On Transformations of Actuarial Valuation Principles

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Abstract. In this paper we determine optimal trading strategies associated with the financial variance and standard deviation principles of Schweizer (1997). These principles take into consideration the possibilities of hedging on the financial market and are derived by an indifference argument, which embeds the traditional (actuarial) variance and standard deviation principles in a financial framework. We also investigate an alternative way of transforming actuarial principles and show that for the standard deviation principle this leads to the financial standard deviation principle. The principles are applied for the valuation and hedging of unit-linked life insurance contracts.

Key words: indifference pricing, variance principle, standard deviation principle, unit-linked insurance, variance optimal martingale measure.

JEL Classification: G10

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1 Introduction

Schweizer (1997) proposes a financial valuation principle that is derived from traditional actuarial premium calculation principles and at the same time takes into consideration the possibility of trading on a financial market. First, an a priori given actuarial valuation principle (which measures risk) is translated into a "measure of preferences". Then, this new measure is used in a so-called indifference argument to define a new financial premium principle, which can be viewed as a financial transformation of the a priori given actuarial principle. The financial counterparts of the actuarial variance and standard deviation principles are called the *financial variance principle* and the *financial standard deviation principle*, respectively.

The traditional variance and standard deviation principles are recalled in Section 2, and it is shown how these premium calculation principles can be translated into measures of preferences by use of an argument which is similar to a zero expected utility argument. In Section 3, we give some preliminaries and recall the definition of the so-called variance optimal martingale measure, which plays an important role for the financial valuation principles. The main results of Schweizer (1997) are reviewed in Section 4 in a simplified framework suitable for pricing of reinsurance contracts that combine traditional insurance risk and financial risk. In his framework, a liquid reinsurance market is not present and the only investment and trading possibilities are given on some financial market. Using the same set-up, Møller (1999b) investigates how the premiums under the financial valuation principles depend on the amount of information available to the reinsurer and obtains upper and lower bounds for these premiums involving only conditional expectations and variances.

We then show in Sections 4.1 and 4.2 how optimal strategies associated with these transformed valuation principles can be determined in the case of the variance principle and the standard deviation principle, thus adding to the existing results. The results presented can basically be viewed as applications of the main results of Schweizer (1997) since most proofs are based on the techniques used there. For the variance principle, the optimal strategy, which maximizes the derived measure of preferences, differs from the *mean-variance hedging strategy* (see e.g. Schweizer (1999)) only by a correction term, which is independent of the contract considered. The optimal strategy under the standard deviation principle is also related to the mean-variance hedging strategy, but in this case, the difference between the two strategies is more complex. In Section 5, we consider an alternative modification of the same principles, which we call the *direct financial transformation*. This approach has the advantage that it does not involve a translation of the actuarial premium calculation principle into a measure of preferences, as it is defined directly in terms of the original actuarial principle. In the case of the standard deviation principle, this approach leads to premiums which are similar to the ones computed by using the financial standard deviation principle. For the variance principle, however, we show that the direct financial transformation does not lead to reasonable premiums. Finally, some applications and numerical results related to unit-linked life insurance contracts are presented in Section 6. The results are compared to the risk-minimizing strategies obtained in Møller (1998, 1999a).

2 The actuarial premium calculation principles

In this section, we first introduce the two classical actuarial premium calculation principles which will be analyzed in the following: the variance and the standard deviation principles. Second, we recall that these valuation principles can be viewed as the solutions to certain simple indifference principles. This serves as a motivation for the results presented in Section 4.

Let H be a claim (or risk) which is to be valuated by an agent, henceforth called a reinsurer. The following actuarial valuation principles are widely used:

$$\tilde{u}_1(H) = \mathrm{E}[H] + a\mathrm{Var}[H], \tag{2.1}$$

$$\tilde{u}_2(H) = \mathrm{E}[H] + a\sqrt{\mathrm{Var}[H]}. \tag{2.2}$$

In the actuarial literature, (2.1) is called the variance principle and (2.2) is the standard deviation principle, see e.g. Goovaerts et al. (1984). The terms $a\mathrm{Var}[H]$ and $a\mathrm{Var}[H]^{1/2}$, respectively, are often called the safety loadings, and we shall refer to a as the safety loading parameter. It is convenient to work with the negative of H, $Y = -H$, which can be interpreted as the amount received by the reinsurer, and we introduce now the following slightly modified versions of the premium principles (2.1) and (2.2):

$$u_1(Y) = \mathrm{E}[Y] - a\mathrm{Var}[Y], \tag{2.3}$$

$$u_2(Y) = \mathrm{E}[Y] - a\sqrt{\mathrm{Var}[Y]}, \tag{2.4}$$

Note that u_i differs from $\tilde{u}_i$, $i = 1,2$, by the sign on the loading factor and by the fact that $\tilde{u}_i$ operates on $-H$. Thus, we shall think of $\tilde{u}_i$ as a measure of risk, whereas u_i is taken as a "measure of preferences". We can indeed obtain the original principles (2.1) and (2.2) from (2.3) and (2.4) by noting that the unique solution $\hat{p}_i$ to the equality

$$u_i(p_i - H) = u_i(0) = 0, \tag{2.5}$$

is given by $\hat{p}_i = \tilde{u}_i(H)$, $i = 1,2$. This way of defining the premium is compatible with the *zero expected utility increase principle*, since it leaves the reinsurer indifferent between selling and not selling the risk; see e.g. Goovaerts et al. (1984) for more details. Note however that the equation (2.5) does not involve a proper utility function. Instead, we simply interpret (2.3) and (2.4) as quantities which describe the preferences of the reinsurer, and which lead to the well-known actuarial pricing principles (2.1) and (2.2).

It is well known and relatively easy to construct examples which show that the actuarial valuation principles $\tilde{u}_1$ and $\tilde{u}_2$ do not satisfy the natural condition

$$H_1 \leq H_2 \ P\text{-a.s.} \ \Rightarrow \tilde{u}_i(H_1) \leq \tilde{u}_i(H_2), \tag{2.6}$$

$i = 1, 2$. Of course, this property would also not be satisfied if we replaced $\tilde{u}_i$ in (2.6) with the modified versions u_i.

3 Preliminaries

In this section we review some technical notions which are needed for the introduction of the financial market in the next section; for unexplained terminology, see Jacod and Shiryaev (1987).

Consider a complete filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions, $\mathcal{F}_0$ is trivial and T is some fixed finite time horizon. Let $X = (X_t)_{0\leq t\leq T}$ be an $\mathbb{R}^d$-valued *continuous* semimartingale with respect to $\mathbb{F}$; X is taken to be the discounted price process for some financial assets. (In Section 6, we consider an example where X is a diffusion process, and hence in particular a continuous semimartingale.)

Before proceeding further, we give an outline of the rest of this section. We first define the *variance optimal martingale measure* $\tilde{P}$, which is here a probability measure that is equivalent to P and (basically) can be characterized by the following properties:

1. It is a martingale measure, i.e. the discounted price process X is a (local) $\tilde{P}$-martingale.
2. The Radon-Nikodym derivative $\frac{d\tilde{P}}{dP}$ with respect to the underlying measure P has minimum variance over all martingale measures.

Then we define a space $\tilde{\Theta}$ of trading strategies which has the important property that the space $G_T(\tilde{\Theta})$ of stochastic integrals $G_T(\vartheta) := \int_0^T \vartheta dX = \sum_{i=1}^d \int_0^T \vartheta^i dX^i$, $\vartheta \in \tilde{\Theta}$, is closed in $L^2(P)$. This choice of space is crucial for Theorem 3.4 below and makes it possible to derive optimal trading strategies in the next section. References are Delbaen and Schachermayer (1996a,b), Schweizer (1996) and Rheinländer and Schweizer (1997).

Let $\mathcal{V}$ denote the linear space spanned by random variables of the form $h^{tr}(X_{T_2} - X_{T_1})$, where $T_1 \leq T_2 \leq T$ are any stopping times such that the stopped process

X^{T_2} is bounded, and h is a bounded $\mathbb{R}^d$-valued $\mathcal{F}_{T_1}$-measurable random variable. Denote by $\mathcal{M}^s(P)$ the space of signed measures $Q \ll P$ with $Q(\Omega) = 1$ and

$$\mathrm{E}\left[\frac{dQ}{dP}f\right] = 0, \tag{3.1}$$

for all $f \in \mathcal{V}$, and by $\mathcal{M}^e(P)$ the set of probability measures $Q \in \mathcal{M}^s(P)$ with $Q \sim P$. Furthermore, define spaces $\mathcal{D}^s$ and $\mathcal{D}^e$ by

$$\mathcal{D}^x = \left\{ \frac{dQ}{dP} \middle| Q \in \mathcal{M}^x(P) \right\}, \tag{3.2}$$

for $x \in \{s, e\}$. The variance optimal martingale measure is defined as follows:

Definition 3.1 *The variance optimal martingale measure $\tilde{P}$ is the unique element of $\mathcal{M}^s(P)$ such that $\tilde{D} := \frac{d\tilde{P}}{dP} \in L^2(P)$ and such that $\tilde{D}$ minimizes $\|D\|_{L^2(P)}$ over all $D \in \mathcal{D}^s \cap L^2(P)$.*

We will be working under Assumption 3.2 below, which ensures the existence of the variance optimal martingale measure $\tilde{P}$ and guarantees that this measure is a probability measure which is equivalent to P, see Delbaen and Schachermayer (1996a, Theorem 1.3).

Assumption 3.2 $\mathcal{D}^e \cap L^2(P) \neq \emptyset$.

