set-up of payments in existing literature on surplus and dividends. However, index-linked payments open for a number of appealing set-ups. An important one is obtained by linking payments directly to the surplus as it will be defined in this paper. The study of such "surplus-linked insurance" has a two-fold motivation: Firstly, it represents a new product combining properties of participating life insurance, pension funding, and unit-linked insurance. Secondly, it seems to represent a good imitation of the behavior of managers. As such it can be used as a management tool as well as a market analysis tool.

Even though payments need not be linked directly to the surplus, the surplus may be an important piece of information included in the information on which the insurance company bases the dividends. It is one of the main purposes of this paper to provide insight in the dynamics of the surplus, and an important step is the classification of surplus into the retrospective surplus and the prospective surplus.

Subjugating life and pension insurance to the market conditions, the appropriate tool seems to be mathematical finance or rather contingent claims analysis. Contingent claims analysis, and in particular option pricing theory, was introduced as a tool for analysis and management of both unit-linked insurance and pension funding in the seventies (see [1] and [2], respectively, and references therein), whereas contingent claims analysis as a tool for analysis and management of participating life insurance has been long in coming and was, to our knowledge, introduced in [3]. A main reason for this delay may be that the link between the payments and the performance of the company in participating life insurance may be provided by statute so vaguely that it is an unreasonable approximation to consider dividends as contractual. Working in a framework of securitization, our main objection to this argument is, of course, that the insurance business and, hereby, the participation in the performance takes place in a competitive market. Thus, the insurance company is forced to adapt e.g. its plans for revising payments to the market conditions. This objection is at the same time the primary argument for applying contingent claims analysis to life and pension insurance at all.

The paper is structured as follows. In Section 2, we recapitulate the framework developed in [6]. In Section 3, we construct the general life and pension insurance contract within that framework and extract some decision problems of a life insurance company. The retrospective and prospective surplus are defined and studied in Section 4. In Section 5, we consider dividend plans in general and surplus-linked dividends in particular.

2 The insurance contract

The basics

In this section we recapitulate the framework developed in [6] and state the main result obtained there. For motivation, details, and examples the reader is asked to confer [6].

We take as given a probability space $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}, P)$. Let $(X_t)_{t\geq 0}$ be a cadlag jump process with finite state space $\mathcal{J} = (1, \ldots, J)$ and associate a marked point process (T_n, Φ_n), where T_n denotes the time of the nth jump of X_t, and Φ_n is the state entered at time T_n, i.e. $X_{T_n} = \Phi_n$. Introduce the counting processes

$$N_t^j = \sum_{n=1}^{\infty} I(T_n \leq t, X_{T_n} = j), \quad j \in \mathcal{J},$$

and the J-dimensional vector

$$N_t = \begin{bmatrix} N_t^1 \\ \vdots \\ N_t^J \end{bmatrix}.$$

Let $(W_t)_{t\geq 0}$ be a standard K-dimensional Brownian motion such that W_t^k and W_t^l are independent for $k \neq l$. The filtration $\mathbf{F}$ formalizes the flow of information generated by X and W.

For a matrix A, we let A^T denote the transpose of A and let $A^{i\cdot}$ and $A^{\cdot i}$ denote the ith row and the ith column of A, respectively. For a vector a, we let $D(a)$ denote the diagonal matrix with the components of a on the principal diagonal and 0 elsewhere. We shall write $\delta^{1\times J}$ and $\delta^{J\times 1}$ instead of the J-dimensional vectors $(\delta, \ldots, \delta)$ and $(\delta, \ldots, \delta)^T$, respectively. For derivatives, we shall use the notation $d_x = \frac{\partial}{\partial x}$ and $d_{xy} = \frac{\partial^2}{\partial x \partial y}$. For a vector a, we let $\int a$ and da mean componentwise integration and componentwise differentiation, respectively.

We introduce an *index* S, an $(I+1)$-dimensional vector of processes, the dynamics of which is given by

$$dS_t = \alpha_t^S dt + \beta_{t-}^S dN_t + \sigma_t^S dW_t, \; S_0 = s_0,$$

where $\alpha^S \in \mathbf{R}^{(I+1)}$, $\beta^S \in \mathbf{R}^{(I+1)\times J}$, and $\sigma^S \in \mathbf{R}^{(I+1)\times K}$ are functions of (t, S_t) and $s_0 \in \mathbf{R}^{I+1}$ is $\mathcal{F}_0$-measurable. The information generated by S is formalized by the filtration $\mathbf{F}^S = \left\{\mathcal{F}_t^S\right\}_{t\geq 0}$, where

$$\mathcal{F}_t^S = \sigma(S_s, \; 0 \leq s \leq t) \subseteq \mathcal{F}_t.$$

We assume that S is a Markov process and that there exist deterministic piecewise continuous functions $\mu^j(t,s)$, $j \in \mathcal{J}$, $s \in \mathbf{R}^{I+1}$ such that N_t^j admits the $\mathcal{F}_t^S$-intensity process $\mu_t^j = \mu^j(t, S_t)$ and we introduce the J-dimensional vectors containing the intensity processes and martingales associated with N,

$$\mu_t = \begin{bmatrix} \mu_t^1 \\ \vdots \\ \mu_t^J \end{bmatrix}, \quad M_t = \begin{bmatrix} M_t^1 \\ \vdots \\ M_t^J \end{bmatrix}.$$

We introduce a *market* Z, an $(n+1)$-dimensional vector $(n \leq I)$ of price processes assumed to be positive. The market Z consists of exactly those entries of S that are marketed, i.e. traded on a given market. We assume that there exists a short rate of interest such that the market comprises a price process Z^0 with the dynamics given by

$$dZ_t^0 = r_t Z_t^0 dt, Z_0^0 = 1.$$

This price process can be considered as the value process of a unit deposited on a bank account at time 0. This entry will be referred to as the riskfree asset, even though r_t is allowed to depend (t, S_t). It should be carefully noted that Z is included in S. Thus, referring to the components of the index S as indices themselves, we can refer to the components of Z as marketed indices.

