Hedging Strategies for Rate of Return Guarantees on Multi-period Assets.

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Abstract

The basis for this paper is the hedging of multi-period rate of return guarantees. We show that the market value of these guarantees can be hedged by trading strategies satisfying Merton’s (1971) self-financing condition. We show expressions for these trading strategies both when interest rates are deterministic and when they are stochastic. We consider the return on two underlying assets, a stock and a money market account. It turns out that the hedging strategies for multi-period rate of return guarantees follows a discontinuous path, which is in contrast to the hedging strategies for traditional European options. Introducing stochastic interest rates changes the hedging strategies quite considerably.

Keywords and phrases: Multi-period rate of return guarantees, self-financing hedging strategies, Heath, Jarrow and Morton term structure of interest rates

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1 Introduction

Investing in a contract, that either be a traditional investment contract, a life insurance contract, or some other form of contract, the return on the investment is usually not known at the initiation of the contract, i.e., there are uncertainty to what return that will prevail. The return on the investment can typically be linked to the development of some underlying asset, that be a stock, an index, an exchange rate, an interest rate sensitive asset etc. The terms of the contact can be set so that if the development of the underlying asset is unfavorable for the investor, he will at least get a minimum guaranteed rate of return on the investment. Such a rate of return guarantee, or for short just guarantee, can have a wide range of different interpretations. The simplest is perhaps one that is only effective at the time when the contract matures, a maturity guarantee. This guarantee secures that the investment bears a minimum guaranteed rate of return over the whole contract period. In this paper we will focus on a more general guarantee, a multi period rate of return guarantee. This guarantee secures that the investment bears a minimum guaranteed rate of return in each of several sub periods over the contract-period. To easier see the difference between these two kinds of guarantees, we can write down the terminal payoff on the maturity guarantee, $\Pi_M$, and on the multi-period guarantee, $\Pi_N$. Assume that the length of both contracts equals $N\Delta$ and that each sub-period is of length $\Delta$:

$$\Pi_M = X e^{\max\left(\sum_{i=1}^{N} \alpha_i, \sum_{i=1}^{N} g_i\right)}$$

$$\Pi_N = X e^{\sum_{i=1}^{N} \max(\alpha_i, g_i)}$$

where $X$ is the initial amount to bear interest, $\alpha_i$ the return on the underlying asset in sub-period $i$, and $g_i$ the minimum guaranteed rate of return in sub-period $i$. It is assumed that the terms of the contract are set so that the whole investment is done at the initiation of the contract and the payout comes at the end of the contract-period. Note that for $N = 1$ the payoff for the multi-period guarantee coincide with the payoff for the maturity guarantee.

Obviously, these guarantees imposes financial risk to the issuer of the contract, and in order to be able to manage this risk, we will in this paper find a set of hedging strategies that can be used to manage this risk. The strategies makes it possible to eliminate all the risk associated with such guarantees.

Multi-period rate of return guarantees are most typically found in life insurance contracts, where the investor, or the policy holder, is, in some countries, by law guaranteed a minimum rate of return each year, i.e., each sub-period equals one year. In life insurance one also has mortality risk and
the possibility for the contract to be terminated prematurely, also known as a surrender option. These more special features regarding life insurance contracts will only interfere with the analysis to be done to find the hedging strategies. We will therefore not devote any time to this issues here, but we will instead focus on multi-period rate of return guarantees per se.

It is easily seen that a maturity guarantee is equivalent to holding the underlying asset and a European put option. Assuming that the market value of the underlying asset is given as a geometric Brownian motion, we have the following expression at time $t = t$ under the equivalent martingale measure $Q$:

$$S_t = S_0 + \int_0^t rS_u du + \int_0^t S_u \sigma_d dW_u$$

where $S_0$ is the market value of the underlying asset at time $t = 0$, $r$ the constant interest rate, $\sigma$ the diffusion term of the underlying asset, and $W_u$ a, possibly multi-dimensional, standard Brownian motion. The market value of the maturity guarantee at time $t = 0$ can be written in the following way:

$$\pi_t = (1 + p_0) X$$

where $p_0$ is the market value of a put option, and at time $t = t$, $t \in [0, \Delta)$ the market value is given by (see e.g. Black and Scholes (1973) and Miltersen and Persson (1999)):

$$p_t = e^{g-r(\Delta-t)} N(-d_2) - \frac{S_t}{S_0} N(-d_1)$$

(1)

The length of the contract is assumed to equal $\Delta$. $N(\cdot)$ is the cumulative normal distribution and $d_1$ and $d_2$ are given by:

$$d_1 = \frac{\ln(S_t/S_0) - g - r(\Delta - t)}{\sigma (\Delta - t)} + \frac{1}{2} \sigma^2 (\Delta - t)$$

$$d_2 = d_1 - \sigma (\Delta - t)$$

Hipp (1996) and Miltersen and Persson (1999) show that the market value of an $N$-period guarantee, $\pi_N$, (for a constant interest rate $r$) can be written as the product of $N$ maturity guarantees:

$$\pi_N = (1 + p_0)^N X$$

Brennan and Schwartz (1976) focus on the pricing of equity-linked life insurance policies with an asset value guarantee. This is basically the same
kind of contract as the maturity guarantee mentioned above. In addition, they also introduce mortality risk. From standard option theory we know that options can be hedged by continuous trade in the underlying asset and a risk free asset, typically a zero-coupon bond. Brennan and Schwartz show that maturity guarantees can be hedged by using the same technique, and they find the hedging strategies. Adjusted to our setup, this gives the following number of units invested in the underlying asset, \( a_t \), and in the zero-coupon bond, \( b_t \), at time \( t = t, \ t \in [0, \Delta) \):

\[
a_t = \frac{X}{S_0} N(d_1) \\
b_t = X e^{\theta t} N(-d_2)
\]

Miltersen and Persson (1999) show that the corresponding hedging strategies when interest rates are stochastic, are given by similar expressions as the one in (2) and (3). In what follows we will for simplicity assume that \( X = 1 \), and \( X \) will therefore not be taken explicitly into account.

