MINIMUM RATE OF RETURN GUARANTEES: THE DANISH CASE

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ABSTRACT. We analyze minimum rate of return guarantees for life insurance (investment) contracts and pension plans with a smooth surplus distribution mechanism. We specifically model the smoothing mechanism used by most Danish life insurance companies and pension funds. The annual distribution of bonus will be based on this smoothing mechanism after taking the minimum rate of return guarantee into account. In addition, based on the contribution method, the customer will receive a final (non-negative) undistributed surplus when the contract matures.

We consider two different methods that the company can use to collect payment for issuing these minimum rate of return guarantee contracts: The direct method where the company gets a fixed percentage fee of the customer's savings each year, e.g., 0.5% in Denmark, and the indirect method where the company gets a share of the distributed surplus. In both cases we analyze how to set the terms of the contract in order to have a fair contract between an individual customer and the company.

Having analyzed the one-customer case we turn to analyzing the case with two customers. We consider the consequences of pooling the undistributed surplus over two inhomogeneous customers. This implies setting up different mechanisms for distributing final bonus (undistributed surplus) between the customers.

1. INTRODUCTION

The historically low interest rates and correspondingly low expected returns on portfolios of risky securities have left Danish pension funds and life insurance companies in a situation where it is difficult for them to find investment opportunities with a return distribution enabling them to meet the guaranteed minimum rate of return they have promised their customers in the past. Standard life insurance practice is to set fairly conservative terms (i.e., premia, annual return, etc.) of a life insurance (or investment) contract based on estimates of the future development of the financial market (and other types of risk including mortality risk) when the contract is initiated and to compensate the customer through a surplus distribution mechanism as time evolves and the true development of the financial market is gradually revealed. This distributed surplus is normally termed *bonus*, cf., e.g, Norberg (1999). In some contracts the surplus is accumulated over the life of the contract and not distributed until the maturity of the contract, in other contracts the surplus is gradually distributed over time. This practice of setting fairly conservative terms initially and compensating the customer with bonus payments as the contract matures is also adapted by most Danish pension funds. Since the terms of the contract cannot be altered by the company during the life of the contract in a way unfavorable for the customer, the initial conservative terms of the contract is de facto a minimum rate of return guarantee issued by the company to the customers. In financial terms the company has issued an option to its customers. In principle, the company could have set the initial terms of the contracts extremely conservative such that the issued option is so far out of the money that it is valueless for all practical purposes. However, competition among these companies has forced them to set less conservative terms for the contracts and hence made

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the option valuable (i.e., less out of the money). Furthermore, the latest development in the financial market has even driven these minimum rate of return guarantees into the money. In the present paper we will try to model the way Danish life insurance investment contracts and pension plans are (or can be) designed to collect this option premium. Moreover, we will try, for fairly realistic parameter values, to find the terms of the contracts such that the company gets a fair option premium for the issued minimum rate of return guarantee.

In the paper we consider a hybrid of the models by Grosen and Jørgensen (1998) and Miltersen and Persson (1998).[1] Grosen and Jørgensen (1998) price minimum rate of return guarantees with profits where the surplus (i.e., profit) is distributed to the customer gradually, based on a so-called *smoothing mechanism*. This way of distributing surplus gives the customer a rate of return on his stake which does not fluctuate very much. They look at a European contract that at maturity gives the customer the amount which has accumulated on his account through both the guaranteed minimum rate of return and the profit paid out during the life of the contract. However, the customer does not receive the undistributed surplus when the contract matures. Grosen and Jørgensen (1998) compare the European contract with an equivalent American contract, i.e., a contract on the same terms except that it also includes a surrender option. They operate with two accounts on the liability side of the insurance company's balance sheet; the customer's account and the bonus reserve (or buffer). On the asset side a given reference portfolio is specified. The return on the reference portfolio (positive or negative) is credited to the bonus reserve. It is the amount of the bonus reserve that determines (via the smoothing mechanism) the amount the customer receives as bonus payment: When the amount of the bonus reserve is a certain percentage above the value of the customer's account, the company distributes some fraction of the bonus reserve. At maturity the company keeps the remaining amount (positive or negative) in the bonus reserve. Hence, this is a way for the company to collect payment for issuing the guarantee, since in some states of nature it collects a payment and in others it must cover the deficit.

Miltersen and Persson (1998) take a somewhat different approach. They also consider minimum rate of return guarantees with a surplus distribution mechanism. Their way of distributing surplus is however different from that of Grosen and Jørgensen (1998). Miltersen and Persson (1998) simply distribute a fraction of the annual excess return (if positive) to the customer. Moreover, they consider contracts which, besides from the amount in the customer's account, also pay out the amount of the bonus reserve (if bonus is positive) at maturity. Hence, the obligation of the company in this type of contract is to issue a (European) call option on the bonus reserve with an exercise price of zero. Opposed to Grosen and Jørgensen (1998) where the company in some states of nature collects payment for issuing the guarantee by keeping the remaining bonus reserve, Miltersen and Persson (1998) must have another way for the company to collect this payment. Therefore, Miltersen and Persson (1998) work with a third account on the liability side—the account whereto payments to the company for issuing the call option (on the bonus reserve) is made. On the asset side, they also specify a given reference portfolio which is used to determine the annually distributed bonus. They price the contract indirectly by finding the terms of the contract (e.g., guaranteed minimum rate of return, cf. Miltersen and Persson (1998)) such that the present value (using an equivalent martingale measure) of the total net payments to the company from the customer up to the date of expiration is zero (i.e., such that the contract is *fair*).

The model in this paper makes use of the surplus distribution mechanism from Grosen and Jørgensen (1998) since this type of smoothing mechanism is often used in practice. Our model considers a contract of the

[1]The idea of merging these two models was inspired by the discussions of Henrik Ramlau-Hansen and Paul Brüniche-Olsen at the conference *Financial Markets in the Nordic Countries*, in Århus, January 14–15, 1999, where both papers were presented.

type where the customer receives a specified annual minimum rate of return, some of the bonus reserve during the life of the contract, and the amount on the bonus reserve (if positive) at maturity (here date T). If the bonus reserve is negative at date T, then the company covers the "deficit" as in Miltersen and Persson (1998). This means that the company has issued a series of options on the annual returns. These are covered by the bonus reserve, hence, the company, de facto, has only issued an option on the final bonus reserve.

We work with three accounts on the liability side of the balance sheet: The customer's account, the bonus reserve, and the company's account. On the asset side we have the value of the customer's investment which the company administrates.