Fixing some time horizon T, we now define an *insurance contract* to be a triplet (S, Z, B), where B is an $\mathcal{F}_t^S$-adapted, cadlag process of finite variation starting at B_0 at time 0 and with dynamics given by

$$dB_t = b_t^c dt - b_{t-}^d dN_t - \Delta B_{T-} d1_{(t \geq T)}, \; t \in (0, T],$$

where $b^c \in \mathbf{R}$ and $b^d \in \mathbf{R}^J$ are functions of (t, S_t) and $\Delta B_T \in \mathbf{R}$ is a function of (T, S_T). Note that the F_t^S-adaptedness of B makes demands on the connection between the coefficients of S and the coefficients of B. The process B is called the *payments* since B_t denotes the cumulative payments from the contract holder to the insurance company over $[0, t]$. To simplify notation, lump sum payments at deterministic times are restricted to time 0 and time T. The insurance contract forms the basis for introduction of two price processes, F and V. F_t is the market price at time t of the contractual payments to the insurance company over $[0, T]$, i.e. past and future premiums less benefits, and V_t is the market price at time t of the contractual payments from the insurance company over $(t, T]$, i.e. future benefits less premiums.

The insurance company receives payments according to the insurance contract (S, Z, B) and we assume that these payments are invested in a value process U. An investment strategy $\theta \in R^{n+1}$ is assumed to generate U, i.e.

$$U_t = \theta_t \cdot Z_t = \sum_{i=0}^{n} \theta_t^i Z_t^i.$$

We require that this strategy is $\mathcal{F}_t^S$-Markovian, i.e. $\theta_t = \theta\left(t, S_{t-}\right)$, selffinancing, i.e. $dU_t = \theta_t \cdot dZ_t$, fulfilling $U_t > 0$ for all t, and complying with whatever institutional requirements there may be. We emphasize that θ, in general, is not a strategy aiming at hedging some contingent claim. Introduce α_t^U, β_t^U, and σ_t^U such that

$$dU_t = U_t \alpha_t^U dt + U_{t-} \beta_{t-}^U dM_t + U_t \sigma_t^U dW_t.$$

The main result

According to the above definitions and assumptions we have that

$$F_t = L_t - V_t,$$

where

$$L_t = U_t \int_{0-}^{t} \frac{1}{U_s} dB_s.$$

Our approach to the price process F is the following: Assuming that the market Z is arbitrage free, we require that also the market (Z, F) be arbitrage free. We use the essential equivalence between arbitrage free markets and existence of a so-called martingale measure, i.e. a measure under which discounted asset prices are martingales. If the no arbitrage condition is fulfilled for (Z, F) we shall speak of

(S, Z, B) as an arbitrage free insurance contract and about V as the corresponding arbitrage free reserve.

Since the market may be incomplete, there may be several martingale measures and, correspondingly, several arbitrage free reserves. Thus, when we talk of *the* arbitrage free reserve, we think of having fixed a martingale measure according to some criterion. Alternatively, one could imagine that there exists only one martingale measure reflecting the market participants' attitudes to risk although this measure, in the incomplete market, is not to be identified by looking at asset prices only. In this case *the* martingale measure could, appropriately, be fixed as the unique measure reflecting the attitudes to risk.

We restrict ourselves to price operators allowing V_t to be written on the form $V(t, S_t)$. This restriction seems reasonable since S is Markov, but it should be noted that this is actually an assumption on the structure of the price operator. It corresponds to the restrictive structure of the measure transformation that we now enter by defining the likelihood process Λ by

$$\begin{aligned} d\Lambda_t &= \Lambda_{t-}\left(\sum_j g^j_{t-} dM^j_t + \sum_k h^k_t dW^k_t\right) \\ &= \Lambda_{t-}\left(g^T_{t-} dM_t + h^T_t dW_t\right), \\ \Lambda_0 &= 1, \end{aligned}$$

where we have introduced

$$g^j_t = g^j(t, S_t),\ h^k_t = h^k(t, S_t),$$

and

$$g_t = \begin{bmatrix} g^1_t \\ \vdots \\ g^J_t \end{bmatrix},\ h_t = \begin{bmatrix} h^1_t \\ \vdots \\ h^K_t \end{bmatrix}.$$