Let $\tilde{\Theta}$ denote the space of $\mathbb{R}^d$-valued $\mathbb{F}$-predictable processes ϑ which are such that $G(\vartheta) = \int \vartheta dX$ is a $\tilde{P}$-martingale and $\int_0^T \vartheta dX \in L^2(P)$, and define

$$G_T(\tilde{\Theta}) := \left\{ \int_0^T \vartheta dX \middle| \vartheta \in \tilde{\Theta} \right\}.$$

It was shown in Delbaen and Schachermayer (1996b) and in Gourieroux, Laurent and Pham (1998), that when X is continuous, Assumption 3.2 implies that $G_T(\tilde{\Theta})$ is closed and is equal to the closure in $L^2(P)$ of the space $\mathcal{V}$, see the remark following Proposition 15 of Rheinländer (1999). This remark also shows that a predictable process ϑ is in $\tilde{\Theta}$ if and only if the process $G(\vartheta)$ is a Q-martingale for any $Q \in \mathcal{M}^e(P)$ with $\frac{dQ}{dP} \in L^2(P)$.

Remark 3.3 Note that the spaces $\mathcal{M}^x(P)$, $x \in \{s, e\}$, depend on the filtration $\mathbb{F}$. Consequently, also the variance optimal martingale is affected by the choice of filtration; we refer to Møller (1999b) for an investigation of this property. □

The following result is now a consequence of the projection theorem for Hilbert spaces:

Theorem 3.4 *Any random variable $H \in L^2(\mathcal{F}_T, P)$ admits a unique decomposition of the form*

$$H = c^H + \int_0^T \vartheta_t^H dX_t + N^H, \tag{3.3}$$

where $\vartheta^H \in \tilde{\Theta}$, $\mathrm{E}[N^H] = 0$, and $\mathrm{E}[N^H \int_0^T \vartheta_t dX_t] = 0$ for all $\vartheta \in \tilde{\Theta}$.

We denote by π the projection in $L^2(P)$ on $G_T(\tilde{\Theta})^\perp$. The following lemma, which is due to Delbaen and Schachermayer (1996a) and Schweizer (1996), relates the variance-optimal martingale measure $\tilde{P}$ to the projection π.

Lemma 3.5 *Under Assumption 3.2, the variance optimal martingale measure exists and is given by*

$$\frac{d\tilde{P}}{dP} = \frac{\pi(1)}{\mathrm{E}[\pi(1)]}.$$

Throughout, we let $\tilde{Z}_T = \frac{d\tilde{P}}{dP}$ and write $\tilde{\mathrm{E}}$ for $\mathrm{E}_{\tilde{P}}$.

4 The financial valuation principles

In this section, we introduce the financial market and recall the crucial indifference argument of Schweizer (1997) that leads to the definition of the fair premium. We consider a financial market consisting of $d+1$ basic traded assets: d stocks with price process $X = (X^1, \dots, X^d)^{tr}$ and a savings account with a price process which is constant and equal to 1. (One can think of X as the discounted value of the stocks expressed in terms of the savings account.) Here, a *self-financing trading strategy* is a $d+1$-dimensional $\mathbb{F}$-adapted process $\varphi = (\vartheta, \eta)$ such that $\vartheta \in \tilde{\Theta}$ and such that the value process defined by $V(\varphi) := \vartheta^{tr} X + \eta$ satisfies

$$V_t(\varphi) = V_0(\varphi) + \int_0^t \vartheta_u dX_u \tag{4.1}$$

for all $t \in [0, T]$. The pair $\varphi_t = (\vartheta_t, \eta_t)$ is the *portfolio* held at t: ϑ_t^i is the number of units of stock number i held, and η_t is the discounted deposit on the savings account at t. In (4.1), $V_0(\varphi)$ is the initial investment at time 0, and $\int_0^t \vartheta dX$ is trading gains from the strategy φ. Thus, (4.1) states that, at each time t, the value of the portfolio φ_t is the initial value plus trading gains, hence no additional in- or out-flow of capital occurs after time 0. A claim H is said to be *attainable* if there exists a self-financing strategy φ such that $V_T(\varphi) = H$ P-a.s., i.e.

$$H = V_0(\varphi) + \int_0^T \vartheta_u dX_u.$$

This implies that the claim H can be replicated perfectly by investing the amount $V_0(\varphi)$ at time 0 and thereafter following the self-financing strategy φ; the initial investment $V_0(\varphi)$ is called the (unique) no-arbitrage price for H.

The idea is now the following: Assume that the reinsurer applies one of the premium calculation principles $\tilde{u}_i$, $i = 1, 2$, or, equivalently, the corresponding valuation functions u_i, $i = 1, 2$ (henceforth called u). The reinsurer is considering the possibility of accepting (insuring) a fraction $\gamma \in I\!R$ of a claim due at time T with discounted value H (here $\gamma < 0$ corresponds to selling the fraction). We denote by c the reinsurer's initial capital (basis capital at time 0) and consider the following possibilities: On the one hand, the reinsurer can choose to accept the risk γH, receive some premium $h(c, \gamma)$ and invest the amount $c + h(c, \gamma)$ on the financial market using a self-financing strategy $\varphi = (\vartheta, \eta)$. This will generate the discounted wealth

$$c + h(c, \gamma) + \int_0^T \vartheta_u dX_u - \gamma H,$$

where we have subtracted the term γH, which is to be paid at time T to the buyer of the contract. However, the reinsurer could also choose not to engage in the risk H and simply invest the initial capital c on the market according to some self-financing strategy $\tilde{\varphi} = (\tilde{\vartheta}, \tilde{\eta})$ and generate the wealth

$$c + \int_0^T \tilde{\vartheta}_u dX_u.$$

The *fair premium* is now defined as the premium which makes the reinsurer indifferent in terms of the valuation function u between the two possibilities of accepting and not accepting the risk. From Schweizer (1997) we have the following formal definition of the fair premium for γ units of the risk H in the presence of a financial market

Definition 4.1 *$h(c, \gamma)$ is called a u-indifference price for γ units of H if it satisfies*

$$\sup_{\vartheta \in \tilde{\Theta}} u\left(c + h(c, \gamma) + \int_0^T \vartheta dX - \gamma H\right) = \sup_{\tilde{\vartheta} \in \tilde{\Theta}} u\left(c + \int_0^T \tilde{\vartheta} dX\right). \tag{4.2}$$

Remark 4.2 This definition of the fair premium specializes to the principle (2.5) in the case where the space of investment strategies is given by $\tilde{\Theta} = \{(0, \ldots, 0)^{tr}\}$, that is, no trading in the stocks is allowed, provided that $u(c + Y) = c + u(Y)$, for $c \in I\!R$. And hence, in the case of the variance or the standard deviation principles, (4.2) generalizes (2.5) to the situation where a financial market is present, since $u_i(c + Y) = c + u_i(Y)$, for $i = 1, 2$. Furthermore, it follows directly from this property and (4.2) that the fair premium $h(c, \gamma)$ will be independent of the initial capital c for the principles (2.3) and (2.4). From a mathematical point of view, the fraction γ in γH is redundant; we could as well work with a claim H. The inclusion of γ is motivated by the application we have in mind here, where the reinsurer participates in a fraction γ of the risk H. The question of how γ should be chosen is not addressed within this context, however. □

Schweizer (1997) proved that in the case of the variance and the standard deviation principles, the solutions to the problem (4.2) can be related to the decomposition (3.3). For completeness, we give here these two results (Theorem 9 and Theorem 12 of Schweizer (1997)). In the case of the variance principle, the solution is:

Theorem 4.3 (Schweizer (1997)) *For any $H \in L^2(P)$ and $\gamma, c \in I\!R$, the u_1-indifference price for γH is*

$$h_1(c,\gamma) = v_1(\gamma H) = \gamma \tilde{\mathrm{E}}\left[H\right] + a\gamma^2 \mathrm{Var}\left[N^H\right]$$

In the case of the standard deviation principle, we have the following result:

Theorem 4.4 (Schweizer (1997)) *For any $H \in L^2(P)$ and $\gamma, c \in I\!R$, the u_2-indifference price for γH is*

$$h_2(c,\gamma) = v_2(\gamma H) = \gamma \tilde{\mathrm{E}}\left[H\right] + a|\gamma| \sqrt{1 - \frac{\mathrm{Var}[\frac{d\tilde{P}}{dP}]}{a^2}} \sqrt{\mathrm{Var}[N^H]},$$

provided that $a^2 \geq \mathrm{Var}[\frac{d\tilde{P}}{dP}]$. If $a^2 < \mathrm{Var}[\frac{d\tilde{P}}{dP}]$, then the u_2-indifference price is undefined.

4.1 The optimal strategy for the variance principle

We will determine the optimal strategy for a reinsurer who is using the principle (4.2) in the case of the variance principle. That is, we determine ϑ^* so that

$$\sup_{\vartheta \in \tilde{\Theta}} u_1 \left(c + h_1(c,\gamma) + \int_0^T \vartheta dX - \gamma H \right) = u_1 \left(c + h_1(c,\gamma) + \int_0^T \vartheta^* dX - \gamma H \right),$$

where $h_1(c,\gamma)$ is the fair premium determined by Theorem 4.3, and express the maximum of u_1 in terms of the decomposition (3.3). The proof uses techniques from Schweizer (1997) and consists in first determining the optimal strategy for fixed expected value m of the trading gains and then maximizing over all $m \in I\!R$. In particular, Theorem 4.3 and 4.4 will follow again from the results given in this and the next subsection.

First note that we can formulate (4.2) equivalently by taking supremum over all elements $g = \int_0^T \vartheta dX$ in the space $G_T(\tilde{\Theta})$, that is $h_1(c,\gamma)$ satisfies

$$\sup_{g \in G_T(\tilde{\Theta})} u_1\left(c + h_1(c,\gamma) + g - \gamma H\right) = \sup_{\tilde{g} \in G_T(\tilde{\Theta})} u_1\left(c + \tilde{g}\right). \tag{4.3}$$

Similarly, we let $g^H = \int_0^T \vartheta^H dX$ denote the term appearing in the decomposition (3.3) for H. The following lemma is crucial for determining the optimal strategy.

Lemma 4.5 *Assume that $1 \notin G_T(\tilde{\Theta})^\perp$. For any $m \in I\!R$, the solution to the problem*

$$\max_{g \in G_T(\tilde{\Theta})} u_1(g - N^H) \quad \textit{subject to } \mathrm{E}[g] = m, \tag{4.4}$$

is given by $g_m = c_m(1 - \pi(1))$ where $c_m = \frac{m}{\mathrm{E}[(1-\pi(1))^2]}$.