Observe that when the bonus payments are linked to the company's own investment portfolio, the company has incentives to lower the volatility of the investment portfolio in order to decrease the risk that the final value of the bonus reserve becomes negative. The customer would recognize this incentive and therefore value the option using the volatility that gives the lowest possible price, that is, using a volatility of zero. A volatility of zero would degenerate our model, and more importantly, it is not what we observe in practice. The reason we do not observe a volatility of zero, is, according to our beliefs, due to competition among the companies. To model competition among insurance companies is however outside the scope of this paper. Instead we assume that the surplus distribution is linked to a certain verifiable reference portfolio with a given volatility. This eliminates the company's incentive to manipulate the investment portfolio.

At initiation of the contract an amount, X, is paid by the customer to the company. This is the amount in the customer's account at date zero. It is invested by the insurance company for the duration of the contract. Besides investing X in the reference portfolio the company has the oppurtunity to set up a hedge portfolio that completely eliminates any risk the company faces as a result of the contract issued to the customer.

First we investigate the one-customer case and characterize fair contracts between the customer and the company. In this case the customer will always have an initial bonus reserve of zero when he enters. At the maturity of the contract the customer will receive the remaining (positive) undistributed surplus.

Secondly we investigate a situation with two customers. The two customers can differ with respect to minimum rate of return guarantees, entry dates, and exit dates. We propose different mechanisms for distributing the final bonus between the customers depending on how the customers differ.

The paper is organized as follows. Section 2 describes the modeling framework. This includes describing the surplus distribution mechanism and examples of possible payment schemes the company can use to collect payment for the contract it has set up with the customer. The terminal condition that sustains a *fair* contract is given in section 3, and the results follow in section 4. Finally, in section 5 we investigate the situation where two customers have one common bonus reserve and compare this to the situation with individual bonus reserves. Section 6 concludes.

2. The Model

In the case with only one customer we use the a brutal simplification of the company's balance sheet. We only include the accounts relevant for determining the customer's contract. That is, we exclude the company's hedge activities from the balance sheet. Hence, the balance sheet can be represented graphically as,

Assets	Liabilities
X	A
	B
	C
X	X

A short description of the different accounts is provided below:

Account A: This is the customer's main account. The initial deposit is credited to this account. In any year the total amount on this account earns the guaranteed minimum rate of return, g, (specified in the contract) and possibly some bonus as interest. Bonus is distributed when the so-called ***buffer ratio*** is above a certain level γ.[2] The buffer ratio is determined by the bonus reserve, B, in a manner equivalent to Grosen and Jørgensen (1998).[3] The contribution from the bonus reserve to the customer's account is also known as distributed surplus.

Account C: This is the account where the company collects the payment for issuing and guaranteeing the contract. In the case of a negative amount in the bonus reserve at the maturity of the contract, the deficit is covered by the company. We consider two different ways of collecting payments, both determined so that the contract is fair.[4] These two methods will be termed the *indirect* and the *direct* method. More about this in subsection 2.1.

Account B: This is the bonus reserve (or undistributed surplus) for the individual customer. It is determined residually in the sense that the annual return of the customer's investment in the reference portfolio (positive or negative) is first distributed to this account. Then the required return to account A and the payments to account C are subtracted from this account. This should become clearer when looking at the model in mathematical terms.

Account X: The account X keeps track of the value of the investment that the company has made in the reference portfolio on behalf of the customer. We assume that the change in this value can be described by a geometric Brownian motion, that is, the value of the investment at date t, $X(t)$, is given by,

$$dX(t) = rX(t)dt + \sigma X(t)dW(t), \quad X(0) = X,$$

under the equivalent martingale measure, Q. r is the instantaneous riskless interest rate, which is assumed to be constant, σ is the volatility of the reference portfolio, and $\{W(t)\}_t$ is a Brownian motion under Q.

We model X under the equivalent martingale measure, Q, since for valuation purposes we would like to use the traditional Harrison-Kreps/Harrison-Pliska approach, cf. Harrison and Kreps (1979) and Harrison and Pliska (1981). This approach relies on assumptions of no-arbitrage and complete markets. Hence, it is possible for the company to dynamically hedge the contract (issued to the customer) completely. That is, one can think of the company's entire investment portfolio, Y, as composed of the investment on behalf of the customer, X, in the reference portfolio and a hedge portfolio, H.

Note that we have implicitly assumed that there are no dividend payments[5] on the reference portfolio since the drift is equal to the short term interest rate.

[2]The size of γ is typically around 10% in Denmark according to regulated solvency rules.

[3]The buffer ratio is defined as the bonus reserve over the sum of the accounts A and C since $A + C$ play the role of the so-called policy account in the Grosen-Jørgensen model (the account on the liability side other than the bonus reserve). That is, the buffer ratio is equal to $\frac{B}{A+C}$.

[4]By fair we simply mean that the present value of the company's net payments from the customer up to date T is equal to 0, i.e., $V_0(C(T) - B^-(T)) = 0$, where $C(T)$ is the total amount on account C at date T, $B^-(T)$ is the potential deficit in the bonus reserve, and $V_0(\cdot)$ denotes the date zero market value operator.

[5]An equivalent interpretation is that potential dividends on the assets included in the reference portfolio are immediately reinvested into the portfolio.

The given specification of the value of reference portfolio implies that the customer's investment in the reference portfolio has a random continuously compounded annual rate of return equal to,

$$\delta(t) = \ln\Big(\frac{X(t)}{X(t-1)}\Big) = (r - \frac{1}{2}\sigma^2) + \sigma\big(W(t) - W(t-1)\big), \tag{1}$$

i.e., $\delta(t) \sim N(r - \frac{1}{2}\sigma^2, \sigma^2)$. Note, moreover that the returns in different years are stochastically independent because increments of the Wiener process are independent.

2.1. **Distributing to the Accounts.** Our method for distributing bonus is somewhat complicated. First, we model the development in the sum of the customer's and the company's accounts. The model incorporates a possible way of collecting payments (for the contract) from the customer to the company.