With conditions on (g, h) we can now change measure from P to Q on (Ω, F_T) by the definition,

$$\Lambda_T = \frac{dQ}{dP}.$$

Upon introducing

$$\begin{aligned} V^j_t &= V\left(t, S_{t-} + \beta^{S\cdot j}_{t-}\right), \\ V^{\mathcal{J}}_t &= \left[V^1_t, \ldots, V^J_t\right], \\ \psi_t &= \frac{1}{2} tr\left(\left(\sigma^S_t\right)^T d_{ss} V_t \sigma^S_t\right), \\ R_t &= b^d_t + V^{\mathcal{J}}_t - V_t 1^{1\times J}, \end{aligned}$$

we can, finally, state the main result of [6]:

Theorem 1 *Assume that the partial derivatives d_tV, d_sV, and $d_{ss}V$ exist and are continuous. Assume that (g,h) can be chosen such that*

$$\alpha_t^Z + \sigma_t^Z h_t + \beta_t^Z D\left(1^{J\times 1} + g_t\right)\mu_t - r_t Z_t = 0. \tag{1}$$

If the arbitrage free reserve on an insurance contract (S,Z,B) can be written on the form $V(t,S_t)$, then $V(t,s)$ solves for some (g,h) subject to (1) the differential equation

$$\begin{aligned} d_tV_t &= b_t^c + r_tV_t - (d_sV_t)^T\left(\alpha_t^S + \sigma_t^S h_t\right) - R_t D\left(1^{J\times 1} + g_t\right)\mu_t - \psi_t, \\ V_{T-} &= \Delta B_T, \end{aligned} \tag{2}$$

and has the representation

$$V_t = E^Q\left(Z_t^0 \int_t^T \frac{1}{Z_u^0} d\left(-B_u\right) \middle| S_t \right). \tag{3}$$

The payments B of an arbitrage free insurance contract fulfills the prospective equivalence relation

$$V_{0-} = 0. \tag{4}$$

The prospective equivalence relation is equivalent to the retrospective equivalence relation

$$E^Q\left(\frac{L_T}{Z_T^0}\right) = 0. \tag{5}$$

3 The general life and pension insurance contract

The first order basis, the first order payments, and the technical basis

In this section we formulate the structure of the general life and pension insurance contract within the framework recapitulated in Section 2.

We introduce a *first order basis*, $\left(\widehat{r}, \widehat{g}, \widehat{h}\right)$, let the first order short rate of interest $\widehat{r}$ drive a first order risk free paper $\widehat{Z}$, and let the first order Girsanov kernel $\left(\widehat{g}, \widehat{h}\right)$ determine a first order measure $\widehat{Q}$. Here, $\widehat{r}$, $\widehat{g}$, and $\widehat{h}$ are functions of (t, S_t). We define a stream of *first order payments* $\widehat{B}$ in the same way as B is defined in Section 2, i.e. linked to (t,S), and we define the *first order reserve* by

$$\widehat{V}_t = E^{\widehat{Q}}\left(\widehat{Z}_t \int_t^T \frac{1}{\widehat{Z}_u} d\left(-\widehat{B}_u\right) \middle| S_t \right).$$

Then, upon introducing

$$\begin{aligned} \widehat{V}_t^j &= \widehat{V}\left(t, S_{t-} + \beta_{t-}^{S\cdot j}\right), \\ \widehat{V}_t^{\mathcal{J}} &= \left[\widehat{V}_t^1, \ldots, \widehat{V}_t^J\right], \\ \widehat{R}_t &= \widehat{b}_t^d + \widehat{V}_t^{\mathcal{J}} - \widehat{V}_t 1^{1\times J}, \\ \widehat{\psi}_t &= \frac{1}{2}\mathrm{tr}\left(\left(\sigma_t^S\right)^T d_{ss}\widehat{V}_t \sigma_t^S\right), \end{aligned}$$

we have, according to Theorem 1, the first order Thiele's differential equation and the first order terminal condition,

$$\begin{aligned} d_t \widehat{V}_t &= \widehat{b}_t^c + \widehat{r}_t \widehat{V}_t - \left(d_s \widehat{V}_t\right)^T \left(\alpha_t^S + \sigma_t^S \widehat{h}_t\right) - \widehat{R}_t D \left(1^{J \times 1} + \widehat{g}_t\right) \mu_t - \widehat{\psi}_t, \\ \widehat{V}_{T-} &= \Delta \widehat{B}_T. \end{aligned} \tag{6}$$

We let $\widehat{B}$ be constrained by the first order prospective equivalence relation,

$$\widehat{V}_{0-} = 0. \tag{7}$$

Actually, this construction of the first order payments corresponds to requiring that $\left(\left(S, \widehat{Z}\right), \widehat{Z}, \widehat{B}\right)$ is an arbitrage free contract.

The first order basis serves solely as a tool for laying down the first order payments at time 0. However, also during the term of the contract, the insurance company needs to valuate the first order payments for different operations. The appropriate conditions for such a valuation depend on what operation is performed. We shall introduce a *technical basis*, (r^*, g^*, h^*) with functions r^*, g^*, and h^* of (t, S_t) for such a valuation of first order payments and define the *technical reserve* as

$$V_t^* = E^{Q^*}\left(Z_t^* \int_t^T \frac{1}{Z_u^*} d\left(-\widehat{B}_u\right) \middle| S_t\right).$$

The technical Thiele's differential equation and the technical terminal condition is now obtained by replacing $\left(\widehat{r}, \widehat{g}, \widehat{h}, \widehat{V}\right)$ in (6) with (r^*, g^*, h^*, V^*). Note that there exists no technical equivalence relation. Only if the first order basis is used as technical basis, the relation $V_{0-}^* = 0$ holds.