Proof: We first note that, by the definition of N^H, we have that $\mathrm{E}[N^H] = \mathrm{E}[N^H g] = 0$ for all $g \in G_T(\tilde{\Theta})$, so that

$$u_1(g - N^H) \;=\; \mathrm{E}[g - N^H] - a\mathrm{Var}[g - N^H] = \mathrm{E}[g] - a\mathrm{Var}[g] - a\mathrm{Var}[N^H],$$

and hence, we have to minimize $||g||^2 := \mathrm{E}[g^2]$ over all $g \in G_T(\tilde{\Theta})$ with $\langle g, 1\rangle := \mathrm{E}[g] = m$. By the projection theorem for Hilbert spaces, any $g \in G_T(\tilde{\Theta})$ admits a unique decomposition

$$g = \alpha(1 - \pi(1)) + \hat{g},$$

where $\alpha \in I\!R$, $\hat{g} \in G_T(\tilde{\Theta})$, and $\hat{g} \perp (1-\pi(1))$ (i.e. $\mathrm{E}[\hat{g}(1-\pi(1))] = \langle \hat{g}, (1-\pi(1))\rangle = 0$). This implies that

$$\mathrm{E}[g^2] = ||g||^2 = \alpha^2||(1 - \pi(1))||^2 + ||\hat{g}||^2,$$

and

$$\mathrm{E}[g] = \langle g, 1\rangle = \alpha\langle(1 - \pi(1)), 1\rangle + \langle \hat{g}, 1\rangle = \alpha||(1 - \pi(1))||^2,$$

since $\hat{g} \perp (1 - \pi(1))$ and $\hat{g} \perp \pi(1)$. Thus the solution to (4.4) is obtained for $\hat{g} = 0$ and $\alpha = \frac{m}{||(1-\pi(1))||^2}$, that is

$$g_m = \frac{m}{||(1 - \pi(1))||^2}(1 - \pi(1)),$$

which is well-defined, since, by assumption, $1 \notin G_T(\tilde{\Theta})^\perp$, so that $||1 - \pi(1)|| > 0$. □

Lemma 4.6 *Assume that $1 \notin G_T(\tilde{\Theta})^\perp$. Let $m \in I\!R$ and let g_m be defined as in Lemma 4.5. Then*

$$\mathrm{Var}[g_m] = \frac{m^2}{\mathrm{Var}[\tilde{Z}_T]}.$$

Proof: This result follows from the proof of Lemma 10 from Schweizer (1997). However, for completeness, we give the proof here. Since $\langle(1-\pi(1)), \pi(1)\rangle = 0$, $\|\pi(1)\|^2 = \mathrm{E}[\pi(1)^2] = \mathrm{E}[\pi(1)]$, and hence $\|1-\pi(1)\|^2 = \mathrm{E}[(1-\pi(1))^2] = 1-\|\pi(1)\|^2$. Direct calculations now show that

$$\mathrm{Var}[g_m] = \mathrm{E}[g_m^2] - m^2 = \frac{m^2}{\|1-\pi(1)\|^2} - m^2 = m^2 \frac{\mathrm{E}[\pi(1)^2]}{\mathrm{E}[(1-\pi(1))^2]}.$$

Recall that $\tilde{Z}_T = \frac{\pi(1)}{\mathrm{E}[\pi(1)]}$. Using the above properties, we have

$$\mathrm{Var}[\tilde{Z}_T] = \frac{\mathrm{E}[\pi(1)^2]}{(\mathrm{E}[\pi(1)])^2} - 1 = \frac{1-\mathrm{E}[\pi(1)]}{\mathrm{E}[\pi(1)]} = \frac{\mathrm{E}[(1-\pi(1))^2]}{\mathrm{E}[\pi(1)^2]},$$

and this ends the proof. □

The next theorem essentially gives the solution to (4.3) in that it contains explicit expressions for the maximum obtainable value of u_1. The proof of this theorem is given after a subsequent corollary, which determines the optimal strategy associated to the financial variance principle, and a remark which relates this strategy to the mean-variance hedging strategy for H.

Theorem 4.7 *For any $H \in L^2(P)$ and $\gamma, c \in I\!R$,*

$$\sup_{g \in G_T(\tilde{\Theta})} u_1\left(c + h_1(c,\gamma) + g - \gamma H\right) = u_1\left(c + h(c,\gamma) + g^* - \gamma H\right), \tag{4.5}$$

where

$$g^* = \gamma g^H + \frac{1+\mathrm{Var}[\tilde{Z}_T]}{2a}(1-\pi(1)) \tag{4.6}$$

Furthermore, the value associated to g^* is

$$u_1\left(c + h(c,\gamma) + g^* - \gamma H\right) = c + h(c,\gamma) - \gamma c^H + \frac{1}{4a}\mathrm{Var}[\tilde{Z}_T] - a\gamma^2 \mathrm{Var}[N^H]. \tag{4.7}$$

Corollary 4.8 *Let $H \in L^2(P)$ and let $1 - \pi(1) = \int_0^T \tilde{\beta} dX$. The optimal strategy ϑ^* for H under the financial variance principle is*

$$\vartheta^* \;=\; \vartheta^H + \frac{1+\mathrm{Var}[\tilde{Z}_T]}{2a}\tilde{\beta}. \tag{4.8}$$

Proof of Corollary 4.8: This is an immediate consequence of Theorem 4.7. □

Remark 4.9 In the solution (4.8), the first term ϑ^H is exactly the *mean-variance hedging strategy* for H, see e.g. Schweizer (1999). The second term is related to the variance optimal martingale measure and the loading factor a and is independent of the claim H. In particular it is seen that as a is increased, the process (4.8) will converge towards the mean-variance hedging strategy. This is intuitively reasonable, since for very large a, (4.5) will essentially amount to minimizing the L^2-distance between H and $c^H + g$. Note also that ϑ^* is a linear combination of the mean-variance hedging strategy and the process related to the variance optimal martingale measure and that this combination does not depend on N^H. We point out that the proof of Theorem 4.7 is very similar to the one of Theorem 4.3. Furthermore, Theorem 4.3 follows directly from (4.7) since $c^H = \tilde{\mathrm{E}}[H]$. □

Proof of Theorem 4.7: First part of the proof is similar to the one of Schweizer (1997, proof of Theorem 9). Since $u_1(x + g - \gamma H) = x + u_1(g - \gamma H)$, we only consider $u_1(g - \gamma H)$, and as in the proof of Lemma 4.5 we find that

$$\begin{aligned} u_1(g-\gamma H) &= \mathrm{E}\left[g-\gamma(c^H+g^H+N^H)\right] - a\mathrm{Var}\left[g-\gamma(c^H+g^H+N^H)\right] \\ &= -\gamma c^H + \mathrm{E}\left[g-\gamma g^H\right] - a\mathrm{Var}\left[g-\gamma g^H\right] - a\mathrm{Var}\left[\gamma N^H\right]. \end{aligned} \tag{4.9}$$

Introducing $g' = g - \gamma g^H$, we now have

$$u_1(g-\gamma H) = -\gamma c^H + u_1(g' - \gamma N^H),$$

and so, we can alternatively maximize $u_1(g' - \gamma N^H)$ over all $g' \in G_T(\tilde{\Theta})$.

We first assume that $1 \notin G_T(\tilde{\Theta})^\perp$ and solve this problem by first maximizing this term under the constraint that $\mathrm{E}[g] = m$ for any $m \in I\!R$ and then maximizing over $m \in I\!R$. The first step follows by Lemma 4.5, and from Lemma 4.6 we find that we should maximize

$$u_1(g_m - \gamma N^H) = m - \frac{a\,m^2}{\mathrm{Var}[\tilde{Z}_T]} - a\gamma^2\mathrm{Var}[N^H] =: f_1(m).$$

Note that $m \mapsto f_1(m)$ is just a negative definite quadratic function, and that its unique maximum is attained for m^* satisfying $f_1'(m^*) = 0$, that is $m^* = \mathrm{Var}[\tilde{Z}_T]/(2a)$. Thus, $u_1(g' - \gamma N^H)$ is maximized for

$$g_{m^*} = \frac{m^*(1-\pi(1))}{\mathrm{E}\left[(1-\pi(1))^2\right]} = \frac{\mathrm{Var}[\tilde{Z}_T](1-\pi(1))}{2a\mathrm{E}\left[(1-\pi(1))^2\right]}.$$

Since

$$\frac{1}{\mathrm{E}\left[(1-\pi(1))^2\right]} = \frac{\mathrm{E}[(1-\pi(1)+\pi(1))^2]}{\mathrm{E}\left[(1-\pi(1))^2\right]} = 1 + \frac{\mathrm{E}[(\pi(1))^2]}{\mathrm{E}\left[(1-\pi(1))^2\right]} = \frac{1+\mathrm{Var}[\tilde{Z}_T]}{\mathrm{Var}[\tilde{Z}_T]},$$

we finally get that

$$g_{m^*} = \frac{1+\mathrm{Var}[\tilde{Z}_T]}{2a}(1-\pi(1)).$$

Hence, the solution to (4.5) is

$$g^* = \gamma g^H + g_{m^*},$$

which proves (4.6). Finally, (4.7) follows by inserting g^* into u_1:

$$\begin{aligned} u_1(g^* - \gamma H) &= -\gamma c^H + \mathrm{E}[g_{m^*}] - a\left(\mathrm{Var}[g_{m^*}] + \gamma^2 \mathrm{Var}[N^H]\right) \\ &= -\gamma c^H + m^* - a(m^*)^2 \frac{1}{\mathrm{Var}[\tilde{Z}_T]} - a\gamma^2 \mathrm{Var}[N^H] \\ &= -\gamma c^H + \frac{\mathrm{Var}[\tilde{Z}_T]}{2a} - a\left(\frac{\mathrm{Var}[\tilde{Z}_T]}{2a}\right)^2 \frac{1}{\mathrm{Var}[\tilde{Z}_T]} - a\gamma^2 \mathrm{Var}[N^H] \\ &= -\gamma c^H + \frac{\mathrm{Var}[\tilde{Z}_T]}{4a} - a\gamma^2 \mathrm{Var}[N^H]. \end{aligned}$$

Now assume that $1 \in G_T(\tilde{\Theta})^\perp$, so that $\pi(1) = 1$, $\mathrm{Var}[\tilde{Z}_T] = 0$ and $\mathrm{E}[g] = 0$ for all $g \in G_T(\tilde{\Theta})$. In this case (4.9) it maximized for $g^* = \gamma g^H$, and it follows immediately that the associated optimal value for u_1 is given by (4.7). This ends the proof. □

4.2 The optimal strategy for the standard deviation principle

In the standard deviation case, we get a result which is similar to Theorem 4.7. This case has also been worked out by Schweizer (1997), and the following theorem can be proven by combining Lemma 10 and 11 and Theorem 12 from Schweizer (1997). Recall that $\tilde{Z}_T = \frac{d\tilde{P}}{dP}$.