Brennan and Schwartz (1979) show, even when allowing for transaction costs and discrete rebalancing of the hedging portfolio, that one still can remove a substantial part of the financial risk of these contracts. Boyle and Hardy (1997) focus on the pricing and hedging of maturity guarantees, both seen from a more standard actuarial point of view and from a financial economics point of view. They also include transaction costs and allow for discrete rebalancing of the hedging portfolio. Grosen and Jorgensen (1997) use American option theory to analyze interest rate guarantees that can be terminated prematurely by the investor. As already mentioned this is also known as a surrender option. They show by numerical calculations that the market value of these “American-style” guarantees can be substantially higher than for their European analogs. Reffs (1998) focuses on an instantaneous rate of return guarantee. This guarantee can be described as a multi-period guarantee where the length of each sub-period approaches 0 and the number of sub-periods approaches infinity. This means that the investment always will bear an instantaneous rate of return that is never below the minimum guaranteed rate of return. The market value of this guarantee is an upper limit for the market value of the multi-period guarantees to be considered here, whereas the maturity guarantee gives a lower limit for the market value. Gerber and Pafumi (1999) focus on a guarantee that includes a modified underlying asset, \( \tilde{S}_t \), which is given by:

\[
\tilde{S}_t = S_t \max \left\{ 1, \max_{0 \leq s \leq t} \frac{K}{S_s} \right\}
\]

where \( K \) is a barrier that can either be a constant or an exponential function of time. This guarantee is also known as a reset guarantee. The idea behind this guarantee is that if the market value of the underlying asset falls below
there will be a continuous transfer of funds so that the market value of the guarantee never falls below $K$, but once the market value of the underlying asset rises, so will the guarantee. In addition to pricing this kind of guarantee, they also find the hedging strategies.

We have not found any work with focus on the hedging of multi-period rate of return guarantees or any kind of multi-period assets. Because these kinds of guarantees can be found in real life, the results we present should be of some interest in regards to risk management. The fundamental difference between the hedging strategies when interest rates are deterministic and when they are stochastic should also make the results interesting from a theoretical point of view.

The paper is organized as follows: In section two we find the hedging strategies for multi-period guarantees. In section three we illustrate the hedging strategies by some numerical examples. In section four we end the paper with some concluding remarks. In addition we have supplied three appendices, where the first gives some relations that are useful for the proofs given in the last two appendices.

2 Multi-period guarantees

Let us in this section turn to the hedging of multi-period guarantees. In order to find the hedging strategies, we will have to modify the closed-form solutions in Miltersen and Persson (1999) so that the market value of the guarantees can be found at any time $t$ in the contract period and not just at the initiation of the contract. We do not give any proofs for these modifications, but they should not be too hard to accept.

We will focus on two different kinds of economies. The first one is a standard Black and Scholes (1973) economy with deterministic interest rates. This is by now a standard framework, and a closer description can be found in Duffie (1988). In the second kind of economy we will allow for stochastic interest rates, and we will let the short term interest rate and the forward rates be described by the model of Heath, Jarrow and Morton (1992). This is a fairly general model for bond pricing. Inspired by Amin and Jarrow (1992), we will combine the two frameworks mentioned above so that we can price and hedge several risky assets within a stochastic interest rate framework. For a comprehensive treatment of these ideas, the reader is referred to Musiela and Rutkowski (1997).

The underlying asset that we used in the introduction can typically be interpreted as a stock. In addition we will also consider an underlying asset that is a pure interest rate sensitive asset to be called a money market account. Under the equivalent martingale measure $Q$, the market value of the money market account can be described as follows:
The money market account is a special case of the stock in that the stock also has a diffusion term. Miltersen and Persson (1999) show the following expression for the return on the money market account in period $n$:

\[
\beta_n = \int_{(n-1)\Delta}^{n\Delta} r_v dv = -\ln F_{n\Delta,t} + \frac{1}{2} \sigma_n^2 + c_n
\]

\[
+ \int_{(n-1)\Delta}^{n\Delta} \int_{(n-1)\Delta}^{n\Delta} \sigma_f(v,u) dudW_v + \int_{(n-1)\Delta}^{n\Delta} \int_{(n-1)\Delta}^{n\Delta} \sigma_f(v,u) dudW_v
\]

where:

\[
F_{n\Delta,t} = \frac{P(t,n\Delta)}{P(t,(n-1)\Delta)}
\]

\[
\sigma_n^2 = \int_0^{(n-1)\Delta} \left( \int_{(n-1)\Delta}^{n\Delta} \sigma_f(v,u) du \right)^2 dv + \int_{(n-1)\Delta}^{n\Delta} \left( \int_{(n-1)\Delta}^{n\Delta} \sigma_f(v,u) du \right)^2 dv
\]

\[
c_n = \int_{(n-1)\Delta}^{n\Delta} \left( \int_{(n-1)\Delta}^{n\Delta} \sigma_f(v,u) du \right) \left( \int_{(n-1)\Delta}^{n\Delta} \sigma_f(v,u) du \right) dv
\]

\[
\sigma_f(v,u)
\]

is defined as in Heath, Jarrow and Morton (1992).