The technical basis plays a role in the operation of reporting to the owners of the company and to the supervising authorities. The insurance company may draw up a statement of accounts at market value if the owners of the company and/or the supervising authorities want a true picture of the company. For this operation the real basis introduced in the following subsection seems to be an obvious choice for technical basis. However, specific conditions for solvency may be formulated under another basis and the supervising authorities may require a presentation of accounts on such a basis. Such a basis could e.g. be the first order basis.

The real basis and the dividends

As opposed to the first order basis $\left(\widehat{r}, \widehat{g}, \widehat{h}\right)$ and the technical basis (r^*, g^*, h^*) we shall speak of (r, g, h) as the real basis. Since the first order basis may differ from the real basis, the first order payments may impose arbitrage possibilities in the real environment. However, the *real payments* B are to be determined such that (S, Z, B) constitutes an arbitrage free insurance contract in the real environment. The real payments are composed by the first order payments and an additional payment stream $\widetilde{B}$ called the *dividends*, i.e.

$$B = \widehat{B} + \widetilde{B}. \tag{8}$$

We want to work within the framework of Section 2, and we are therefore interested in index-linked dividends. The index to which dividends are linked may be the same index as the index to which also the first order payments are linked. However, we may also augment this index by further state variables.

The formulas of Theorem 1 then reads the real martingale measure constraint, the real Thiele's differential equation, the real expected value representation, and the real prospective and retrospective equivalence relations. If the dividends are designed in such a way that the contract (S, Z, B) is arbitrage free, i.e. the real equivalence relation holds, we shall simply say that the dividends are arbitrage free. We shall be interested in designing the dividends such that they are index-linked and arbitrage free.

The dividends rectifies a possible imbalance between the first order basis and the real basis in the sense that we get from putting (8) into (3) and (4),

$$E^Q\left(\int_{0-}^{T} \frac{1}{Z_t^0} d\tilde{B}_t\right) = -E^Q\left(\int_{0-}^{T} \frac{1}{Z_t^0} d\hat{B}_t\right). \tag{9}$$

The sign of $E^Q\left(\int_{0-}^{T} \frac{1}{Z_t^0} d\hat{B}_t\right)$ decides whether an insurance contract has positive or negative dividends in expectation. In particular, if the real basis is used as first order basis, then the expected dividends become zero. In this case, the dividends given by $\tilde{B} = 0$ would, obviously, be arbitrage free, and the *unrevised* contract would be the appropriate name for this particular construction.

In participating life insurance the dividends are restricted to be to the policy holders advantage, i.e. $\tilde{B}$ must be a non-increasing process with $\tilde{B}_0 \leq 0$. From (9) it is seen that there will exist arbitrage free dividend plans to the contract holders advantage, only if

$$E^Q\left(\int_{0-}^{T} \frac{1}{Z_t^0} d\hat{B}_t\right) \geq 0. \tag{10}$$

On the other hand, if (10) is fulfilled, an arbitrage free dividend plan can easily be devised. We conclude that (10) is a necessary and sufficient condition on the relation between the first order basis and the real basis for existence of an arbitrage free dividend plan. The interpretation is that the insurance company cannot come up with dividends to the policy holders advantage arbitrage freely if the first order payments are to the policy holders advantage already from the beginning. But if they are not, there will exist a continuum of arbitrage free dividend plans.

A delicate decision problem

When designing a life insurance product we face a delicate decision problem. First of all, we have to decide on a first order basis. Given this first order basis, we need to decide on a dividend plan such that the insurance contract becomes arbitrage free. One can think of many dividend plans, and amongst them some quite obscure ones. To mention a few one could pay out $\tilde{B}_0 = -E^Q\left(\int_{0-}^{T} \frac{1}{Z_t^0} d\hat{B}_t\right)$ as a deterministic lump sum payment at time 0 and hereby get through the revision of payments at time of issue. If the policy holder is bored by this plan one could simply toss a coin to see whether the policy holder should receive a deterministic lump sum at time 0 or not. The size of the lump sum would depend on the market's attitude to toss-up. Usually, however, the policy holder is more interested in gambling on the financial market, and if we somehow let the revision of payments depend on the portfolio U, e.g. through the process L_t, also the strategy θ becomes a part of the decision problem.

We want to design products which in these decision aspects imitate the manager of a life insurance company. The problem is to come up with an appropriate index which, on one hand, contains the important information on which the manager bases the decisions and, on the other hand, is mathematical tractable, i.e. not with "too many" state variables. We shall in the next section study, thoroughly, the notion of surplus, since this seems to be the all-important piece of information on the basis of which the manager makes the decisions concerning dividends. The surplus introduced in the next section depends on the technical basis. Hereby determination of dividends is added to the list of operations for which a technical bases must be specified.

The delicate decision is made subject to two basic constraints. Firstly, we have the arbitrage condition

$$V_{0-} = 0,$$

which appropriately could be called the market constraint. Secondly, we have the legislative constraints. They could e.g. simply put bounds on the first order rate of interest. More interesting are possible constraints on the relation between the dividends and the surplus. If such a relation is included in the legislative constraints, it is of course important that the insurance company and the supervising authorities agree upon what a surplus is and, possibly, on which technical basis the surplus should be based.