We distribute the guaranteed minimum return (determined by the minimum rate of return guarantee, g) and possibly an extra amount depending primarily on the size of the bonus reserve to the sum of the two accounts $(A+C)$. More specifically, the accounts together receive either the guaranteed minimum return, $(e^g - 1)(A+C)(t-1)$, at date t or a certain fraction, $\alpha+\rho$, where $\alpha+\rho \in [0,1]$, of the excess bonus reserve (bonus above the optimal buffer level, $\gamma(A+C)$), whichever amount is the larger. Hence, the sum of accounts A and C is compounded (continuously) at the following rate[6]

$$\max\Big\{g, \ln\Big(1 + (\alpha+\rho)\Big(\frac{B(t-1)}{(A+C)(t-1)} - \gamma\Big)\Big)\Big\}. \tag{2}$$

In order to see that our method for distributing to the sum of accounts A and C in fact distributes a certain fraction, $\alpha+\rho$, of the excess bonus reserve in the case where the return is greater than the guaranteed minimum return, consider the following: The desired level of the bonus reserve at date t is $\gamma(A+C)(t)$, $t \leq T$. If $g < \ln\Big(1 + (\alpha+\rho)\Big(\frac{B(t-1)}{(A+C)(t-1)} - \gamma\Big)\Big)$, then the sum of accounts A and C develops as[7]

$$\begin{aligned}(A+C)(t) &= (A+C)(t-1)e^{\ln\big(1+(\alpha+\rho)\big(\frac{B(t-1)}{(A+C)(t-1)}-\gamma\big)\big)}\\ &= (A+C)(t-1)\Big(1 + (\alpha+\rho)\big(\frac{B(t-1)}{(A+C)(t-1)} - \gamma\big)\Big)\\ &= (A+C)(t-1) + (\alpha+\rho)\big(B(t-1) - \gamma(A+C)(t-1)\big).\end{aligned} \tag{3}$$

Since $\gamma(A+C)(t-1)$ is the targeted level of the bonus reserve and $B(t-1)$ is the actual level, the difference, $B(t-1) - \gamma(A+C)(t-1)$, is the excess bonus mentioned above.[8]

After having determined the method for distributing the return from the buffer account to the sum of the accounts A and C, we will now specify the method for distributing between the two accounts. The development of the amount in the customer's account is modeled similarly to the sum of the accounts A and C. Account A receives the guaranteed minimum return or a fraction of the excess bonus reserve, whichever is the greater. This fraction, however, is smaller than for $A+C$. Only the fraction α is distributed to the account A. Moreover, a percentage fee, ξ, is subtracted from the customer's rate of return. This fee, which will be referred to as the rate of payment fee, is introduced as a method for

[6]Grosen and Jørgensen (1998) use an annually compounded rate of $\max\{g_a, r_a\}$, where $r_a = (\alpha+\rho)\big(\frac{B(t-1)}{(A+C)(t-1)} - \gamma\big)$. This is equivalent to the continuously compounded rate from expression (2), since $1 + r_a = e^{r_c} \Leftrightarrow r_c = \ln(1 + r_a)$. Here a and c denote annual and continuous compounding, respectively.

[7]This idea is borrowed from Grosen and Jørgensen (1998).

[8]Note that equation (3) only makes sense if $\Big(1 + (\alpha+\rho)\big(\frac{B(t-1)}{(A+C)(t-1)} - \gamma\big)\Big) > 0$ otherwise $A+C$ can change sign from date $t-1$ to date t and we may start chasing a negative optimal buffer level. Fortunately, this will never happen when we have the minimum rate of return guarantee, g, since this prevents us from ever emptying the sum of the accounts A and C totally. That is, whenever $\Big(1 + (\alpha+\rho)\big(\frac{B(t-1)}{(A+C)(t-1)} - \gamma\big)\Big) \leq 0$ we know that expression (2) will be equal to g even though the second term in the max is not well-defined. We have taken that into account in our computer implementation of the model.

collecting payment for the contract. Hence, the rate of return on account A is,

$$\max\Big\{g, \ln\Big(1+\alpha\Big(\frac{B(t-1)}{(A+C)(t-1)}-\gamma\Big)\Big)\Big\}-\xi.$$

That is, we have modeled two ways that the company can collect payment (into the account C) for issuing the guarantee. Either the contract can be specified with a positive ξ or a positive ρ. When the contract is specified with a positive ξ (and $\rho = 0$), we say that the company uses the *direct* method for collecting payment for the contract, whereas when the contract is specified with a positive ρ (and $\xi = 0$), we say that the company uses the *indirect* method for collecting payment for the contract. Of course the indirect and the direct method, can be combined, setting $\rho > 0$ and $\xi > 0$ at the same time. However, analyzing the effects of this is not the purpose of the present paper. We use the term *direct* in the case where ξ is positive since in this case the payment is collected directly from account A. In particular, a certain fraction, ξ, of the amount in the customer's account is paid to the company.

Observe that payment for the contract is made over time. No up-front premium is paid.

A simple subtraction of A from the value of $A+C$ gives us the amount in the company's account, that is, the amount the company collects for issuing the contract.

Let us summarize: The development in $A+C$, A, and C from year to year can be written as,

$$(A+C)(t) = (A+C)(t-1)e^{\max\left\{g,\ln\left(1+(\alpha+\rho)\left(\frac{B(t-1)}{(A+C)(t-1)}-\gamma\right)\right)\right\}}, \qquad \alpha,\rho \in [0,1], \alpha+\rho \in [0,1], \tag{4}$$

$$A(t) = A(t-1)e^{\max\left\{g,\ln\left(1+\alpha\left(\frac{B(t-1)}{(A+C)(t-1)}-\gamma\right)\right)\right\}-\xi}, \qquad \xi \in [0,1], \tag{5}$$

and

$$C(t) = (A+C)(t) - A(t), \tag{6}$$

where $t \in \{1,\ldots,T\}$ and T is the maturity date. Moreover, note that if both ξ and ρ are zero, then the company does not collect any premium and hence contracts cannot be fair.