The return on the stock in period $n$ can be written as:

\[
\delta_n = \int_{(n-1)\Delta}^{n\Delta} \left( r_v - \frac{1}{2} \sigma_s(v)^2 \right) dv + \int_{(n-1)\Delta}^{n\Delta} \sigma_s(v) dW_v
\]

$c_n$ is the covariance between the rate of return on the money market account in period $n$ and $n-1$. From this we can clearly see that the rate of return on both the money market account and on the stock in one period is dependent on the return in the previous period. Since we use a continuous path for the interest rate, this seems intuitive. A high interest rate at the end of one period will typically be followed by a high interest rate in the beginning of the next period. Let us first concentrate on the case with deterministic interest rates.

### 2.1 Deterministic interest rates

When interest rates are deterministic and the stock price follows a geometric Brownian motion, there will be no correlation between the rate of return in the different periods. We know that as the time elapses and the stock price evolves, the return on the stock in one period will not be affected by events in
earlier periods. The return in the previous periods has already materialized and will obviously not be affected by changes in the stock price in later periods. We let the top-script $d$ indicate that we are working with the guarantee on the stock return in a deterministic interest rate environment.

Let $n^d$ be the realized return in the first $n$ periods and let it be given by:

$$n^d = \prod_{i=1}^{n} \max \left( \frac{S_{i\Delta}}{S_{(i-1)\Delta}}, e^{\mu_i} \right)$$

Let further $\pi^d,n$ be the market value of the guarantees in the remaining $n$ periods and let it be given by:

$$\pi^d,n = \prod_{i=1}^{n} \left( N(d_{1,0}) + e^{\mu_i} F_{i\Delta,t} N(-d_{2,0}) \right)$$

where:

$$d_{1,t} = \frac{\ln(S_t/S_0) - \mu - \frac{1}{2} \sigma^2}{\sigma} + \frac{1}{2} \sigma \tau$$

and

$$d_{2,t} = d_{1,t} - \frac{1}{2} \sigma \tau$$

where we have used that $\sigma = \int_{1}^{T} \sigma_d(v)dv$.

From this we get the following expression for the market value of an $N$-period guarantee at time $t = T$, dependent on the present period (denoted in the top-script by $T$):

$$\pi_{N,T,t}^d = (T-1)^{d-N} \left( \frac{S_t}{S_{(T-1)\Delta}} \right) N(d_{1,t}) e^{\mu_t} P(t, \tau \Delta) N(-d_{2,t}) \pi_{d-N,T,t}^d$$

For $d_{1,t}$ and $d_{2,t}$, we have to replace $\ln(S_t/S_0)$ by $\ln(S_t/S_{(T-1)\Delta})$, $t \in [(T-1), T]$.

From this we propose the following hedging strategy for period $\tau$:

**Proposition 1.** To get a self-financing hedging strategy in period $\tau$, we can hold the following number of units of the stock:

$$d_{N,0}^{d,T} = (T-1)^{d-N} \left( \frac{1}{S_{(T-1)\Delta}} \right) N(d_{1,t}) e^{\mu_t} P(t, \tau \Delta) N(-d_{2,t}) \pi_{d-N,T,t}^d$$

and the following number of units of the bond maturing at time $t = \tau \Delta$:

$$b_{N,0}^{d,T} = (T-1)^{d-N} \left( e^{\mu_t} N(-d_{2,t}) \right) \pi_{d-N,T,t}^d$$

**Proof.** In period $\tau$ the market value of the guarantee in the previous and the remaining periods are constants. The proof is therefore equivalent to the proof for the maturity guarantee and follows by straightforward calculations. \[\square\]
2.2 Stochastic interest rates

As already mentioned, considering a stochastic interest rate environment, the rate of return in one period is dependent on the rate of return in earlier periods. For the case with deterministic interest rates, only two assets are needed to replicate the market value of a multi-period guarantee, since the only uncertainty comes from the development of the stock price.\(^1\) As we will see below, the uncertainty about the market value of \(P(\tilde{n}\Delta, N\Delta), \tilde{n} \in \{1,2,\ldots,N-1\}\), at time \(t \in [0,\Delta]\) forces us to use more than one bond to hedge the market value of the guarantee in the first \(N-1\) periods. The only bond that has a known value at time \(t = n\Delta, n \in \{1,2,\ldots,N\}\) for all \(t \in [0,n\Delta]\), is the one that matures at time \(t = n\Delta\).

Although we only will consider two-period guarantees, this is sufficient to emphasize the impact done by relaxing the restriction about deterministic interest rates. It turns out that the trading strategies for the guarantee on the return on the money market account and on the stock return are quite similar. Considering the similarities between the pricing formulas for these two guarantees, this hardly comes as a surprise.