4 The notion of surplus

The retrospective surplus

We define the *retrospective surplus* $\overleftarrow{F}_t^*$ corresponding to a technical basis (r^*, g^*, h^*) by

$$\begin{aligned} \overleftarrow{F}_t^* &= L_t - V_t^*, \qquad t \geq 0, \\ \overleftarrow{F}_{0-}^* &= 0, \end{aligned}$$

and start out by noting that the real equivalence principle can be expressed in terms of the retrospective surplus: Since

$$E^Q\left(\frac{\overleftarrow{F}_T^*}{Z_T^0}\right) = E^Q\left(\frac{L_T}{Z_T^0}\right),$$

we conclude from (5) that the dividends are arbitrage free if and only if

$$E^Q\left(\frac{\overleftarrow{F}^*_T}{Z^0_T}\right)=0. \tag{11}$$

Using Ito's formula on $\overleftarrow{F}^*_t$, we get

$$\begin{aligned}
d\overleftarrow{F}^*_t &= dL_t - dV^*_t \\
&= b^c_t dt - b^d_{t-} dN_t + L_{t-}\frac{dU_t}{U_{t-}} - \left(d_t V^*_t + \left(d_s V^*_t\right)^T \alpha^S_t + \psi^*_t\right) dt \\
&\quad - \left(V^{*J}_t - V^*_{t-}1^{1\times J}\right) dN_t - \left(d_s V^*_t\right)^T \sigma^S_t dW_t.
\end{aligned}$$

Subtraction of the technical Thiele's differential equation and some rearrangements give the form

$$\begin{aligned}
d\overleftarrow{F}^*_t &= \overleftarrow{F}^*_{t-}\left(\alpha^U_t dt + \beta^U_{t-} dM_t + \sigma^U_t dW_t\right) \\
&\quad + \left(\left(\alpha^U_t - r^*_t\right) V^*_t + \left(d_s V^*_t\right)^T \sigma^S_t h^*_t + R^*_t D\left(g^*_t\right)\mu_t\right) dt \\
&\quad + \left(V^*_t \sigma^U_t - \left(d_s V^*_t\right)^T \sigma^S_t\right) dW_t + \left(V^*_{t-}\beta^U_{t-} - R^*_{t-}\right) dM_t + d\widetilde{B}_t.
\end{aligned} \tag{12}$$

It can now be shown, by Ito's formula, that $\overleftarrow{F}^*_t$ can be written on the form

$$\overleftarrow{F}^*_t = \int_{0-}^t \frac{U_t}{U_s}\left(d\left(C^*_s + \widetilde{B}_s\right) - \sigma^*_s \sigma^U_s ds\right), \tag{13}$$

where C^* is a process defined by

$$C^*_0 = -V^*_{0-}, \tag{14}$$

$$dC^*_t = \alpha^*_t dt + \beta^*_{t-} dM_t + \sigma^*_t dW_t,\ t>0, \tag{15}$$

with

$$\begin{aligned}
\alpha^*_t &= \left(\alpha^U_t - r^*_t\right) V^*_t + \left(d_s V^*_t\right)^T \sigma^S_t h^*_t + R^*_t D\left(g^*_t\right)\mu_t, \\
\beta^*_t &= V^*_t \beta^U_t - R^*_t, \\
\sigma^*_t &= V^*_t \sigma^U_t - \left(d_s V^*_t\right)^T \sigma^S_t.
\end{aligned}$$

The term $\sigma^*_t \sigma^U_t dt$ in (13) is a correction term stemming from the dependence between the diffusion increment of dU_t and the diffusion increment of dC^*_t.

Apart from the term $d\widetilde{B}_t$, (12) and (15) are written on semimartingale form under the measure P. Since (11) is a relation under the measure Q, we shall derive a corresponding semimartingale form under Q,

$$\begin{aligned}
d\overleftarrow{F}^*_t &= \overleftarrow{F}^*_{t-}\left(r_t dt + \beta^U_{t-} dM^Q_t + \sigma^U_t dW^Q_t\right) \\
&\quad + \left(\left(r_t - r^*_t\right) V^*_t + \left(d_s V^*_t\right)^T \sigma^S_t\left(h^*_t - h_t\right) + R^*_t D\left(g^*_t - g_t\right)\mu_t\right) dt \\
&\quad + \left(V^*_t \sigma^U_t - \left(d_s V^*_t\right)^T \sigma^S_t\right) dW^Q_t + \left(V^*_{t-}\beta^U_{t-} - R^*_{t-}\right) dM^Q_t + d\widetilde{B}_t,
\end{aligned} \tag{16}$$

and

$$dC_t^* = \alpha_t^{*Q} dt + \beta_{t-}^* dM_t^Q + \sigma_t^* dW_t^Q \tag{17}$$

with

$$\alpha_t^{*Q} = (r_t - r_t^*) V_t^* + (d_s V_t^*)^T \sigma_t^S (h_t^* - h_t) + R_t^* D (g_t^* - g_t) \mu_t. \tag{18}$$

Allowing, for a moment, of diffusion payments, we note that the retrospective surplus can be considered as the retrospective reserve of an insurance contract with payments given by $C^* + \widetilde{B}$ minus the correction term $\int \sigma_s^* \sigma_s^U ds$. An appealing interpretation of this payment process is to consider the process $C^* - \int \sigma_s^* \sigma_s^U ds$ as the premium payments, in general positive or negative, and the process $\widetilde{B}$ as the benefit process, in general positive or negative. The payments of this contract start out with a lump sum payment at time 0 of $C_0^* + \widetilde{B}_0 = B_0 - V_0^*$ and develop according to dC^* (15) and $d\widetilde{B}$, including a lump sum payment at time T of $\Delta \widetilde{B}_T$. The relation (11) is simply the retrospective equivalence relation of this contract.