Theorem 4.10 *Assume that* $a^2 > \mathrm{Var}[\tilde{Z}_T]$. *For any* $H \in L^2(P)$ *and* $\gamma, c \in \mathbb{R}$,

$$\sup_{g \in G_T(\tilde{\Theta})} u_2\left(c + h(c,\gamma) + g - \gamma H\right) = u_2\left(c + h(c,\gamma) + g^* - \gamma H\right), \tag{4.10}$$

where

$$g^* = \gamma g^H + \frac{1 + \mathrm{Var}[\tilde{Z}_T]}{a\sqrt{1 - \frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}} |\gamma| \sqrt{\mathrm{Var}[N^H]}(1 - \pi(1)). \tag{4.11}$$

Furthermore, the value associated to g^* *is*

$$\begin{aligned} u_2\left(c + h(c,\gamma) + g^* - \gamma H\right) &= c + h(c,\gamma) - \gamma c^H \\ &\quad - a\sqrt{\mathrm{Var}[\gamma N^H]}\sqrt{1 - \frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}. \end{aligned} \tag{4.12}$$

Proof: As in the proof of Theorem 4.7, $u_2(x+g-\gamma H) = x + u_2(g-\gamma H)$, so that we only need to consider $u_2(g-\gamma H)$. Similarly,

$$\begin{aligned} u_2(g-\gamma H) &= \mathrm{E}\left[g-\gamma(c^H+g^H+N^H)\right] - a\sqrt{\mathrm{Var}\left[g-\gamma(c^H+g^H+N^H)\right]} \\ &= -\gamma c^H + \mathrm{E}\left[g'\right] - a\sqrt{\mathrm{Var}\left[g'\right] + \mathrm{Var}\left[\gamma N^H\right]}, \end{aligned} \tag{4.13}$$

where $g' = g - \gamma g^H$. This problem is very similar to the one considered in the case of the variance principle. We consider here only the situation where $1 \notin G_T(\tilde{\Theta})^\perp$; the case $1 \in G_T(\tilde{\Theta})^\perp$ can be treated as in the proof of Theorem 4.7. From the proof of Lemma 4.5 we have that subject to the constraint $\mathrm{E}[g] = m$, $\mathrm{Var}[g]$ is minimized by $g_m = \frac{m(1-\pi(1))}{\mathrm{E}[(1-\pi(1))^2]}$, and from Lemma 4.6 we have that $\mathrm{Var}[g_m] = m^2/\mathrm{Var}[\tilde{Z}_T]$. Thus, (4.13) is maximized by maximizing over $m \in I\!R$ the function f_2 defined by

$$u_2(g_m - \gamma N^H) = m - \sqrt{\frac{a^2}{\mathrm{Var}[\tilde{Z}_T]}m^2 + a^2\mathrm{Var}[\gamma N^H]} =: f_2(m). \tag{4.14}$$

This is a simple maximization problem, and it follows for example by Schweizer (1997, Lemma 11) that f_2 attains its unique maximum for

$$m^* = \sqrt{\frac{a^2\mathrm{Var}[\gamma N^H]}{C(C-1)}},$$

where $C = \frac{a^2}{\mathrm{Var}[\tilde{Z}_T]}$, provided that $\mathrm{Var}[\gamma N^H] > 0$ and $C > 1$. Thus, (4.13) attains its maximum for

$$\begin{aligned} g_{m^*} &= \frac{m^*(1-\pi(1))}{\mathrm{E}\left[(1-\pi(1))^2\right]} \\ &= \sqrt{\frac{a^2\mathrm{Var}[\gamma N^H]}{\frac{a^2}{\mathrm{Var}[\tilde{Z}_T]}\left(\frac{a^2}{\mathrm{Var}[\tilde{Z}_T]}-1\right)}}\,\frac{1+\mathrm{Var}[\tilde{Z}_T]}{\mathrm{Var}[\tilde{Z}_T]}(1-\pi(1)) \\ &= \frac{1+\mathrm{Var}[\tilde{Z}_T]}{a\sqrt{1-\frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}}|\gamma|\sqrt{\mathrm{Var}[N^H]}(1-\pi(1)), \end{aligned}$$

and this shows (4.11). To see (4.12), note that

$$\begin{aligned} u_2(g^*-\gamma H) &= -\gamma c^H + u_2(g_{m^*} - \gamma N^H) \\ &= -\gamma c^H + m^* - \sqrt{C(m^*)^2 + a^2\mathrm{Var}[\gamma N^H]} \\ &= -\gamma c^H + \sqrt{\frac{a^2\mathrm{Var}[\gamma N^H]}{C(C-1)}} - \sqrt{C\frac{a^2\mathrm{Var}[\gamma N^H]}{C(C-1)} + a^2\mathrm{Var}[\gamma N^H]} \\ &= -\gamma c^H - a\sqrt{\mathrm{Var}[\gamma N^H]}\sqrt{1-\frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}. \end{aligned}$$

This ends the proof. □

Remark 4.11 In Theorem 4.4 it was only assumed that $a^2 \geq \mathrm{Var}[\tilde{Z}_T]$. In Theorem 4.10, however, we need to assume that $a^2 > \mathrm{Var}[\tilde{Z}_T]$ in order to guarantee that the supremum (4.10) is attained for an element $g \in G_T(\tilde{\Theta})$. To see this, consider the case where $a^2 = \mathrm{Var}[\tilde{Z}_T]$, so that the function (4.14) is of the form

$$f_2(m) = m - \sqrt{m^2 + y},$$

where $y > 0$. In this case, $f_2(m) < 0$ for all $m \in I\!R$ and $f_2(m) \to 0$ for $m \to \infty$, so that the supremum is not attained for any $m \in I\!R$, and hence, (4.13) does not attain the supremum for any $g \in G_T(\tilde{\Theta})$. However, the supremum can be approximated e.g. by choosing a sequence $(g_{m_k})_{k \in I\!N}$, where g_m is defined in Lemma 4.5 and where $m_k \to \infty$ for $k \to \infty$. In this case, we obtain

$$\sup_{g \in G_T(\tilde{\Theta})} u_2\left(c + h(c,\gamma) + g - \gamma H\right) = c + h(c,\gamma) - \gamma c^H,$$

which extends (4.12) to the case where $a^2 = \mathrm{Var}[\tilde{Z}_T]$. For later use, we also note that when $a^2 < \mathrm{Var}[\tilde{Z}_T]$ then $f_2(m) \to \infty$ for $m \to \infty$, so that

$$\sup_{g \in G_T(\tilde{\Theta})} u_2\left(c + h(c,\gamma) + g - \gamma H\right) = \infty. \ \Box$$

As in the case of the variance principle, we obtain an explicit expression for the optimal strategy immediately as a straightforward consequence of the theorem:

Corollary 4.12 *Assume that $a^2 > \mathrm{Var}[\tilde{Z}_T]$. Let $H \in L^2(P)$ and $1 - \pi(1) = \int_0^T \tilde{\beta} dX$. Then, the optimal strategy ϑ^* for H under the financial standard deviation principle is*

$$\vartheta^* = \vartheta^H + \frac{1 + \mathrm{Var}[\tilde{Z}_T]}{a\sqrt{1 - \frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}} \sqrt{\mathrm{Var}[N^H]}\, \tilde{\beta}. \tag{4.15}$$

Remark 4.13 Note that the factor on $\tilde{\beta}$ in (4.15) now depends on $\mathrm{Var}[N^H]$, whereas, for the variance principle, it was independent of N^H, see (4.8). This difference is a consequence of the fact that the standard deviation principle involves maximization of a function which includes the square-root of a sum of the variance of the trading gain and the variance of N^H. For the variance principle, this complex dependence is not present, since N^H is orthogonal to all trading gains $g \in G_T(\tilde{\Theta})$. $\Box$

We close this section with an investigation of the condition $a^2 > \mathrm{Var}[\tilde{Z}_T]$ in the case of a standard Black-Scholes market. In this case there is only one martingale measure, and hence this is trivially the variance optimal martingale measure. Let

ν denote the market price of risk; the Radon-Nikodym derivative of the variance optimal martingale measure with respect to P is then given by

$$\tilde{Z}_T = \exp\left(-\nu W_T - \frac{1}{2}\nu^2 T\right).$$

It is seen that

$$\tilde{Z}_T^2 = \exp\left(-2\nu W_T - \frac{1}{2}(2\nu)^2 T\right)\exp\left(\nu^2 T\right),$$

hence $\mathrm{Var}[\tilde{Z}_T] = \exp(\nu^2 T) - 1$. Thus, the standard deviation principle can be applied if $a^2 > \exp(\nu^2 T) - 1$, or equivalently

$$\sqrt{\frac{\ln(1+a^2)}{T}} > |\nu|.$$

If for example $T = 1$ and $\nu = 1/5$ (e.g. risk-free interest rate $r = 0.05$, rate of return on the stock $\alpha = 0.10$ and standard deviation $\sigma = 0.25$), then the standard deviation principle is well-defined provided that $a > 0.2020$.