As mentioned, account B is determined residually. It starts off at zero value, (the individual bonus reserve is always zero at the time of entry, since the customer has not built any reserve yet). At the end of each year, the return on the customer's investment in the reference portfolio is added to account B while the amount going to $A+C$ (according to equation (4)) is withdrawn. That is, everything is first put into account B and then amounts are distributed to A and C according to equations (4)–(6). We can write,

$$B(t) = B(t-1) + \underbrace{X(t) - X(t-1)}_{\text{return on the assets}} - (A+C)(t) + (A+C)(t-1). \tag{7}$$

According to equation (1) we have that the value of the customer's investment portfolio can be given recursively as

$$X(t) = X(t-1)e^{\delta(t)}. \tag{8}$$

The initial values of the different accounts are,

$$X(0) = X,$$
$$A(0) = X,$$
$$B(0) = 0,$$

and

$$C(0) = 0.$$

This allows us to rewrite equations (4), (5), and (8) as,

$$\begin{aligned}(A+C)(t) &= (A+C)(0)e^{\sum_{i=1}^{t} \max\left\{g, \ln\left(1+(\alpha+\rho)\left(\frac{B(i-1)}{(A+C)(i-1)}-\gamma\right)\right)\right\}} \\ &= Xe^{\sum_{i=1}^{t} \max\left\{g, \ln\left(1+(\alpha+\rho)\left(\frac{B(i-1)}{(A+C)(i-1)}-\gamma\right)\right)\right\}},\end{aligned} \tag{9}$$

$$A(t) = Xe^{\sum_{i=1}^{t} \max\left\{g, \ln\left(1+\alpha\left(\frac{B(i-1)}{(A+C)(i-1)}-\gamma\right)\right)\right\}} e^{-t\xi}, \tag{10}$$

and

$$X(t) = Xe^{\sum_{i=1}^{t} \delta(i)}. \tag{11}$$

Recall that in the case where the bonus reserve is negative at date T, a transfer of $B^-(T)$ takes place from account C to B where $B^-(T)$ denotes the amount of the potential deficit. That is, we have that the value of the company's account at date T is, $C(T) - B^-(T)$.

3. Pricing the Contract Fairly

We will use the same method of pricing the contract as Miltersen and Persson (1998). We assume that the insurance market is characterized by being competitive. This competitiveness forces abnormal profits to be zero. Since the company's profits are collected in the account C, abnormal profit equal to zero is equivalent to the present value of future (total) profits being zero, i.e., $V_0\big(C(T) - B^-(T)\big) = 0$. Here $V_t(\cdot)$ denotes the date t market value operator, i.e.,

$$V_t\big(Z(T)\big) = e^{-r(T-t)}E_t^Q\big[Z(T)\big],$$

where $E_t^Q[\cdot]$ denotes the conditional expectation under the equivalent martingale measure, Q, given the information at date t and $Z(T)$ is a (stochastic) payoff at date T.

The present value of the future profits must be zero, otherwise if, e.g.,

$$V_0\big(C(T) - B^-(T)\big) > 0,$$

then another company could offer a contract with better terms for the customer and still have $V_0\big(C(T) - B^-(T)\big) > 0$. This mechanism of the market will eventually drive $V_0\big(C(T) - B^-(T)\big)$ to zero.

At any date t, $t \leq T$, we have that

$$X(t) = A(t) + B(t) + C(t),$$

since the usual accounting principle has to apply (i.e., sum of assets equals sum of liabilities). Writing the account B as the difference between its positive and negative part $B = B^+ - B^-$, we get

$$X(t) = A(t) + B^+(t) - B^-(t) + C(t).$$

For $t = T$ we have, $X(T) = A(T) + B^+(T) + C(T) - B^-(T)$. The use of the market value operator and equation (11) yield

$$V_0\big(Xe^{\sum_{i=1}^{T} \delta(i)}\big) = V_0\big(A(T) + B^+(T)\big) + V_0\big(C(T) - B^-(T)\big),$$

which implies by the competitive market argument that we have,

$$V_0\big(Xe^{\sum_{i=1}^{T} \delta(i)}\big) = V_0\big(A(T) + B^+(T)\big). \tag{12}$$

Since the date 0 market value of investing X in the reference portfolio has to equal X in order to preclude arbitrage,[9] we end up with the following requirement for a fair contract

$$X = V_0\big(A(T) + B^+(T)\big) \quad \Leftrightarrow \quad 1 = V_0\Big(\frac{A(T)}{X}\Big) + V_0\Big(\frac{B^+(T)}{X}\Big). \tag{13}$$

This final condition determines the relation between:

(i) The annual minimum rate of return guarantee, g, the fraction of the bonus reserve distributed to the customer, α, and the indirect payment fee, ρ, for the contracts offered by an insurance company using the *indirect* method, and

(ii) The annual minimum rate of return guarantee, g, the fraction of the bonus reserve distributed to the customer, α, and the rate of (direct) payment fee, ξ, for the contracts offered by an insurance company using the *direct* method.

Note that we assume that the company is able to invest in a portfolio, Y, which completely replicates the payoff, $A(T) + B^+(T)$, to the customer at maturity of the contract. That is, the value of this portfolio can be expressed as

$$Y(t) = V_t\big(A(T) + B^+(T)\big).$$

Since the hedge portfolio, H, can be expressed as $H(t) = Y(t) - X(t)$ we have,

$$H(0) = Y(0) - X = V_0\big(A(T) + B^+(T)\big) - X = 0,$$

where the last equality follows from Equation (13). Thus, it is costless for the company to set up the hedge.

The theoretically correct representation (in accounting sense) of the company's balance sheet (in the single customer case) should besides the customer's investment in the reference portfolio include the hedge portfolio on the asset side. Moreover, the liability side of the balance sheet should simply consist of the current market value of the future obligations to the customer. The market value of the future obligations can be expressed as

$$L(t) = V_t\big(A(T) + B^+(T)\big).$$

Hence, at any given date, t, the bookkeeping condition is fulfilled as the following (true and fair) balance sheet shows

Assets	Liabilities
$X(t)$	$L(t)$
$H(t)$	
$Y(t)$	$Y(t)$

4. Results

We use numerical methods to find the terms of the contract so that condition (13) is fulfilled. Bonus is distributed when the buffer ratio is above 10% (i.e., $\gamma = 0.1$).

The payout from the contract at maturity is, $A(T) + B^+(T)$. It is determined using Monte Carlo simulation. Specifically, we simulate the amount in the different accounts A, B, C, and X, thereby finding the value of $A(T) + B^+(T)$. We use the assumption of a constant interest rate r to discount the

[9]The reference portfolio is a traded asset.