The prospective surplus

We define the *prospective surplus* $\overrightarrow{F}_t^*$ corresponding to a technical basis (r^*, g^*, h^*) by

$$\begin{aligned} \overrightarrow{F}_t^* &= V_t - V_t^*, \qquad t \geq 0, \\ \overrightarrow{F}_{0-}^* &= 0, \end{aligned} \tag{19}$$

and start out by noting that the real equivalence principle can be expressed in terms of the prospective surplus: Since

$$\overrightarrow{F}_0^* = V_0 - V_0^*,$$

we conclude from (4) that the dividends are arbitrage free if and only if

$$\overrightarrow{F}_0^* = B_0 - V_0^*. \tag{20}$$

Using Ito's formula on $\overrightarrow{F}_t^*$, we get

$$\begin{aligned} d\overrightarrow{F}_t^* &= \left(d_t V_t + d_s V_t \alpha_t^S + \psi_t\right) dt + \left(V_t^{\mathcal{J}} - V_{t-} 1^{1\times J}\right) dN_t + (d_s V_t)^T \sigma_t^S dW_t \\ &\quad - \left(d_t V_t^* + d_s V_t^* \alpha_t^S + \psi_t^*\right) dt - \left(V_t^{*\mathcal{J}} - V_{t-}^* 1^{1\times J}\right) dN_t - (d_s V_t^*)^T \sigma_t^S dW_t. \end{aligned}$$

Subtraction of the real Thiele's differential equation, addition of the technical Thiele's differential equation, and some rearrangements give the form

$$\begin{aligned} d\overrightarrow{F}_t^* &= r_t \overrightarrow{F}_t^* dt + \left((r_t - r_t^*) V_t^* + \left((d_s V_t^*)^T \sigma_t^S h_t^* - (d_s V_t)^T \sigma_t^S h_t\right)\right) dt \\ &\quad + \left(R_t^* D(g_t^*) - R_t D(g_t)\right) \mu_t dt \\ &\quad - (d_s V_t^* - d_s V_t)^T \sigma_t^S dW_t - \left(R_{t-}^* - R_{t-}\right) dM_t + d\widetilde{B}_t. \end{aligned} \tag{21}$$

Apart from the term $d\tilde{B}_t$, (21) is written on semimartingale form under the measure P. To compare with the dynamics of the retrospective surplus, we shall also derive the semimartingale form under the measure Q,

$$\begin{aligned} d\overrightarrow{F}_t^* &= r_t \overrightarrow{F}_t^* dt \\ &+ \left((r_t - r_t^*) V_t^* + (d_s V_t^*)^T \sigma_t^S (h_t^* - h_t) + R_t^* D (g_t^* - g_t) \mu_t \right) dt \quad (22) \\ &+ (d_s V_t - d_s V_t^*)^T \sigma_t^S dW_t^Q + \left(R_{t-} - R_{t-}^* \right) dM_t^Q + d\tilde{B}_t. \end{aligned}$$

Since $F_t = \overleftarrow{F}_t^* - \overrightarrow{F}_t^*$, we know that $\frac{\overleftarrow{F}_t^* - \overrightarrow{F}_t^*}{Z_t^0}$ is an $\mathcal{F}_t^S$-martingale under the measure Q (see [6]). Writing the retrospective surplus on the form (13), this can be used to derive an appealing representation of the prospective surplus. It follows that

$$\begin{aligned} \frac{\overleftarrow{F}_t^* - \overrightarrow{F}_t^*}{Z_t^0} &= E^Q \left(\left. \frac{\overleftarrow{F}_T^*}{Z_T^0} \right| \mathcal{F}_t^S \right) \\ &= E^Q \left(\left. \frac{1}{Z_T^0} \int_{0-}^t \frac{U_T}{U_s} \left(d\left(C_s^* + \tilde{B}_s \right) - \sigma_s^* \sigma_s^U ds \right) \right| \mathcal{F}_t^S \right) \\ &\quad + E^Q \left(\left. \frac{1}{Z_T^0} \int_t^T \frac{U_T}{U_s} \left(d\left(C_s^* + \tilde{B}_s \right) - \sigma_s^* \sigma_s^U ds \right) \right| \mathcal{F}_t^S \right) \\ &= E^Q \left(\left. \frac{U_T}{Z_T^0} \right| \mathcal{F}_t^S \right) \int_{0-}^t \frac{1}{U_s} \left(d\left(C_s^* + \tilde{B}_s \right) - \sigma_s^* \sigma_s^U ds \right) \\ &\quad + \int_t^T E^Q \left(\left. \frac{1}{U_s} \left(d\left(C_s^* + \tilde{B}_s \right) - \sigma_s^* \sigma_s^U ds \right) E^Q \left(\left. \frac{U_T}{Z_T^0} \right| \mathcal{F}_s^S \right) \right| \mathcal{F}_t^S \right) \\ &= \frac{U_t}{Z_t^0} \int_{0-}^t \frac{1}{U_s} \left(d\left(C_s^* + \tilde{B}_s \right) - \sigma_s^* \sigma_s^U ds \right) \\ &\quad + \int_t^T E^Q \left(\left. \frac{1}{U_s} d\left(C_s^* + \tilde{B}_s \right) \frac{U_s}{Z_s^0} \right| \mathcal{F}_t^S \right) \qquad (23) \\ &= \frac{\overleftarrow{F}_t^*}{Z_t^0} + E^Q \left(\left. \int_t^T \frac{1}{Z_s^0} d\left(C_s^* + \tilde{B}_s \right) \right| \mathcal{F}_t^S \right), \end{aligned}$$