5 Alternative financial transformations

In Section 4 we considered the financial variance and financial standard deviation principles as they were defined by Schweizer (1997). Either principle is derived from an actuarial premium principle $\tilde{u}$ by first changing sign on the loading factor to obtain a function u that measures the reinsurer's preferences. It is not immediately clear how similar modifications should be made for other premium principles, for example the so-called Esscher-transform principle. An alternative idea is, therefore, to define a new premium calculation principle directly in terms of the actuarial premium principle $\tilde{u}$ by

$$\hat{u}(H) = \inf_{g \in G_T(\tilde{\Theta})} \tilde{u}(H - g). \tag{5.1}$$

We shall refer to $\hat{u}$ as the *direct financial transformation* of $\tilde{u}$. The interpretation of the principle (5.1) is the following: For a given actuarial valuation principle, we look for the self-financing trading strategy ϑ which leads to the smallest possible premium for the claim $H - \int_0^T \vartheta dX$ using the original premium principle $\tilde{u}$.

Note that when H is an attainable claim, i.e. $H = c^H + g^H$ for some $c^H \in \mathbb{R}$ and $g^H \in G_T(\tilde{\Theta})$, then the linearity of $G_T(\tilde{\Theta})$ implies that (5.1) can be rewritten as

$$\hat{u}(H) = \inf_{g \in G_T(\tilde{\Theta})} \tilde{u}(c^H + g^H - g) = \inf_{g \in G_T(\tilde{\Theta})} \tilde{u}(c^H + g).$$

In particular, we ask the questions: Does the principle (5.1) assign the no-arbitrage price c^H to an attainable claim? Is the principle equivalent to the indifference transformation principle proposed by Schweizer (1997) in the cases of the standard deviation and the variance principle?

Throughout this section, we work under the standing Assumption 3.2.

5.1 The variance principle

In the situation where $\tilde{u}$ is equal to $\tilde{u}_1$ we have the following negative result

Theorem 5.1 *The direct financial transformation of the variance principle is*

$$\hat{u}_1(H) = \tilde{\mathrm{E}}[H] + a\mathrm{Var}[N^H] - \frac{\mathrm{Var}[\tilde{Z}_T]}{4a}. \tag{5.2}$$

The associated optimal strategy $\hat{\vartheta}$ is given by (4.8).

Proof: From Theorem 3.4, we have the decomposition

$$H = c^H + g^H + N^H,$$

where we have used the notation $g^H = \int_0^T \vartheta^H dX$. Furthermore, it was shown in the proof of Theorem 4.7 that

$$u_1(g - H) = -c^H + \mathrm{E}[g - g^H] - a\mathrm{Var}[g - g^H] - a\mathrm{Var}[N^H]. \tag{5.3}$$

Since $\tilde{u}_1(H - g) = -u_1(g - H)$, minimizing $\tilde{u}_1(H - g)$ is equivalent to maximizing $u_1(g - H)$ over $g \in G_T(\tilde{\Theta})$, and hence, we find by (4.7) and (5.3) that

$$\begin{aligned}
\inf_{g \in G_T(\tilde{\Theta})} \tilde{u}_1(H - g) &= -\sup_{g \in G_T(\tilde{\Theta})} u_1(g - H) \\
&= -u_1(g^* - H) = c^H + a\mathrm{Var}[N^H] - \frac{\mathrm{Var}[\tilde{Z}_T]}{4a},
\end{aligned}$$

where g^* is also given by Theorem 4.7. This proves (5.2). Since also $-u_1(g^* - H) = \tilde{u}_1(H - g^*)$, we find that the optimal strategy $\hat{\vartheta}$ is exactly equal to the strategy determined by Corollary 4.8. This completes the proof. □

Remark 5.2 It follows from Theorem 5.1 that the valuation principle $\hat{u}_1$ differs from the financial valuation principle of Schweizer (1997) by the additional term $-\mathrm{Var}[\tilde{Z}_T]/(4a)$. This has the consequence that $\hat{u}_1$ is only consistent with absence of arbitrage if $\mathrm{Var}[\tilde{Z}_T] = 0$, that is, if the measure P is a martingale measure. In fact, for the claim $H = 0$, we get $\hat{u}_1(0) = -\mathrm{Var}[\tilde{Z}_T]/(4a)$, and clearly this claim should have the price 0. The same applies to any claim on the form $H = \int_0^T \vartheta dX$. □

5.2 The standard deviation principle

In the situation where $\tilde{u}$ is equal to $\tilde{u}_2$ we have the following result

Theorem 5.3 *The direct financial transformation of the standard deviation principle is*

$$\hat{u}_2(H) = \tilde{\mathrm{E}}[H] + a\sqrt{1 - \frac{\mathrm{Var}[\frac{d\tilde{P}}{dP}]}{a^2}}\sqrt{\mathrm{Var}[N^H]}, \tag{5.4}$$

provided that $a^2 \geq \mathrm{Var}[\frac{d\tilde{P}}{dP}]$. *If* $a^2 < \mathrm{Var}[\frac{d\tilde{P}}{dP}]$, *then the direct financial transformation is undefined. The associated optimal strategy* $\hat{\vartheta}$ *is given by (4.15).*

Proof: The result follows directly from the proof of Theorem 4.10 and Remark 4.11 by using arguments similar to the ones in the proof of Theorem 5.1. □

Remark 5.4 For the standard deviation principle, the direct financial transform is equivalent to the indifference valuation principle proposed by Schweizer (1997), see Theorem 4.3. This gives an alternative characterization of this indifference pricing principle. □

5.3 A generalization

As shown above in Section 5.1 the direct financial transform (5.1) did not lead to a pricing principle with reasonable properties in the case of the variance principle in that the new principle would assign negative prices to attainable claims with no-arbitrage price 0. We also proved that the direct transform of the standard deviation principle was in fact identical to the indifference pricing principle examined in Section 4. In this section, we consider for every $\rho > 0$ principles

$$\tilde{v}_\rho(H) = \mathrm{E}[H] + a\,(\mathrm{Var}[H])^\rho\,, \tag{5.5}$$

noting that $\rho = 1$ and $\rho = \frac{1}{2}$ are the variance principle and the standard deviation principle, respectively. We basically show here by a simple argument that if P is not a martingale measure then the direct transform of $\tilde{v}_\rho$ will assign negative prices to any attainable claim with no-arbitrage price 0 if $\rho \neq \frac{1}{2}$. This implies that the standard deviation principle is the only principle from the class (5.5) that can be transformed directly into a pricing principle which is consistent with the unique no-arbitrage prices for attainable claims.

Assume that there exists $g = \int_0^T \vartheta dX \in G_T(\tilde{\Theta})$, with the property that $\mathrm{E}[g] < 0$; recall that g is the trading gain from some self-financing strategy $\vartheta \in \tilde{\Theta}$. Of course, if there exists a strategy with $\mathrm{E}[g] \neq 0$, then we can always get a strategy such that $\mathrm{E}[g] < 0$ by multiplying the strategy with -1 if the expected value is positive. Furthermore, we note that this automatically implies that $\mathrm{Var}[g] > 0$. To see this, assume that $\mathrm{Var}[g] = 0$, i.e. that g is constant and equal to $\mathrm{E}[g]$ P-a.s. Now, Assumption 3.2 guarantees the existence of an equivalent martingale measure Q

such that $\mathrm{E}^Q[g] = 0$, since $g \in G_T(\tilde{\Theta})$. However, since $g = \mathrm{E}[g]$ a.s., this shows that $\mathrm{E}[g] = 0$, which contradicts our assumption. Hence, $\mathrm{Var}[g] > 0$.

For this strategy ϑ, we define for $x \in \mathbb{R}_+$, $g_x := xg$, which is the trading gain for the strategy $x\vartheta$. Then

$$\tilde{v}_\rho(g_x) = \mathrm{E}[g_x] + a\left(\mathrm{Var}[g_x]\right)^\rho = x\left(\mathrm{E}[g] + ax^{2\rho-1}\left(\mathrm{Var}[g]\right)^\rho\right). \tag{5.6}$$

Consider first the case where $\rho < \frac{1}{2}$. In this case, (5.6) immediately shows that

$$\tilde{v}_\rho(g_x) \to -\infty \text{ for } x \to \infty \text{ if } \rho < \frac{1}{2},$$

since $\mathrm{E}[g] < 0$ and since $x^{2\rho-1} \to 0$ for $x \to \infty$. This result implies that the direct financial transform is not well defined in the case where $\rho < \frac{1}{2}$, since

$$\inf_{g \in G_T(\tilde{\Theta})} \tilde{v}_\rho(H - g) = -\infty,$$

when H is attainable. Now assume that $\rho > \frac{1}{2}$. From (5.6) it is seen that $\tilde{v}_\rho(g_x) < 0$ for $x \in (0, x^*)$ where

$$x^* = \left(\frac{-\mathrm{E}[g]}{a(\mathrm{Var}[g])^\rho}\right)^{\frac{1}{2\rho-1}}.$$

This shows that there exist self-financing strategies $\vartheta \in \tilde{\Theta}$ so that $\tilde{v}_\rho(\int_0^T \vartheta dX) < 0$ and this has the consequence that the direct financial transform (5.1) will assign a strictly negative price to the trivial claim $H = 0$. By the above calculations combined with Theorem 4.10 and Remark 4.11, we obtain:

Theorem 5.5 *Assume in addition to Assumption 3.2 that* $\mathrm{E}[g] \neq 0$ *for some* $g \in G_T(\tilde{\Theta})$. *Consider for* $\rho > 0$ *the principles (5.5) and let* $H = c^H + \int_0^T \vartheta^H dX$ *be an attainable claim with no-arbitrage price* c^H. *Then*

$$\inf_{g \in G_T(\tilde{\Theta})} \tilde{v}_\rho(H - g) = \begin{cases} -\infty & \text{if } \rho < \frac{1}{2}, \\ -\infty & \text{if } \rho = \frac{1}{2} \text{ and } a^2 < \mathrm{Var}[\tilde{Z}_T], \\ c^H & \text{if } \rho = \frac{1}{2} \text{ and } a^2 \geq \mathrm{Var}[\tilde{Z}_T]. \end{cases}$$

Furthermore, if $\rho > \frac{1}{2}$, *then* $\inf_{g \in G_T(\tilde{\Theta})} \tilde{v}_\rho(H - g) < c^H$.