2.2.1 The money market account

By a slightly modification of the expression for the market value of the guarantee on the money market account in Miltersen and Persson (1999), we get the following expression for \(t \in [0,\Delta]\) (\(\beta\) now indicates that we are considering the money market account):

\[
\pi_{2,t}^{\beta,1} = \frac{M_t}{M_0} N(a_1, b_1, \rho) + \frac{M_t}{M_0} F_{2\Delta,t} e^{\sigma_1 - \rho \sigma_1} N(a_2, b_2, -\rho) + P(t, \Delta)e^{\sigma_1} N(a_3, -b_3, -\rho) + P(t, 2\Delta)e^{\sigma_1 + \sigma_2} N(a_4, b_4, \rho)
\]

\(^1\)The only bond that is needed is the one that matures at time \(t = N\Delta\), although in each period we have used the bond that matures at the end of the period.
where:

\[ a_1 = \frac{-\ln(M_t/M_0) + g_1 + \ln(P(t, \Delta)) - \frac{1}{2} \sigma_{\beta_1,t}^2}{\sigma_{\beta_1,t}} \]

\[ b_1 = \frac{g_2 + \ln(F_{2\Delta,t}) - \frac{1}{2} \sigma_{\beta_2,t}^2 - \rho \sigma_{\beta_1,t}}{\sigma_{\beta_2,t}} \]

\[ \sigma_{\beta_n,t}^2 = \int_t^{(n-1)\Delta} \left( \int_{(n-1)\Delta}^{n\Delta} \sigma_f(v,u)dv \right)^2 du + \int_0^{n\Delta} \left( \int_{(n-1)\Delta}^{n\Delta} \sigma_f(v,u)du \right)^2 dv \]

\[ \rho = \frac{c_{2,t}}{\sigma_{\beta_1,t} \sigma_{\beta_2,t}} \]

\[ c_{2,t} = \int_0^\Delta \left( \int_0^\Delta \sigma_f(v,u)du \right) \left( \int_0^{2\Delta} \sigma_f(v,u)du \right) dv \]

\[ n \in \{1, 2\} \]

\[ a_2 = a_1 + \rho \sigma_{\beta_2,t} \quad a_3 = a_1 + \sigma_{\beta_1,t} \quad a_4 = a_1 + \rho \sigma_{\beta_2,t} + \sigma_{\beta_1,t} \]

\[ b_2 = b_1 + \sigma_{\beta_2,t} \quad b_3 = b_1 + \rho \sigma_{\beta_2,t} \quad b_4 = b_1 + \rho \sigma_{\beta_1,t} + \sigma_{\beta_2,t} \]

We now propose the following hedging strategy for the first period:

**Proposition 2.** To get a self-financing hedging strategy in the first period, we can hold the following number of units of the money market account:

\[ \alpha_{2,t}^{1,1} = \frac{\partial \pi_{2,t}^{1,1}}{\partial M_t} = \frac{1}{M_0} N(-a_1, -b_1, \rho) + F_{2\Delta,t} e^{g_1 - \rho g_3} e^{\sigma_{\beta_1,t} \sigma_{\beta_2,t}} N(-a_2, b_2, -\rho) \]

and the following number of units of the bond that matures at time \( t = \Delta \):

\[ b_{2,t}^{1,1} = \frac{\partial \pi_{2,t}^{1,1}}{\partial P(t, \Delta)} = e^{g_1} N(a_3, -b_3, -\rho) - \frac{M_t/M_0}{P(t, \Delta)} F_{2\Delta,t} e^{g_2 - \rho g_4} e^{\sigma_{\beta_1,t} \sigma_{\beta_2,t}} N(-a_2, b_2, -\rho) \]

and the following number of units of the bond that matures at time \( t = 2\Delta \):

\[ y_{2,t}^{1,1} = \frac{\partial \pi_{2,t}^{1,1}}{\partial P(t, 2\Delta)} = e^{(g_1 + g_2)} N(a_4, b_4, \rho) + \frac{M_t/M_0}{P(t, 2\Delta)} F_{2\Delta,t} e^{g_2 - \rho g_4} e^{\sigma_{\beta_1,t} \sigma_{\beta_2,t}} N(-a_2, b_2, -\rho) \]

**Proof.** Since the stock is a more general product than the money market account, the proof is just a special case of that given for the guarantee on the stock return in appendix C, and the reader is referred to this proof. □
We cannot suppress $F_{2,t}$'s dependence of the two bonds when calculating the hedging strategies. This can easily be checked by using Youngs theorem.

Now, let us turn to the second period. The return in the first period has already materialized, and in the second period it can be treated as a constant. It then follows that the market value of the guarantee at time $t = t, t \in [\Delta, 2\Delta)$, is given by:

$$\pi_{3,t}^{2,t} = \frac{M_t}{M_{2,t}} N(d_{1,t}) + P(t, 2\Delta) e^{\sigma t} N(-d_{2,t})$$

where $S_t / S_0$ in $d_{1,t}$ and $d_{2,t}$ has to be replaced by $M_t / M_{\Delta}, t \in [\Delta, 2\Delta)$, and $\sigma$ by $\sigma_{3,t}$.