and we get the following representation of the prospective surplus

$$\overrightarrow{F}_t^* = E^Q \left(\left. Z_t^0 \int_t^T \frac{1}{Z_s^0} d\left(-C_s^* - \tilde{B}_s \right) \right| \mathcal{F}_t^S \right). \qquad (24)$$

In (23) the correction term has disappeared since we work with Ito-integrals which are based on forward increments.

This shows that the prospective surplus can be considered as the prospective reserve of an insurance contract with payments given by $C^* + \tilde{B}$ (allowing for diffusion payments) and (19) is simply the prospective equivalence relation of this contract.

A deterministic differential equation for the statewise prospective surplus can be obtained in the same way as it is done for the prospective reserve in [6]. The only extension is the allowance for diffusion payments. Upon introducing

$$\begin{aligned}
\overrightarrow{F}_t^{*\mathcal{J}} &= V\left(t, S_{t-} + \beta_{t-}^{S\cdot j}\right) - V^*\left(t, S_{t-} + \beta_{t-}^{S\cdot j}\right), \\
\overrightarrow{R}_t^* &= \widetilde{b}_t^d + \overrightarrow{F}_t^{*\mathcal{J}} - \overrightarrow{F}_t^* 1^{1\times J}, \\
\overrightarrow{\psi}_t^* &= \frac{1}{2} tr\left(\left(\sigma_t^S\right)^T d_{ss}\overrightarrow{F}_t^* \sigma_t^S\right),
\end{aligned}$$

$\overrightarrow{F}_t^*$ solves the differential equation

$$\begin{aligned}
d_t\overrightarrow{F}_t^* &= \alpha_t^{*Q} + \widetilde{b}_t^c + r_t\overrightarrow{F}_t^* - \left(d_s\overrightarrow{F}_t^*\right)^T\left(\alpha_t^S + \sigma_t^S h_t\right) \\
&\quad - \overrightarrow{R}_t^* D\left(1^{J\times 1} + g_t\right)\mu_t - \overrightarrow{\psi}_t^*, \qquad (25) \\
F_{T-} &= \Delta\widetilde{B}_T.
\end{aligned}$$

As an alternative to the real Thiele's differential equation and the real prospective equivalence relation, (25) may be used together with the prospective equivalence relation (20) and (14) to determine an arbitrage free dividend plan.

5 Dividends

The contribution plan and the second order basis

We start this section by considering a simple dividend plan, called the *contribution plan*. It plays an important role in the definition of a notion of participating life insurance, the *second order basis*. The contribution plan amounts to arranging $\widetilde{B}$ such that the discounted retrospective surplus $\frac{\overleftarrow{F}_t^*}{Z_t^0}$ becomes a zero mean Q-martingale, i.e.

$$\begin{aligned}
\widetilde{B}_0 &= V_{0-}^*, \\
\widetilde{b}_t^c - \widetilde{b}_t^d D\left(1^{J\times 1} + g_t\right)\mu_t &= -\alpha_t^{*Q}. \qquad (26)
\end{aligned}$$

Since all terms of α_t^{*Q} are functions of (t, S_t), the contribution plan is index-linked. Furthermore, since $E^Q\left(\frac{\overleftarrow{F}_T^*}{Z_T^0}\right) = 0$, the contribution plan is arbitrage free by construction. Note that also the discounted prospective surplus $\frac{\overrightarrow{F}_t^*}{Z_t^0}$ is a zero mean Q-martingale under the contribution plan.

The equation (26) is actually one equation with two unknowns ($\widetilde{b}_t^c$ and $\widetilde{b}_t^d$) and we have a whole set of contribution plans. Usually, *the* contribution plan is considered as the special case where $\widetilde{b}_t^d = 0$, i.e.

$$\widetilde{b}_t^c = -\left(\left(r_t - r_t^*\right)V_t^* + \left(d_s V_t^*\right)^T \sigma_t^S\left(h_t^* - h_t\right) + R_t^* D\left(g_t^* - g_t\right)\mu_t\right). \qquad (27)$$