6 Applications to unit-linked life insurance

In this section we apply the financial valuation principles to the valuation of unit-linked life insurance contracts. With a unit-linked life insurance contract, benefits depend explicitly on the price of some specified assets; for more details see Aase and Persson (1994) and the references therein. In Møller (1998), risk-minimizing hedging strategies were determined for a portfolio of unit-linked life insurance contracts. By applying these strategies, the insurer can reduce the combined financial and insurance risk (as measured by the variance under a specific martingale measure of future losses) inherent in these contracts. We employ the basic set-up of that paper, and this will allow us to draw on the results obtained there.

6.1 The basic model

In the following, all elements are defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, and T is a fixed finite time horizon. We briefly review the basic model of Møller (1998). Consider a life insurance portfolio consisting of n policy-holders aged y and denote by $T_1, \ldots, T_n$ the (unknown) remaining lifetimes. For simplicity, it is assumed that $T_1, \ldots, T_n$ are i.i.d. with survival function ${}_tp_y = \exp(-\int_0^t \mu_{y+\tau} d\tau)$, where μ is a deterministic continuous function (called the hazard rate). The process $N_t = \sum_{i=1}^n 1_{\{T_i \leq t\}}$ counts the number of deaths up to time t and $(n - N_t)$ the number of survivors.

Consider in addition a financial market consisting of two basic traded assets whose price processes are given by the dynamics

$$dS_t = \alpha(t, S_t) S_t dt + \sigma(t, S_t) S_t \, dW_t, \tag{6.1}$$

$$dB_t = r(t, S_t) B_t \, dt, \tag{6.2}$$

$S_0 > 0$, $B_0 = 1$, where $W = (W_t)_{0 \leq t \leq T}$ is a standard Brownian motion on the time interval $[0, T]$ and is assumed to be stochastically independent of N. Let $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ be the P-augmentation of the natural filtration generated by (N, W), that is $\mathcal{F}_t = \mathcal{F}_{t+}^0 \vee \mathcal{N}$, where $\mathcal{F}_t^0 = \sigma\{(N_u, W_u); u \leq t\}$ and $\mathcal{N}$ is the σ-algebra generated by all P-null-sets. The discounted stock price process is $X := S/B$, and we let $\lambda = \frac{\alpha - r}{\sigma^2 X}$. It is assumed that the functions α, r, σ are bounded and satisfy certain Lipschitz conditions, which, in particular, ensure the existence of a unique solution to (6.1), see e.g. Karatzas and Shreve (1991, Theorem 5.2.9). Furthermore, we assume that r and σ are non-negative and that σ is uniformly bounded away from 0.

Finally, we recall that the process $\tilde{M}$ defined by

$$\tilde{M}_t = N_t - \int_0^t (n - N_u) \mu_{y+u} du$$

is a P-martingale and is independent of W.

The filtration $\mathbb{F}$ describes the amount of information which is available to the insurer. With the present construction, the insurer has access to current information concerning the number of deaths within the insurance portfolio as well as to the development of the asset prices.

6.2 The variance optimal martingale measure

In the present set-up, the so-called market price of risk process

$$\nu_t := (\alpha(t,S_t) - r(t,S_t))/\sigma(t,S_t)$$

is bounded, and hence we can define a new measure $\hat{P} \in \mathcal{M}^e(P)$ by

$$D := \frac{d\hat{P}}{dP} = \exp\left(-\int_0^T \nu_u dW_u - \frac{1}{2}\int_0^T \nu_u^2 du\right). \tag{6.3}$$

The measure $\hat{P}$ defined by (6.3) is known from the literature as the *minimal martingale measure*, see e.g. Schweizer (1995). For later use, introduce the likelihood process

$$Z_t := \mathrm{E}[D|\mathcal{F}_t] = \exp\left(-\int_0^t \nu_u dW_u - \frac{1}{2}\int_0^t \nu_u^2 du\right).$$

In general, the variance optimal martingale measure $\tilde{P}$ will differ from $\hat{P}$. As shown in Grandits and Rheinländer (1999), this would for instance be the case if the process ν would be a function of the triple (t, S_t, N_t). However, by exploiting the results of Pham, Rheinländer and Schweizer (1998, Section 4.3), we find that the two measures actually coincide in the present set-up: Since our model is a special case of what they call "an almost complete diffusion model", their argument justifies that D can be written on the form

$$D = D_0 + \int_0^T \tilde{\zeta}_u dX_u. \tag{6.4}$$

But Lemma 1 of Schweizer (1996) then implies that $\frac{d\tilde{P}}{dP} = D$, so that indeed $\tilde{P} = \hat{P}$. By Lemma 3.5, the density D can also be written as

$$\frac{d\tilde{P}}{dP} = \frac{\pi(1)}{\mathrm{E}[\pi(1)]} = \frac{1 - \int_0^T \tilde{\beta}_u dX_u}{\mathrm{E}[\pi(1)]},$$

and by equating the two expressions for D we find that $\mathrm{E}[\pi(1)] = \frac{1}{D_0}$ and $\tilde{\beta} = -\frac{\tilde{\zeta}}{D_0}$. The integrand $\tilde{\zeta}$ in (6.4) can now be determined as in Pham, Rheinländer and Schweizer (1998, Proposition 10), where $\tilde{\zeta}$ is expressed in terms of the solution to a second order PDE; note however that the present set-up differs slightly from their framework in that our coefficients α, r and σ depend on (t, S_t) instead of (t, X_t). In the special case where α, r and σ are functions of t only, we have that $D_0 = \tilde{\mathrm{E}}[D] = \exp(\int_0^T \nu^2(u)du)$, and

$$\tilde{\zeta}_t = -Z_t\lambda_t \exp\left(\int_t^T \nu^2(u)du\right) = -\frac{\nu(t)}{\sigma(t)X_t}\tilde{Z}_t, \tag{6.5}$$

where we have introduced the $\tilde{P}$-martingale $\tilde{Z}_t = \tilde{\mathrm{E}}[D|\mathcal{F}_t]$ and used that $\lambda_t = \frac{\nu(t)}{\sigma(t)X_t}$. Furthermore, when ν does not depend on S, it follows from (6.3) that

$$\tilde{Z}_t = Z_t \exp\left(\int_t^T \nu^2(u)du\right).$$

It was shown in Møller (1998, Section 2.3) that $\tilde{M}$ is a $\tilde{P}$-martingale and that $(\tilde{M}, X)$ are stochastically independent under $\tilde{P}$.

6.3 The unit-linked pure endowment contract

We consider the discounted payoff

$$H = B_T^{-1} g(S_T)(n - N_T), \tag{6.6}$$

where g is some continuous function such that $\mathrm{E}[(g(S_T)B_T^{-1})^2] < \infty$. With this construction the benefit is linked to the financial asset S in that each of the $(n-N_T)$ survivors receives the amount $g(S_T)$ at time T. In addition, we introduce the unique no-arbitrage price process for $g(S_T)$ given by $F^g(t, S_t) := \tilde{\mathrm{E}}[B_T^{-1}B_t g(S_T)|\mathcal{F}_t]$ and assume that $F^g \in C^{1,2}$. We denote by F_s^g the partial derivative with respect to s and require that the function F_s^g is bounded, that is, that there exists a constant K_0 such that $|F^g(t,s)| \leq K_0$ for all $(t,s) \in [0,T] \times [0,\infty)$.

In order to apply the financial valuation principles, we need the decomposition (3.3) for the claim H. This decomposition can be expressed in terms of the Kunita-Watanabe decomposition (under $\tilde{P}$) for the $\tilde{P}$-martingale $\tilde{V}_t := \tilde{\mathrm{E}}[H|\mathcal{F}_t]$; see e.g. Schweizer (1999) for a general version of this result. The martingale $\tilde{V}$ was studied in Møller (1998) for the unit-linked contract (6.6), and it was shown there that

$$\tilde{V}_t = (n - N_t)\,{}_{T-t}p_{y+t} B_t^{-1} F^g(t, S_t). \tag{6.7}$$

Furthermore, by Møller (1998, Lemma 4.1), the Kunita-Watanabe decomposition for $\tilde{V}$ is

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_u^H dX_u + \int_0^t \nu_u^H d\tilde{M}_u, \tag{6.8}$$

where (ξ^H, ν^H) are given by

$$(\xi_t^H,\ \nu_t^H) = \left((n - N_{t-})\,{}_{T-t}p_{y+t} F_s^g(t, S_t),\ -B_t^{-1} F^g(t, S_t)\,{}_{T-t}p_{y+t}\right). \tag{6.9}$$

From Schweizer (1999, Theorem 4.6) we now obtain the following expression for the integrand ϑ^H in the decomposition (3.3):

$$\vartheta_t^H = (n - N_{t-})\,{}_{T-t}p_{y+t} F_s^g(t, S_t) + \tilde{\zeta}_t \int_0^{t-} \frac{1}{\tilde{Z}_u} B_u^{-1} F^g(u, S_u)\,{}_{T-u}p_{y+u} d\tilde{M}_u. \tag{6.10}$$

Furthermore, by using (3.3), (6.8), (6.10) and the product rule, it follows that

$$N^H = \tilde{Z}_T \int_0^T \frac{1}{\tilde{Z}_u} \nu_u^H d\tilde{M}_u = -\tilde{Z}_T \int_0^T \frac{1}{\tilde{Z}_u} B_u^{-1} F^g(u, S_u)\,{}_{T-u}p_{y+u} d\tilde{M}_u.$$