	α										
ξ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0025	0.0015	0.0018	0.0022	-0.0004	-0.0009	-0.0026	-0.0036	-0.0062	-0.0090	-0.0101	-0.0118
0.0050	0.0145	0.0146	0.0154	0.0142	0.0139	0.0126	0.0122	0.0114	0.0096	0.0088	0.0073
0.0075	0.0231	0.0228	0.0237	0.0234	0.0228	0.0223	0.0220	0.0210	0.0199	0.0192	0.0181
0.0100	0.0295	0.0296	0.0299	0.0299	0.0296	0.0292	0.0290	0.0283	0.0278	0.0271	0.0264
0.0125	0.0354	0.0350	0.0354	0.0357	0.0354	0.0351	0.0345	0.0343	0.0337	0.0329	0.0327
0.0150	0.0399	0.0398	0.0404	0.0407	0.0402	0.0402	0.0398	0.0395	0.0389	0.0385	0.0381
0.0175	0.0442	0.0446	0.0447	0.0448	0.0448	0.0446	0.0441	0.0440	0.0438	0.0433	0.0429
0.0200	0.0487	0.0485	0.0488	0.0488	0.0488	0.0486	0.0484	0.0480	0.0479	0.0475	0.0471
0.0225	0.0525	0.0523	0.0526	0.0527	0.0527	0.0525	0.0523	0.0521	0.0518	0.0516	0.0514
0.0250	0.0560	0.0561	0.0562	0.0564	0.0561	0.0562	0.0560	0.0557	0.0554	0.0553	0.0552

TABLE 1. Values of g for different choices of ξ and α; $\sigma = 0.1$, $T = 10$, $\rho = 0$, and $r = 0.05(1 - 0.26)$.

payoff back to date 0, i.e., we find

$$V_0\big(A(T) + B^+(T)\big) = e^{-rT} E^Q\big[A(T) + B^+(T)\big].$$

This simulation procedure gives the value of the contract issued to the customer for a specific combination of the parameters g, α, ρ, and ξ. We have to search for combinations of parameter values which fulfill requirement (13). This is done through the use of a modified Newton-Raphson algorithm. In order to simplify requirement (13) we assume that $X = 1$. This assumption implies that we are searching for parameter values such that

$$V_0\big(A(T) + B^+(T)\big) = 1. \tag{14}$$

We analyze a few different cases. First we look for values of g (using Newton-Raphson on g) such that requirement (14) is satisfied for different combinations of α and ξ, and α and ρ.

With a choice of $r = 0.05(1 - 0.26) = 0.037$[10] and $\sigma = 0.1$ we find the values of g for different values of α and ξ (i.e., using the direct method, $\rho = 0$). The values are given in table 1. As an illustration consider a company that wants to offer a contract with an α equal to 20 percent and a rate of payment fee of 0.75 percent. What guaranteed minimum rate of return can the company offer in this case? That is, what value of g makes the contract fair? According to our calculations the contract with the features mentioned is fair when the company offers a minimum rate of return guarantee, g, of 2.37 percent.

In table 2 values of g for different choices of α and ρ are given (i.e., using the indirect method). Remember that we have assumed that the company cannot distribute more than 100 percent of the bonus reserve to accounts A and C, that is, $\alpha + \rho \leq 1$. Therefore only half of the table is full. Looking through the column with $\alpha = 0.2$ in table 2, we see that a minimum rate of return guarantee near the one offered for $\xi = 0.0075$ in the direct case (i.e., $g = 2.37$ percent) can be given for a ρ between 20 and 30 percent.

In order to find the magnitude of ξ or ρ that the company should claim for a contract with a guaranteed minimum rate of return as high as the ones that exist in Denmark today (i.e., 3 percent and 5 percent) we have done the simulation for g fixed at these values. The search algorithm has found values of ξ and ρ, respectively, so that the contract is fair. The results are depicted in figures 1 and 2, respectively.

We observe, in figure 1, that for a minimum rate of return guarantee, g, of 3 percent, ξ is about 1 percent regardless of the size of α. Similarly in the case of a minimum rate of return guarantee of 5 percent—here ξ is around 2.1 percent for the different values of α. The higher the minimum rate of

[10] Approximately the present (after-tax) short interest rate in Denmark.

	α										
ρ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1000	0.0132	0.0174	0.0163	0.0134	0.0110	0.0082	0.0057	0.0024	-0.0010	-0.0034	—
0.2000	0.0204	0.0232	0.0226	0.0211	0.0188	0.0168	0.0144	0.0125	0.0105	—	—
0.3000	0.0239	0.0260	0.0257	0.0245	0.0227	0.0211	0.0193	0.0172	—	—	—
0.4000	0.0263	0.0278	0.0277	0.0266	0.0252	0.0235	0.0220	—	—	—	—
0.5000	0.0278	0.0290	0.0289	0.0280	0.0268	0.0255	—	—	—	—	—
0.6000	0.0290	0.0299	0.0299	0.0291	0.0281	—	—	—	—	—	—
0.7000	0.0298	0.0305	0.0306	0.0299	—	—	—	—	—	—	—
0.8000	0.0305	0.0311	0.0311	—	—	—	—	—	—	—	—
0.9000	0.0311	0.0315	—	—	—	—	—	—	—	—	—
1.0000	0.0316	—	—	—	—	—	—	—	—	—	—

TABLE 2. Values of g for different choices of ρ and α; $\sigma = 0.1$, $T = 10$, $\xi = 0$, and $r = 0.05(1 - 0.26)$.

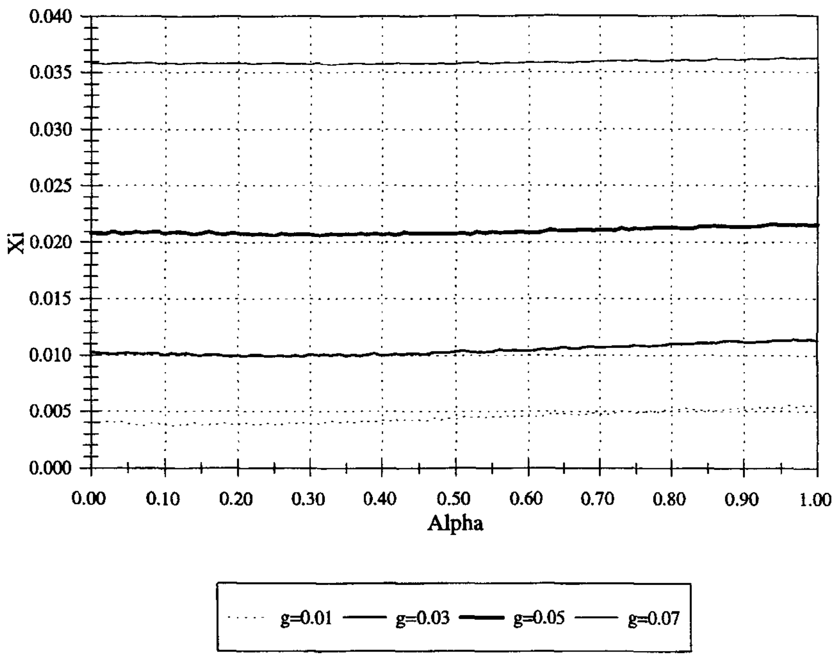

FIGURE 1. Corresponding values of α and ξ for four different values of g; $T = 10$, $\rho = 0$, $\sigma = 0.1$, and $r = 0.05(1 - 0.26)$.