We now propose the following hedging strategy:

**Proposition 3.** To get a self-financing hedging strategy in the second period, we can hold the following number of units of the money market account:

$$\alpha_{2,t}^{3,t} = \frac{\partial \pi_{3,t}^{2,t}}{\partial S_t} = \frac{\pi}{M_{\Delta}} N(d_{1,t})$$

and the following number of units of the bond that matures at time $t = 2\Delta$:

$$\beta_{2,t}^{3,t} = \frac{\partial \pi_{3,t}^{2,t}}{\partial P(t, 2\Delta)} = \frac{\pi}{\sigma} e^{\sigma t} N(-d_{2,t})$$

**Proof.** Since the stock is a more general product than the money market account, the proof is just a special case of that given for the guarantee on the stock return in appendix B, and the reader is referred to this proof.

### 2.2.2 The stock market account

By modifying the expression for the market value of the guarantee on the return on the stock in Miltersen and Persson (1999), we get the following expression for $t \in [0, \Delta)$ ($\delta$ now indicates that we are considering the stock):

$$\pi_{2,t}^{\delta,t} = \frac{S_t}{S_0} N(-\bar{a}_1, -\bar{b}_1, \rho) + \frac{S_t}{S_0} F_{2,t} e^{\sigma t} \rho \delta_{t, t} N(-\bar{a}_2, \bar{b}_2, -\rho) + P(t, \Delta) e^{\delta t} N(\bar{a}_3, -\bar{b}_3, -\rho) + P(t, 2\Delta) e^{\delta t + \delta_{2,t}} N(\bar{a}_4, \bar{b}_4, \rho)$$

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where:

\[
\begin{align*}
\tilde{a}_1 &= -\frac{\ln(S_t/S_0) + g_1 - \ln(P(t, \Delta)) - \frac{1}{2} \sigma_{\delta, t}^2}{\sigma_{\delta, t}}, \\
\tilde{b}_1 &= \frac{g_2 + \ln(k_{2, t}) - \frac{1}{2} \sigma_{\delta, t}^2}{\sigma_{\delta, t}} - \tilde{\rho} \sigma_{\delta, t}, \\
\sigma_{\delta, t}^2 &= \sigma_{\delta, t}^2 + 2k_1 + \sigma_{\delta, t}^2, \\
\sigma_{\delta, t}^2 &= \sigma_{\delta, t}^2 + 2k_3 + \sigma_{\delta, t}^2, \\
\sigma_{\delta, t}^2 &= \int_t^\Delta \sigma_s^2(v) dv \\
\tilde{\rho} &= \frac{c_{2, t} + k_2}{\sigma_{\delta, t}^2}, \\
k_1 &= \int_t^\Delta \sigma_s(v) \int_v^\Delta \sigma_f(v, u) du dv, \\
k_2 &= \int_t^\Delta \sigma_s(v) \int_v^{2\Delta} \sigma_f(v, u) du dv, \\
k_3 &= \int_t^{2\Delta} \sigma_s(v) \int_v^\Delta \sigma_f(v, u) du dv
\end{align*}
\]

\[
\begin{align*}
\tilde{a}_2 &= \tilde{a}_1 + \tilde{\rho} \sigma_{\delta, t}, \\
\tilde{a}_3 &= \tilde{a}_1 + \sigma_{\delta, t}, \\
\tilde{a}_4 &= \tilde{a}_1 + \tilde{\rho} \sigma_{\delta, t} + \sigma_{\delta, t}, \\
\tilde{b}_2 &= \tilde{b}_1 + \sigma_{\delta, t}, \\
\tilde{b}_3 &= \tilde{b}_1 + \tilde{\rho} \sigma_{\delta, t}, \\
\tilde{b}_4 &= \tilde{b}_1 + \tilde{\rho} \sigma_{\delta, t} + \sigma_{\delta, t}
\end{align*}
\]

We propose the following hedging strategy for the two-period guarantee on the stock return in the first period:

**Proposition 4.** To get a self-financing hedging strategy in the first period, we can hold the following number of units of the stock:

\[
\begin{align*}
a_{2, t}^{\delta, 1} &= \frac{\partial \eta_{2, t}^{\delta, 1}}{\partial S_t} = \frac{1}{S_0} N(-\tilde{a}_1, -\tilde{b}_1, \tilde{\rho}) + \frac{F_{2, \Delta, t}}{S_0} e^{g_2 - \tilde{\rho} \sigma_{\delta, t} \sigma_{\delta, t}} N(-\tilde{a}_2, -\tilde{b}_2, -\tilde{\rho})
\end{align*}
\]

and the following number of units of the bond that matures at time \( t = \Delta \):

\[
\begin{align*}
b_{2, t}^{\delta, 1} &= \frac{\partial \eta_{2, t}^{\delta, 1}}{\partial P(t, \Delta)} = e^{g_1} N(\tilde{a}_3, -\tilde{b}_3, -\tilde{\rho}) - \frac{S_t}{P(t, \Delta)} F_{2, \Delta, t} e^{g_2 - \tilde{\rho} \sigma_{\delta, t} \sigma_{\delta, t}} N(-\tilde{a}_2, -\tilde{b}_2, -\tilde{\rho})
\end{align*}
\]
and the following number of units of the bond that matures at time $t = 2\Delta$:

$$y^A_{2,t} = \frac{\partial \pi^A_{2,t}}{\partial P(t, 2\Delta)} = e^{(g_1 + g_2)} N(\tilde{\alpha}_4, \tilde{\beta}_4, \tilde{\rho})$$

$$+ \frac{S_t/S_0}{P(t, 2\Delta)} F_{2\Delta,t} e^{g_2 - \rho \sigma_{s_1,t} \sigma_{s_2,t}} N(-\tilde{\alpha}_2, \tilde{\beta}_2, -\tilde{\rho})$$