6.4 Risk-minimizing hedging strategies

The main results of Møller (1998) are briefly reviewed. In that paper, the criterion of risk-minimization, which is due to Föllmer and Sondermann (1986), is applied. This criterion essentially amounts to minimizing at any time t the variance (under some suitably chosen equivalent martingale measure) of future losses defined as the amount to be paid at time T reduced by future trading gains. The risk-minimizing strategy is determined by first computing the so-called intrinsic value process (6.7) and then determining the Kunita-Watanabe decomposition (6.8) of this process. In this way, $\tilde{V}$ is decomposed into an integral with respect to X which represents the hedgeable part of H and a martingale $L^H = \int \nu^H d\tilde{M}$ which is orthogonal to X and which represents the risk inherent in H that cannot be hedged away. It follows from Møller (1998, Theorem 4.4) that the risk-minimizing hedging strategy $\varphi^* = (\xi^*, \eta^*)$ for (6.6) is given by

$$\begin{aligned}
\xi_t^* &= (n - N_{t-})_{T-t}p_{y+t}F_s^g(t, S_t), \\
\eta_t^* &= (n - N_t)_{T-t}p_{y+t}B_t^{-1}F^g(t, S_t) - \xi_t^* X_t,
\end{aligned}$$

and that the minimum obtainable variance associated with φ^* is

$$\begin{aligned}
R_0^* &:= \operatorname{Var}_{\tilde{P}}\left[H - \int_0^T \xi_t^* dX_t\right] \\
&= \operatorname{Var}_{\tilde{P}}\left[L_T^H\right] = n\,_Tp_y \int_0^T \tilde{\mathrm{E}}\left[(F^g(t, S_t)B_t^{-1})^2\right]{}_{T-t}p_{y+t}\,\mu_{y+t}dt. \qquad (6.11)
\end{aligned}$$

In Møller (1998, Section 6) the quantities $\tilde{V}_0$ and R_0^* are evaluated numerically in the situation where the benefit is of the form $g(S_T) = \max(S_T, K)$ (K is a minimum guarantee) for various choices of volatility σ and guarantee K.

6.5 The variance of N^H

In this situation, we have

$$\operatorname{Var}[N^H] = n\,_Tp_y \int_0^T \mathrm{E}\left[\frac{Z_t}{\tilde{Z}_t}\left(F^g(t, S_t)B_t^{-1}\right)^2\right]{}_{T-t}p_{y+t}\,\mu_{y+t}dt. \qquad (6.12)$$

We give here only an idea of how this result can be proved and refer to Møller (1999b) for a rigorous argument. Note that (6.12) specializes to (6.11) in the special case where $P = \tilde{P}$, that is, when the physical measure P is a martingale measure.

It follows already from Theorem 3.4 that $\mathrm{E}[N^H] = 0$, and hence

$$\begin{aligned}
\operatorname{Var}[N^H] &= \mathrm{E}[(N^H)^2] \\
&= \mathrm{E}\left[\left(\tilde{Z}_T \int_0^T \frac{1}{\tilde{Z}_u}\nu_u^H d\tilde{M}_u\right)^2\right] = \tilde{\mathrm{E}}\left[\tilde{Z}_T\left(\int_0^T \frac{1}{\tilde{Z}_u}\nu_u^H d\tilde{M}_u\right)^2\right].
\end{aligned}$$

We let $\tilde{L} = \int \frac{\nu^H}{\tilde{Z}} d\tilde{M}$ and apply Itô's formula to the process $\tilde{Z}\tilde{L}^2$ (see e.g. Jacod and Shiryaev (1987) for a version that applies in this generality). After some rearrangement of terms, we arrive at

$$\tilde{Z}_T \tilde{L}_T^2 = \int_0^T \tilde{L}_{t-}^2 d\tilde{Z}_t + 2\int_0^T \tilde{Z}_t \tilde{L}_{t-} d\tilde{L}_t + \int_0^T \tilde{Z}_t \left(\frac{\nu_t^H}{\tilde{Z}_t}\right)^2 dN_t.$$

The first two terms are integrals with respect to the $\tilde{P}$-martingales $\tilde{Z}$ and $\tilde{L}$, respectively, and hence, they are *likely* to be $\tilde{P}$-martingales. However, we can only guarantee that they are so-called *local* $\tilde{P}$-martingales, and, in particular, this implies that their expected values under $\tilde{P}$ need not be equal to 0. It therefore requires some additional work to show that this is actually the case! But *provided* that the two processes are indeed $\tilde{P}$-martingales, we have now obtained that

$$\mathrm{Var}[N^H] = \tilde{\mathrm{E}}\left[\int_0^T \frac{(\nu_t^H)^2}{\tilde{Z}_t} dN_t\right].$$

And *provided* that also the local $\tilde{P}$-martingale $\int \frac{(\nu^H)^2}{\tilde{Z}} d\tilde{M}$ is a $\tilde{P}$-martingale, this can be rewritten as

$$\begin{aligned}
\mathrm{Var}[N^H] &= \tilde{\mathrm{E}}\left[\int_0^T \frac{(\nu_t^H)^2}{\tilde{Z}_t}(n - N_{t-})\mu_{y+t} dt\right] \\
&= \int_0^T \tilde{\mathrm{E}}\left[\frac{(\nu_t^H)^2}{\tilde{Z}_t}\right] \tilde{\mathrm{E}}\left[(n - N_{t-})\right] \mu_{t+y} dt \\
&= \int_0^T \mathrm{E}\left[\frac{Z_t}{\tilde{Z}_t}(F^g(t, S_t)B_t^{-1})^2\right] ({}_{T-t}p_{y+t})^2 \mathrm{E}[(n - N_t)]\mu_{y+t} dt \\
&= n\, {}_Tp_y \int_0^T \mathrm{E}\left[\frac{Z_t}{\tilde{Z}_t}(F^g(t, S_t)B_t^{-1})^2\right] {}_{T-t}p_{y+t}\mu_{y+t} dt.
\end{aligned}$$

The second equality follows by the Fubini theorem and the $\tilde{P}$-independence between S and N, and the third equality is obtained by applying the definition of the measure $\tilde{P}$ and using the explicit expression for ν^H given in (6.9). This verifies (6.12) under the additional assumption that the local $\tilde{P}$-martingales involved are true $\tilde{P}$-martingales.

6.6 The financial variance principle

The fair premium for (6.6) under the financial variance principle can now be determined by Theorem 4.3 and is given by

$$v_1(H) = n\,{}_Tp_y \left(F^g(0, S_0) + a\int_0^T \mathrm{E}\left[\frac{Z_t}{\tilde{Z}_t}(F^g(t, S_t)B_t^{-1})^2\right] {}_{T-t}p_{y+t}\mu_{y+t} dt\right). \quad (6.13)$$

Here, the first term is the number of survivors $n\,{}_Tp_y$ times the market value $F^g(0, S_0)$ at time 0 of the benefit $g(S_T)$. The second term is more difficult to interpret. However, as noted in Section 6.5, it specializes to the variance of $\int_0^T \nu^H d\tilde{M}$

when $P = \tilde{P}$. Note also that the premium is here proportional to n; Møller (1999a, Section 2.2) contains a discussion on this choice of premium for unit-linked life insurance contracts. The optimal strategy ϑ^* for the seller of H can be obtained by applying Corollary 4.8 and (6.10);

$$\begin{aligned}\vartheta_t^* &= (n - N_{t-})_{T-t}p_{y+t}F_s^g(t,S_t)\\ &\quad+\tilde{\zeta}_t\int_0^{t-}\frac{1}{\tilde{Z}_u}B_u^{-1}F^g(u,S_u)_{T-u}p_{y+u}d\tilde{M}_u + \frac{1+\mathrm{Var}[\tilde{Z}_T]}{2a}\tilde{\beta}_t. \qquad (6.14)\end{aligned}$$

The first term in (6.14) is recognized as the risk-minimizing strategy for H under $\tilde{P}$, see Møller (1998). The second term is a "correction term", which is related to the seller's loss; we also refer to Møller (1998) for an interpretation of the integral with respect to $\tilde{M}$ in the martingale case. The third term is independent of the claim H and is related to the quadratic criterion applied.

Explicit expressions for $\tilde{\zeta}$, $\tilde{\beta}$ and $\frac{Z_t}{\tilde{Z}_t}$ are given in Section 6.2 for the case where the coefficients α, r and σ are functions of t only.

6.7 The financial standard deviation principle

The fair premium under the financial standard deviation principle is found by using Theorem 4.4, which applies provided that $a^2 \geq \mathrm{Var}[\frac{d\tilde{P}}{dP}]$. In this case, the fair premium under the financial standard deviation principle is given by

$$v_2(H) = n_Tp_yF^g(0,S_0) + \tilde{a}\left(n_Tp_y\int_0^T \mathrm{E}\left[\frac{Z_t}{\tilde{Z}_t}(F^g(t,S_t)B_t^{-1})^2\right]{}_{T-t}p_{y+t}\mu_{y+t}dt\right)^{\frac{1}{2}},$$

where we have introduced

$$\tilde{a} = a\sqrt{1-\frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}.$$

The optimal strategy for the seller of H is obtained from Corollary 4.12 and is now given by

$$\begin{aligned}\vartheta_t^* &= (n - N_{t-})_{T-t}p_{y+t}F_s^g(t,S_t) + \tilde{\zeta}_t\int_0^{t-}\frac{1}{\tilde{Z}_u}B_u^{-1}F^g(u,S_u)_{T-u}p_{y+u}d\tilde{M}_u\\ &\quad+\frac{1+\mathrm{Var}[\tilde{Z}_T]}{\tilde{a}}\left(n_Tp_y\int_0^T \mathrm{E}\left[\frac{Z_t}{\tilde{Z}_t}(F^g(t,S_t)B_t^{-1})^2\right]{}_{T-t}p_{y+t}\mu_{y+t}dt\right)^{\frac{1}{2}}\tilde{\beta}_t.\end{aligned}$$

6.8 Numerical results

We consider a numerical example with the same parameters as the ones used in the numerical example of Møller (1998); for the insurance portfolio we take $y = 45$, $T = 15$, $n = 100$ and

$$\mu_{y+t} = 0.0005 + 0.000075858 \cdot 1.09144^{y+t}, \quad t \geq 0. \qquad (6.15)$$

We apply a standard Black-Scholes market with parameters $S_0 = B_0 = 1$, $\alpha = 0.10$, $r = 0.06$ and $\sigma = 0.25$, and consider in addition the cases of low volatility ($\sigma = 0.15$) and high volatility ($\sigma = 0.35$). Furthermore, we take $g(S_T) = \max(S_T, K)$, which is known as a unit-linked contract with guarantee, and consider various choices of guarantee K. It follows by the well-known Black-Scholes formula that

$$F^g(t, S_t) = Ke^{-r(T-t)}\Phi\left(-z_t + \sigma\sqrt{T-t}\right) + S_t\Phi(z_t), \tag{6.16}$$

where Φ is the standard normal distribution function and

$$z_t = \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Furthermore, the first partial derivative is $F^g_s(t, S_t) = \Phi(z_t)$.