return guarantee is, the more the company has to claim (ξ is higher) to be able to honour the contract. We observe that α does not have any significant influence on the size of the rate of payment fee, ξ, necessary to retain a fair contract. We explain this by the way the payment scheme is constructed: The customer pays a certain percentage of the amount of his account each year. A higher α results in a larger amount in account A, however, the fee for the contract is calculated on the basis of this larger value, and therefore, as it turns out, ξ is more or less independent of the size of α. This is a convenient

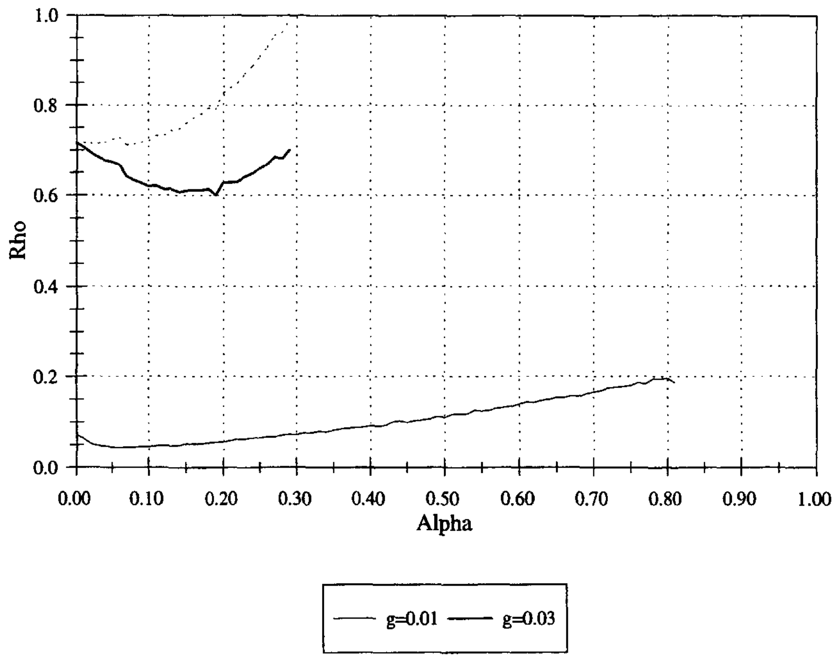

FIGURE 2. Corresponding values of α and ρ for two different values of g; $T = 10$, $\xi = 0$, $\sigma = 0.1$, and $r = 0.05(1 - 0.26)$.

feature since the company does not have to worry about fine tunning the size of alpha. A closer look at figure 1, however, indicates that the rate of payment fee required by the company might be marginally increasing in α for small values of the minimum rate of return guarantee, g. To further justify this claim we have also depicted the curves for $g = 0.01$ and $g = 0.07$. Notice how the curve for $g = 0.07$ is literally horizontal, and as g gets smaller the curves become more and more increasing in α—even though this effect is only marginal. An explanation for this is that for small values of the minimum rate of return guarantee, g, there is a greater possibility of distributing more than g, and hence the size of the bonus reserve will be more sensitive to the chosen α. The larger the value of α, the more of the excess bonus is distributed and the larger the probability of ending up with a negative bonus reserve. Therefore a larger rate of payment fee, ξ, is needed as α increases.

In figure 2 we are only able to depict corresponding values of α and ρ for a minimum rate of return guarantee of 3 percent and a limited range of αs. This is because for a minimum rate of return guarantee of 5 percent there is no way, even by setting $\rho = 1$, that the company can collect payment enough for the contract when they use the indirect method (see table 2). This is also the case for $g = 0.03$ and $\alpha \geq 0.3$. That is, for α greater than 0.3, even the highest possible ρ (i.e., $\rho = 1 - \alpha$) does not provide the company with enough payments to make the contract fair (see table 2). In figure 2 we have also depicted the curve for $g = 0.01$, here we have fair contracts for α as high as 0.8.

First consider the curve for $g = 0.03$. Above the curve we have depicted the sum $\alpha + \rho$ (the dotted curve). We see that this curve is non-decreasing in α; for $\alpha \in [0, 0.1]$ the curve is more or less constant

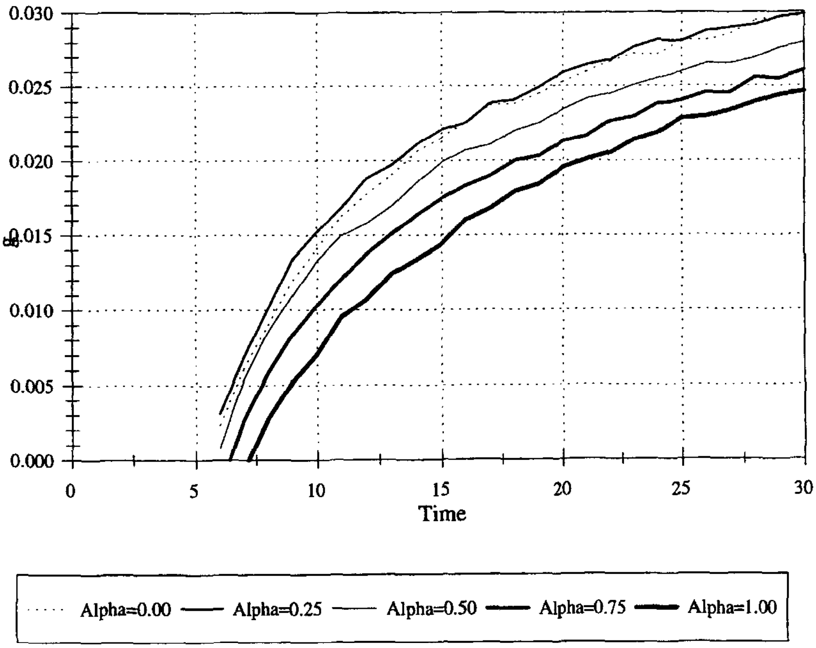

FIGURE 3. Corresponding values of T and g for five different values of α; $\xi = 0.005$, $\rho = 0$, $\sigma = 0.1$, and $r = 0.05(1 - 0.26)$.