Proof. See appendix C. □

Let us now turn to the second period and see how the market value of the guarantee can be hedged. In the second period, the return in the first period is a constant, and it then follows that the value of the guarantee is the same as a maturity guarantee multiplied by $1^{\pi^6}$, i.e.:

$$\pi^A_{2,t} = 1^{\pi^6} \left( \frac{S_t}{S_\Delta} N(d_{1,t}) + P(t, 2\Delta) e^{g_2} N(-d_{2,t}) \right)$$

where $S_t/S_0$ in $d_{1,t}$ and $d_{2,t}$ must be replaced by $S_t/S_\Delta$, $t \in [\Delta, 2\Delta)$, and $\sigma_d$ by $\sigma_{s_{1,t}}$.

We now propose the following hedging strategy:

**Proposition 5.** To get a self-financing hedging strategy in the second period, we can hold the following number of units of the stock:

$$a^A_{2,t} = \frac{\partial \pi^A_{2,t}}{\partial S_t} = 1^{\pi^6} \frac{1}{S_\Delta} N(d_{1,t})$$

and the following number of units of the bond that matures at time $t = 2\Delta$:

$$b^A_{2,t} = \frac{\partial \pi^A_{2,t}}{\partial P(t, 2\Delta)} = 1^{\pi^6} e^{g_2} N(-d_{2,t})$$

Proof. Since $1^{\pi^6}$ is a constant in the second period, the proof is identical to the proof for the maturity guarantee and is given in appendix B. □

### 3 Numerical examples

To get some more intuition behind these hedging strategies, we will now present some numerical examples where we show the market value of the stock, the money market account, the guarantees, and how many units we must hold of the underlying asset and the bond(s). First we consider the case with deterministic interest rates. The following parameters are used ($g_1 = g_2 = g$):
Figure 1: Hedging strategies when the guarantee is effective in the first period.

\[ S_0 = 1 \quad g = \ln(1.04) \quad \sigma_s = 0.20 \quad \mu = 0.12 \]

\[ r = 0.05 \quad \Delta = 1 \quad N = 2 \]

From figure 1 we can see that as we approach the end of the first period, the number of stocks held turns to zero. By conditioning on that \( e^g > S_\Delta \), and taking the limit as \( t \) approaches \( \Delta \), this can easily be seen. Even though we arrive at time \( t = \Delta \) with no stocks, we leave by a positive investment in the stock, i.e., there has to be an instantaneous shift in the hedging strategy. We can explain this by that at the end of the first period the uncertainty about the return in the first period is almost fully resolved. At the beginning of the second period we start all over again, and to be prepared for the uncertainty, we hold a mixture of the stock and the bond.

Let us now extend the example to a stochastic interest rate environment. The following volatility structure will be used:

\[ \sigma_s(t) = \sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ \sigma_f(v, u) = \sigma e^{-\kappa(u-v)} \begin{pmatrix} \varphi \\ \sqrt{1 - \varphi^2} \end{pmatrix} \]

where \( \sigma, \sigma, \kappa, \) and \( \varphi \) are constants. This is the same model as in Miltersen and Persson (1999) and corresponds to the model by Vasicek (1977). We will use the following parameters (\( f(v, u) \) is the forward rate from time \( t = v \) to \( t = u, u > v \)): 459
\[ \dot{\sigma} = 0.20 \quad \sigma = 0.03 \quad \mu = 0.12 \quad \kappa = 0.1 \]
\[ \phi = -0.5 \quad g = \ln(1.04) \quad S_0 = 1 \quad M_0 = 1 \]
\[ r_0 = 0.05 \quad f(0,1) = 0.05 \quad f(0,2) = 0.05 \]

Figure 2: Hedging strategies when the guarantee is effective in the first period (\( MMA \) is the market value of the money market account).

In figure 2 we can see that the guarantee is effective in the first period, and at the end of the first period the number of units held of the money market account turns to zero, whereas at the end of the second period the number of units held of the bond turns to zero. In figure 3 the guarantee is effective in both periods. In the first period we can see the same pattern as in figure 2. From figure 4 we can see that when the guarantee is not effective in the first period, there will be held a short position in the bond that matures at time \( t = \Delta \) and a long position in the one that matures at time \( t = 2\Delta \). However, the total amount invested in the bonds at the end of the first period equals zero. From figure 2 - 4 we can also see that there is no instantaneous shift at time \( t = \Delta \) in the number of units held in the bond that matures at time \( t = 2\Delta \). This can be explained by the fact that when interest rates are stochastic, in addition to hedging the market value of the guarantee, we also have to secure that we are able to buy the right amount of bonds at time \( t = \Delta \). It can also be seen by taking the limit as \( t \) approaches \( \Delta \):

\[
\lim_{t \to \Delta} y_{2,t}^\gamma = b_{2,t}^\gamma, \quad \gamma \in \{\beta, \delta\}
\]
This means that the total amount at the end of the first period that is invested in the underlying asset and the bond that matures at time \( t = \Delta \) will equal the amount invested in the underlying asset at the beginning of the second period.