If there exists a vector $\left(\tilde{r},\tilde{g},\tilde{h}\right)$ such that (27) holds with (r,g,h) replaced by $\left(\tilde{r},\tilde{g},\tilde{h}\right)$ we call $\left(\tilde{r},\tilde{g},\tilde{h}\right)$ *the second order basis.* This means that the second order basis for a given dividend plan is the basis which, playing the role as real basis, turns the dividend plan into the contribution plan. It is seen that the basis (r,g,h) candidate to the second order basis only if the insurance contract actually follows the contribution plan. If the contract does not follow the contribution plan, then the real basis (r,g,h) does not candidate to the second order basis. The word candidate is appropriate here, since even if the insurance company actually follows the contribution plan, $\tilde{b}_t$ can be obtained from (27) with other bases $\left(\tilde{r},\tilde{g},\tilde{h}\right)$ than (r,g,h) ((27) is in this connection one equation with three unknowns). This recognition of the second order basis as a decision variable has to our knowledge not, previously, been described in theoretical literature, although it is well-known in practice. Sometimes the triplet of bases is completed by naming the real basis by *the third order basis.*

In Section 3 it was seen that there will exist arbitrage free dividend plans to the contract holders advantage if and only if

$$E^Q\left(\int_{0-}^{T}\frac{1}{Z_t^0}d\widehat{B}_t\right)\geq 0.$$

In the actuarial literature on participating life insurance one normally works with the much stricter requirement that, in particular, the contribution plan has to be arbitrage free and to the policy holders advantage. This is obtained if and only if

$$\begin{aligned} -V_{0-}^* &\geq 0, \\ \alpha_t^{*Q} &\geq 0. \end{aligned}$$

One way of achieving $\alpha_t^{*Q}\geq 0$ is by having all the components of α_t^{*Q} greater or equal to zero, i.e.

$$\begin{aligned} \left(r_t-r_t^*\right)V_t^* &\geq 0, \\ R_t^*D\left(g_t^*-g_t\right)\mu_t &\geq 0, \\ \left(d_sV_t^*\right)^T\sigma_t^S\left(h_t^*-h_t\right) &\geq 0. \end{aligned}$$

These are well-known relations (perhaps except for the last one which is a consequence of our diffusion type of indices).

Surplus-linked insurance

The apparently appealing contribution plan has considerable drawbacks in our framework. Here we refer to the fact that, under the contribution plan, the contract holder does not participate in the game of investment. The contribution plan leads to the same dividends independently of the investment strategy underlying U, namely the dividends corresponding to pure investment in the riskfree asset. This, on the other hand, explains why the contribution plan in some actuarial literature (e.g. [4] and [5]), naturally, can be taken as canonical: Only investment in the riskfree asset is taken into consideration there. In practice, however, the insurance companies hold large positions in risky assets and this circumstance is integrated in our set-up.

We propose another plan which we shall call the surplus-linked dividend plan. We add the retrospective surplus and the prospective surplus to the index and let the dividend plan be linked to this augmented index. It is clear from Section 4 that the augmented index is really an index since both the retrospective and the prospective surplus possess the dynamics of an index and the coefficients appearing in them are functions of (t, S_t).

We now go on by specifying the functional dependence of the dividends on the retrospective surplus and the prospective surplus. One can think of various constructions but we shall go directly to a continuous affine form given by

$$\begin{aligned}
\widetilde{B}_0 &= 0, \\
\widetilde{b}^d_t &= 0, \\
\widetilde{b}^c_t &= -\left(p_t + q_t \overleftarrow{F}^*_t\right) \text{ or } \widetilde{b}^c_t = -\left(p_t + q_t \overrightarrow{F}^*_t\right), \\
\Delta \widetilde{B}_T &= 0,
\end{aligned} \tag{28}$$

where p_t and q_t are specified functions of (t, S_t). If we are in the case of participating life insurance, (28) should be modified such that $\widetilde{b}^c_t \leq 0$, e.g. by

$$\widetilde{b}^c_t = -\left(p_t + q_t \overleftarrow{F}^*_t\right)^+ \text{ or } \widetilde{b}^c_t = -\left(p_t + q_t \overrightarrow{F}^*_t\right)^+. \tag{29}$$

Here the options structure of products in participating life insurance can be recognized. We shall refer to the form in (28) as the pension funding form and the form in (29) as the participating life insurance form.

There is a variety of candidates for the functions p_t and q_t. A simple form would be to let p_t and q_t be deterministic functions. Other examples are $q_t = \frac{q'_t}{V^*_t}$ or $q_t = \frac{q'_t}{R^*_t \mu_t}$, q'_t being a deterministic function. Hereby, we measure the surplus per technical reserve or per sum at risk. Such formulations could be motivated by solvency regulations.

As mentioned in Section 4, only the retrospective surplus (not the prospective surplus) is effected by risky investments. Thus, only dividends linked to the retrospective surplus meets the drawback of the contribution plan mentioned in the introduction of this subsection. We propose also dividends linked to the prospective reserve in order to make the theory more complete.

Theorem 1 is a constructive tool for determination of parameters (p, q) leading to arbitrage free dividends. For applications of the differential equation in Theorem 1 to a concrete example, see [7], where also the bonus plan, i.e. the plan for how the dividends are reflected in payments, is studied in details.

The forms (28) and (29) are examples of surplus-linked dividend plans. This continuous affine form is simple and is closely connected to the way gains and losses are typically amortized in pension funds. It is left to the reader to think of other forms. E.g. one could formulate the decision of dividends as a control problem and get various optimal surplus-linked dividend plans corresponding to various objective functions.

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