and for $\alpha \in [0.1, 0.3]$ it is strictly increasing. The curve for combinations of α and ρ that corresponds to $g = 0.03$ can also be divided into two parts, one for each of the intervals of α just mentioned. This curve is decreasing in α in the first interval and non-decreasing in α in the second. We interpret the curve in the following way: For small values of α (i.e., $\alpha \in [0, 0.1]$) there is a high probability that the customer's account only grows at the minimum rate of return guarantee, g. Hence, the company receives almost all the surplus distributed to accounts A and C. However, the contract has to be fair and since there is no distribution of extra funds to the customer, the conditions between the customer and the company are almost the same for all $\alpha \in [0, 0.10]$, hence $\alpha + \rho$ must be more or less constant. This explains why the curve for ρ is decreasing for small αs. For higher αs the customer starts getting more than the minimum rate of return guarantee and, at the same time, the probability that the bonus reserve will be negative is increased, hence the company must have a higher payment for the contract, i.e., a higher ρ.

For $g = 0.01$ we have a more traditional picture. That is, the higher α is, the higher is the share of the distributed bonus, ρ, that must be distributed to the company for the contract to remain fair.

Note that compared to the direct payment method, the indirect payment method does not have the same convenient feature that the rate of payment fee is more or less independent of α.

Because of the limitations in the contract design and therefore also the available menu of fair contracts when the indirect method of payment is used, we will not investigate further into this payment method. Moreover, the direct payment method is more in line with the way real-life Danish contracts are designed.

All the tables and curves considered so far are for contracts with a maturity of 10 years. To see the influence of maturity on the guaranteed minimum rate of return, we have depicted values of g for varying T's in figure 3. We have drawn the curves for five different values of α. These curves are derived using the direct method of payment with $\xi = 0.005$[11], $\sigma = 0.1$, and $r = 0.037$. There are two features of figure 3 which should be emphasized:

(i) For $\alpha \neq 0$ and T fixed we have that g decreases as α rises. This effect can be explained in the following way: As α increases, more and more of the bonus is distributed and this increases the probability of a negative terminal bonus reserve (i.e., a higher option value). Therefore in order for the contract to be fair, a lower minimum rate of return guarantee will be offered.

(ii) The minimum rate of return guarantee, g rises as T increases for a fixed value of α. There are two effects explaining why g increases in T. The first effect follows from equations (9) and (10), since we have, for $\rho = 0$,

$$A(T) = Xe^{\sum_{i=1}^{T} \max\{g, \ln(1+\alpha(\frac{B(i-1)}{(A+C)(i-1)}-\gamma))\}}e^{-T\xi},$$

and

$$C(T) = Xe^{\sum_{i=1}^{T} \max\{g, \ln(1+\alpha(\frac{B(i-1)}{(A+C)(i-1)}-\gamma))\}}\left(1-e^{-T\xi}\right).$$

That is, as T increases, a larger share of the amount distributed to the accounts A and C is distributed to account C and, hence, a higher minimum rate of return guarantee can be offered. The second effect is that the bonus reserve increases with time since the targeted buffer size increases with the sum of the accounts A and C, and, hence, the probability that the bonus reserve will end up being negative at the maturity of the contract decreases with the maturity of the contract.

Finally, observe that for $\alpha = 0$ and $\alpha = 0.25$, the minimum rate of return guarantee of 3 percent, which has been offered until recently in Denmark, is consistent with a fair contract when the maturity of the contract is around 30 years.

5. Pooled Bonus Reserve: The Two-Customer Case

Most insurance companies and pension funds today do not keep track of the individual customers' bonus reserves. Normal practise for the companies is to place the different customers' bonus reserves in one pool. From this pooled bonus reserve the company then distributes bonus to the customers. The group of customers is, however, not homogeneous, e.g., the customers have different minimum rate of return guarantees and/or different maturities. One of the questions that raise great debate in these days is whether this practise causes a redistribution of bonus from one group of customers (with similar contracts) to another group of customers (with similar contracts). In Denmark the big issue is whether the group of customers with a 3 percent guarantee (new contracts) is treated unfairly compared to the group of customers with a 5 percent guarantee (old contracts). In order to analyze the question in a simple setting, we consider the case with only two customers, customer 1 and customer 2. We compare the situation where the customers have individual bonus reserves with the case where there is only one pooled bonus reserve and bonus is distributed to the customers using some criteria that the company controls. The way bonus is distributed is known by the customer at the date he enters into the contract. Whether it is possible for a company to alter the way of distributing bonus during the life of the contracts is a question that we will leave to qualified people to answer. Here we will assume that the company cannot change the bonus distribution mechanism.

[11]Most insurance companies in Denmark collect a 0.5 percent fee for administrative costs etc.

We consider five different scenarios. The first scenario considered is the base case where the customers are identical, that is, they have the same minimum rate of return guarantee and so on. In scenario two we look at the isolated effect of the customers having different minimum rate of return guarantees. In the third scenario we analyze the isolated effect of different maturities. More specifically, we consider the case where the two customers engage in a contract at the same date but the contracts have different maturities. The effect of different maturities is also investigated in scenario four. However, this time the two customers enter into a contract at different dates but their contracts expire at the same date. The last scenario considers the combined effect of different minimum rate of return guarantees and different maturities, in particular it combines scenario 2 and 4. Each scenario consists of two different parts:

Part (a) In this part we consider the question of how the bonus is redistributed as a result of pooling. The rate of payment fee, ξ, that the company gets as direct payment for issuing the guarantee is calculated individually for the two customers in such a way that their contracts are fair with individual bonus reserves. That is, for individual rate of payment fees, ξ_1 and ξ_2, that make the individual contracts fair, we compare the value of the customer's contract in the case of an individual bonus reserve to the case with a pooled bonus reserve and consider who benefits from the use of a pooled bonus reserve instead of individual bonus reserves.

The values of ξ_1 and ξ_2 that make the contracts fair (using individual bonus reserves, i.e., using the method from section 3) are the ξs which solve

$$X = V_0(A_1 + B_1^+) \tag{15}$$

and

$$X = V_0(A_2 + B_2^+), \tag{16}$$

where the subscripts on the accounts A and B refer to the two customers. Recall that we have set $X = 1$ so the value of each contract equals 1 (at the date of entry).