It should be mentioned that the interest rates we have simulated are under the probability measure \( Q \) and the stock price under the objective probability measure \( P \). This does not turn out to be a problem in this example because we only want to present one development in the interest rates and the stock price, and then show what the hedging strategy looks like.

4 Conclusions

In this paper we have found self-financing hedging strategies for multi-period rate of return guarantees. We found that introducing stochastic interest rates changed the hedging strategies quite considerably, but given stochastic interest rates, there were not much difference between the strategies for the guarantee on the money market account and the stock. One also has to use more than one bond to hedge a multi-period guarantee in a stochastic interest rate environment. At time \( t = i\Delta, i \in \{1, 2, \ldots, N - 1\} \) there will have to be an instantaneous shift in the hedging strategy.
Figure 4: Hedging strategies when the guarantee is effective in the second period.

References


A Appendix

In this appendix we will supply some relations that turns out to be useful in the calculations done in appendix B and C.

First we can notice that under the equivalent martingale measure $Q$, the bond price can be expressed in the following way:

$$P(t,\Delta) = P(0,\Delta) + \int_0^t r_v P(v,\Delta)dv + \int_0^t P(v,\Delta)\sigma_p(t)dW_v$$

where:

$$\sigma_p(t) = -\int_t^{\Delta} \sigma_f(t,u)du$$

In the following $\sigma_p(t)$ will apply for the bond that matures at time $t = \Delta$ and $\sigma_f(t)$ for the one that matures at time $t = 2\Delta$. 
For two variables, $x$ and $y$, and two continuously differentiable functions, $w$ and $z$, the cumulative binormal distribution is given by:

$$N(w; z; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{w} \int_{-\infty}^{z} e^{-\frac{x^2+2\rho xy+y^2}{2(1-\rho^2)}} \, dx \, dy$$
From this we get:

\[
\frac{\partial N(w; z; \rho)}{\partial w} = \frac{1}{2\pi \sqrt{1 - \rho^2}} \int_{-\infty}^{\infty} e^{-\frac{(z - \mu w)^2}{2(1 - \rho^2)}} dx
\]

\[
= \frac{e^{-w^2/2}}{\sqrt{2\pi (1 - \rho^2)}} N\left(\frac{z - \rho w}{\sqrt{1 - \rho^2}}\right)
\]

**B Appendix**

In this appendix we will prove that the hedging strategy for the maturity guarantee on the stock return in the Amin & Jarrow economy is self-financing and that it hedges the market value of the guarantee.

Both \(a_t = a(t, S_t, P(t, \Delta))\) and \(b_t = b(t, S_t, P(t, \Delta))\) are twice continuously differentiable on \([0, \infty) \times \mathbb{R} \times \mathbb{R}\). Since \(S_t\) and \(P(t, \Delta)\) are Itô processes, it follows that \(a_t\) and \(b_t\) also are Itô processes (see e.g. Øksendal (1995)). Define \(\mu_a, \sigma_a\) and \(\mu_b, \sigma_b\) as the drift term and the diffusion term of \(a_t\) and \(b_t\) respectively. Application of Itô's lemma to the hedging portfolio now yields:

\[
a_t S_t + b_t P(t, \Delta) = a_0 S_0 + b_0 P(0, \Delta) + \int_0^t a_{\tau} dS_\tau + \int_0^t b_{\tau} dP(\tau, \Delta)
\]

\[
+ \int_0^t S_\tau da_{\tau} + \int_0^t P(\tau, \Delta)db_{\tau} + \int_0^t da_{\tau} dS_\tau + \int_0^t db_{\tau} dP(\tau, \Delta)
\]

**Condition 1.** For the hedging strategy for the maturity guarantee to be self-financing, the sum of the last line in \((4)\) has to equal zero (see e.g. Duffie (1988)).

By Itô's lemma the market value of the guarantee can be expressed as:

\[
\pi_{1,t}^\delta = \pi_{1,0}^\delta + \int_0^t \left[ \frac{\partial \pi_{1,\tau}^\delta}{\partial \tau} + \frac{\partial \pi_{1,\tau}^\delta}{\partial S_\tau} r_\tau S_\tau + \frac{1}{2} \frac{\partial^2 \pi_{1,\tau}^\delta}{\partial S_\tau^2} S_\tau^2 \sigma_s(\tau)^2 \right]
\]

\[
+ \frac{\partial \pi_{1,\tau}^\delta}{\partial P(\tau, \Delta)} r_\tau P(\tau, \Delta) + \frac{1}{2} \frac{\partial^2 \pi_{1,\tau}^\delta}{\partial P(\tau, \Delta)^2} P(\tau, \Delta)^2 \sigma_p(\tau)^2
\]

\[
+ \frac{\partial^2 \pi_{1,\tau}^\delta}{\partial S_\tau \partial P(\tau, \Delta)} S_\tau \sigma_s(\tau) \sigma_p(\tau) d\tau
\]

\[
+ \int_0^t \left[ \frac{\partial \pi_{1,\tau}^\delta}{\partial S_\tau} r_\tau \sigma_s(\tau) + \frac{\partial \pi_{1,\tau}^\delta}{\partial P(\tau, \Delta)} P(\tau, \Delta) \sigma_p(\tau) \right] dW_\tau
\]