	Guarantee (K)	$\tilde{\mathrm{E}}[H]$	$\mathrm{Var}[N^H]$	(std. dev.)
$\sigma = 0.15$	0	0.8796	0.224	–
	$0.5\exp(rT)$	0.8996	0.224	(0.0004)
	$\exp(rT)$	1.0807	0.238	(0.0004)
($M = 500{,}000$)	$2\exp(rT)$	1.7993	0.379	(0.0003)
$\sigma = 0.25$	0	0.8796	0.415	–
	$0.5\exp(rT)$	0.9580	0.422	(0.0015)
	$\exp(rT)$	1.2066	0.460	(0.0015)
($M = 1{,}000{,}000$)	$2\exp(rT)$	1.9161	0.671	(0.0015)
$\sigma = 0.35$	0	0.8796	0.873	–
	$0.5\exp(rT)$	1.0255	0.883	(0.005)
	$\exp(rT)$	1.3213	0.940	(0.005)
($M = 5{,}000{,}000$)	$2\exp(rT)$	2.0511	1.197	(0.005)

Table 1: *Values for* $\tilde{\mathrm{E}}[H]$, $\mathrm{Var}[N^H]$ *for one policy-holder and various choices of volatility* σ *and guarantee* K.

With $\sigma = 0.25$, we find that $\nu = 0.16$ and hence it follows from the investigation at the end of Section 4 that the financial standard deviation principle is only well-defined for $a > 0.6842$. Since this value is probably too high for applications, we have chosen only to apply the financial variance principle in our numerical example.

We set out by computing the fair premium (6.13). Thus we need to determine the variance of N^H, which here simplifies to

$$n_T p_y \mathrm{E}\left[\int_0^T e^{-\nu^2(T-t)}\left(F^g(t, S_t)e^{-rt}\right)^2 {}_{T-t}p_{y+t}\,\mu_{y+t}dt\right], \tag{6.17}$$

since $\frac{\tilde{Z}_t}{Z_t} = e^{-\nu^2(T-t)}$. The evaluation of this variance is very similar to the computation of (6.11). Note however, that the two quantities differ by the factor $e^{-\nu^2(T-t)}$

and in that (6.17) involves a P-expectation whereas R_0^* involves expectation with respect to a martingale measure. We apply here the same numerical method as in Møller (1998), that is, we use Monte Carlo simulation for S and discretize the integral in (6.17) by using the summed Simpson rule, see e.g. Schwarz (1989). Throughout, we use the step size $\Delta t = 1/100$. In Table 1 we have listed the quantities $\tilde{V}_0 = \tilde{\mathrm{E}}[H]$, which have been computed directly from (6.15) and (6.16) without the use of simulation, and the estimates of $\mathrm{Var}[N^H]$ for various choices of σ and K. This table also gives standard errors of the estimates of $\mathrm{Var}[N^H]$ and the number M of simulated paths used. Since the premium under the financial variance principle is linear in n, we have furthermore fixed $n = 1$. In Table 2 we have fixed $\sigma = 0.25$ and $K = \exp(rT)$ and computed the fair premium for various choices of safety loading parameter a. These numbers illustrate the impact of a on the fair premium, which attains values from 1.211 to 2.127, and this corresponds to a relative loading (computed as $(v_1(H) - \tilde{\mathrm{E}}[H])/\tilde{\mathrm{E}}[H]$) between 0.004 and 0.76.

Safety loading (a)	0.01	0.1	0.25	0.5	1	2
Premium	1.211	1.253	1.322	1.437	1.667	2.127

Table 2: *The fair premium under the financial variance principle for $n = 1$, $\sigma = 0.25$, $K = \exp(rT)$, and various choices of safety loading a.*

We consider in the rest of this section an insurance portfolio with $n = 100$ and present some simulation results for N, S and ϑ^*. We take $a = 0.25$ and fix $\sigma = 0.25$ and $K = \exp(rT)$. Figure 1 shows a possible realization for the counting process N. The first death occurs after approximately 1.5 years, and the total number of deaths is 12, which is close to the expected number $\mathrm{E}[N_T] = n(1 - {}_{15}p_{45}) = 100 \cdot (1 - 0.8796) \approx 12$. Figure 2 gives a possible realization for the stock price process S; for comparison we have included the deterministic savings account $B_t = e^{rt}$. The value of the stocks falls below the savings account only shortly after time 0 and after approximately 5 years. Finally, Figure 3 shows the optimal investment strategy (6.14) corresponding to the outcome of the insurance portfolio from Figure 1 and the stock price process from Figure 2. We draw attention to some interesting features: The drop in the price process close to time 5 is reflected in the optimal strategy, which falls from 70 at time 4 to 55 at time 5. The optimal number of stocks increases to 90 at time 10, which is close to the conditional expected number of survivors, $(100 - N_{10})_5p_{55} = 90 \cdot 0.9408 \approx 89$; this can be partly explained by the fact that the value of the stock is at a level, where the probability of falling below the guarantee $K = e^{rT}$ is very small. After time 10, deaths occurring within the insurance portfolio are clearly visible in the strategy, which shows sharp jumps downwards in connection with each death. These jumps can be described explicitly by considering a jump time τ for the process N: Letting

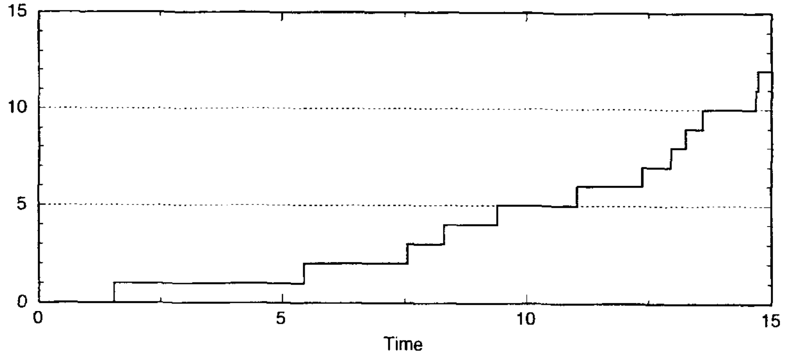

Figure 1: *Simulation of the process N in the case $n = 100$.*

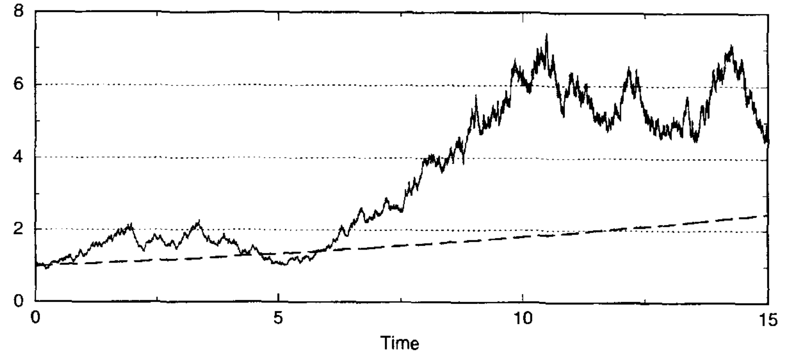

Figure 2: *Simulation of the price process S (solid line) and the deterministic savings account B (dashed line).*

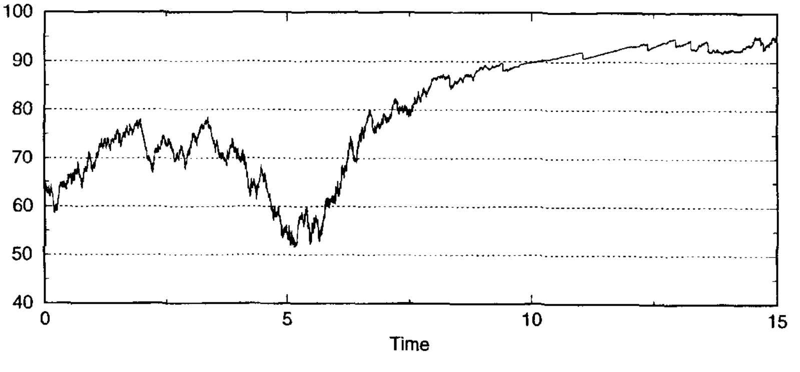

Figure 3: *The optimal trading strategy corresponding to the realizations in Figures 1 and 2.*

$\vartheta^*_{t+} := \lim_{h\searrow 0} \vartheta^*_{t+h}$, we obtain from (6.14)

$$\vartheta^*_{\tau+} - \vartheta^*_{\tau} = -{}_{T-\tau}p_{y+\tau} F^g_s(\tau, S_\tau) + \tilde{\zeta}_\tau \frac{1}{\tilde{Z}_\tau} B_\tau^{-1} F^g(\tau, S_\tau)\,{}_{T-\tau}p_{y+\tau}$$

on the set $\{\tau < T\}$. Here, both terms are negative since $\tilde{\zeta} < 0$. Furthermore, this shows that the jump for ϑ^* is big when the value of S is big, since in this case $F^g_s \approx 1$ and $F^g(t, S_t) \approx S_t$.

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