From condition 1 we get the following expression for the market value of the hedging portfolio:

\[
\theta_t = a_0 S_0 + b_0 P(0, \Delta) + \int_0^t \left( a_{\tau} r_\tau S_\tau + b_{\tau} r_\tau P(\tau, \Delta) \right) d\tau
\]

\[
+ \int_0^t \left( a_{\tau} S_\tau \sigma_s(\tau) + b_{\tau} P(\tau, \Delta) \sigma_p(\tau) \right) dW_\tau
\]

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Condition 2. By the unique decomposition property for Itô processes, we know that (5) and (6) must have the same drift and diffusion term if the hedging strategy is to hedge the market value of the guarantee, i.e., $(5) - (6) = 0$, $\forall t \in [0, \Delta]$ a.s.

We can now state a proof for that the hedging strategy is self-financing and hedges the market value of the guarantee.

Proof. Calculate the derivatives and check that condition 1 and 2 are satisfied. \hfill \Box

C Appendix

In this appendix we will prove that the hedging strategy in proposition 4 is self-financing and that it hedges the market value of the guarantee.

Both $a_t = a(t, S_t, P(t, \Delta), P(t, 2\Delta))$, $b_t = b(t, S_t, P(t, \Delta), P(t, 2\Delta))$ and $y_t = y(t, S_t, P(t, \Delta), P(t, 2\Delta))$ are twice continuously differentiable on $[0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and are thereby Itô processes.

An application of Itô's lemma gives the following expression for the market value of hedging portfolio:

$$a_{2,t}^\Delta S_t + b_{2,t}^\Delta P(t, \Delta) + y_{2,t}^\Delta P(t, 2\Delta) = \ldots$$

An application of Itô's lemma yields the following expression for the market value of the guarantee:

$$a_{2,0}^\Delta S_0 + b_{2,0}^\Delta P(0, \Delta) + y_{2,0}^\Delta P(0, 2\Delta) + \int_0^t a_{2,t}^\Delta dS_t + \int_0^t b_{2,t}^\Delta dP(t, \Delta) + \int_0^t y_{2,t}^\Delta dP(t, 2\Delta) + \int_0^t S_t \, d\bar{a}_{2,\tau}^\Delta + \int_0^t P(\tau, \Delta) \, d\bar{b}_{2,\tau}^\Delta + \int_0^t P(\tau, 2\Delta) \, d\bar{y}_{2,\tau}^\Delta + \int_0^t d\bar{a}_{2,\tau}^\Delta dS_t + \int_0^t d\bar{b}_{2,\tau}^\Delta dP(\tau, \Delta) + \int_0^t d\bar{y}_{2,\tau}^\Delta dP(\tau, 2\Delta)$$

Condition 3. For the hedging strategy in proposition 4 to be self-financing, the sum of the last two lines in (7) has to equal zero.

An application of Itô's lemma yields the following expression for the market value of the guarantee:
\[ \pi_{x_1}^{\delta,1} = \pi_{x_0}^{\delta,1} + \int_0^t \left[ \frac{\partial \pi_{x_1}^{\delta,1}}{\partial \tau} d\tau + \frac{1}{2} \frac{\partial^2 \pi_{x_1}^{\delta,1}}{\partial \tau^2} (dS_\tau)^2 + \frac{\partial \pi_{x_1}^{\delta,1}}{\partial P(\tau, \Delta)} dP(\tau, \Delta) \right. \\
+ \left. \frac{1}{2} \frac{\partial^2 \pi_{x_1}^{\delta,1}}{\partial \tau \partial Y(\tau, \Delta)^2} (dP(\tau, \Delta))^2 + \frac{\partial^2 \pi_{x_1}^{\delta,1}}{\partial S_\tau \partial P(\tau, \Delta)} dS_\tau dP(\tau, \Delta) \right] (8) \\
+ \frac{\partial^2 \pi_{x_1}^{\delta,1}}{\partial P(\tau, \Delta) \partial P(\tau, 2\Delta)} dP(\tau, \Delta) dP(\tau, 2\Delta) \\
+ \frac{\partial^2 \pi_{x_1}^{\delta,1}}{\partial P(\tau, \Delta)} dP(\tau, \Delta) dP(\tau, 2\Delta) \]

From condition 3 we get the following expression for the market value of the hedging portfolio:

\[ \theta_t = a_{x_0}^{\delta,1} S_0 + b_{x_0}^{\delta,1} P(0, \Delta) + y_{x_0}^{\delta,1} P(0, 2\Delta) \\
+ \int_0^t \left[ a_{x_1}^{\delta,1} dS_\tau + b_{x_1}^{\delta,1} dP(\tau, \Delta) + y_{x_1}^{\delta,1} dP(\tau, 2\Delta) \right] (9) \]

**Condition 4.** By the unique decomposition property for Itô processes, we know that (8) and (9) must have the same drift and diffusion term if the hedging strategy is to hedge the market value of the guarantee, i.e., (8) − (9) = 0, \( \forall t \in [0, \Delta] \) a.s.

We can now prove that the hedging strategy in proposition 4 is self-financing and that it hedges the market value of the guarantee.

**Proof.** Calculate the derivatives and check that condition 3 and 4 are satisfied. \( \square \)