Part (b) In this part we consider the question of how bonus is redistributed when we use a common rate of payment fee, ξ, for all customers. That is, for the value of ξ that makes the sum of the contracts fair, we again compare the values of the contracts in the pooled bonus case to the individual bonus case. The common ξ is found as the ξ which solves

$$2X = V_0(A_1 + A_2 + B^+), \tag{17}$$

where B represents the pooled bonus reserve.

We calculate the values of the contracts in the case of a pooled bonus reserve using a method for distributing terminal bonus that depends on which scenario we are in. The method for distributing the terminal bonus reserve depends mainly on the individual customer's time of entry and exit. The general principle for distributing the terminal bonus reserve is to distribute bonus according to the fraction of the assets, X, that each customer has contributed to. This fraction depends on the times of entry and exit of the customers. A detailed description is provided below for each scenario separately. In each of the scenarios below α is equal 0.25 and each customer deposits $X = 1$ with the company at the time of entry.

Scenario 1: In this case the two customers, 1 and 2, have the same minimum rate of return guarantee, $g_1 = g_2 = 3$ percent, and they enter into and exit the contracts at the same time. Maturity is 10 years. In the case of a pooled bonus reserve, bonus is distributed equally among the two customers since they have contributed equally to the assets, that is, we distribute half of the terminal bonus to each of the two customers. Since the terms of the two contracts are the same and the customers receive equal shares of the terminal bonus, the values of the contracts based on pooled bonus should be equal to the values found using individual bonus reserves.

Scenario 2: In this scenario we change the value of the minimum rate of return guarantee, so that customer 1 has $g_1 = 5$ and customer 2 has $g_2 = 3$ percent. All other parameters remain unchanged. Since the entry and exit dates are the same for customers 1 and 2, they receive an equal fraction of the terminal bonus reserve (as in scenario 1). That is, we find the values of the contracts using a pooled bonus reserve as, $V_0(A_1 + \frac{1}{2}B^+)$ and $V_0(A_2 + \frac{1}{2}B^+)$.

Scenario 3: In this scenario we go back to the case where the customers have the same minimum rate of return guarantee, $g_1 = g_2 = 3$ percent. Moreover, the customers enter at the same date. However, their contracts have different maturities. We set the maturity for customer 1 equal to 20 years, i.e., $T_1 = 20$, and the maturity for customer 2 equal to 10 years, $T_2 = 10$. Since they enter at the same date and both pay X at the beginning, this amount will have grown equally for both at the time customer 2 exits. That is, at date $T_2 = 10$ they have helped build the same fraction of the bonus reserve in the case of the pooled bonus reserve. More specifically one half each. Note, however, that because customer 2 leaves the company before customer 1, we have to adjust the bonus account (and the asset side) at date $T_2 = 10$ when customer 2 exits. Customer 2 receives half of the bonus reserve (if positive) at this date. The rest is kept in the bonus reserve and whatever amount (if positive) there is in the bonus reserve at date $T_1 = 20$ goes to customer 1. That is, the contract values for customers 1 and 2, respectively, are calculated as:

Customer 1:

$$V_0(A_1(T_1) + B^+(T_1)). \tag{18}$$

Customer 2:

$$V_0(A_2(T_2) + \frac{1}{2}B^+(T_2)). \tag{19}$$

At date $T_2 = 10$ we also have to adjust the asset side in order to maintain the bookkeeping equality (assets = liabilities). This is done in the simulations by withdrawing customer 2's amount, $A_2(T_2) + \frac{1}{2}B^+(T_2)$, from the account on the asset side. The amount on the asset side at date T_2, when customer 2 has exited, is therefore $2Xe^{\sum_{i=1}^{T_2}\delta(i)} - A_2(T_2) - \frac{1}{2}B^+(T_2)$.

Scenario 4: In this scenario customer 1 enters at date 0 and exits at date 20, i.e., $T_1 = 20$. However, customer 2 enters 10 years later than customer 1, that is, at date $T = 10$. His contract has a maturity of 10 years, $T_2 = 10$, hence he exits at the same time as customer 1. All other parameters remain the same.

At the time of entry of customer 2, date $T = 10$, customer 1 has already built a bonus reserve, $B(T)$. In the case where the final pooled bonus reserve is positive, we distribute bonus to the two customers in such a way that customer 1 in principle receives all the bonus built up to date $T = 10$ (discounted forward at the rate at which the reference portfolio grows). The bonus built in the period from date $T = 10$ to date $T_1 = 20$ is distributed among the two customers relative to their fractions of the total amount the company has invested on their behalf in the reference portfolio at date $T = 10$ when customer 2 enters. The two customers' portfolio weights at date $T = 10$ are given by β and $1 - \beta$, respectively, where β is defined as

$$\beta = \frac{Xe^{\sum_{i=1}^{T}\delta(i)}}{X + Xe^{\sum_{i=1}^{T}\delta(i)}} = \frac{e^{\sum_{i=1}^{T}\delta(i)}}{1 + e^{\sum_{i=1}^{T}\delta(i)}}.$$

Sc.1a	$(\xi_1, \xi_2) = (0.0099, 0.0099)$	
	Individual bonus	Pooled bonus
Customer 1	1.0008	1.0012
Customer 2	0.9992	1.0012
Sum	2.0000	2.0024

TABLE 3. Ind. ξ's, $g_1 = g_2 = 0.03$, and $T_1 = T_2 = 10$.

The fraction, ϵ, of the total bonus reserve at date $T_1 = 20$ that originates from the period up to date $T = 10$, is given by

$$\epsilon = \frac{B(T)e^{\sum_{i=T+1}^{T_1}\delta(i)}}{B(T_1)}.$$

Note that we have discounted $B(T)$ forward to date T_1.

Using these variables the (pooled) terminal bonus at date $T_1 = 20$ is distributed according to the following,

Customer 1:

$$\left(\min\{\epsilon + (1-\epsilon)\beta, 1\}\right)^+ B^+(T_1).$$

Customer 2:

$$\left(\min\{(1-\epsilon)(1-\beta), 1\}\right)^+ B^+(T_1).$$

We have used the $\min(\cdot,\cdot)$ operator in combination with the $(\cdot)^+$ operator in order to make sure that the company does not distribute more than the total terminal bonus.

Scenario 5: We make one change from scenario 4. We look at the case where the customers have different minimum rate of return guarantees. In particular, $g_1 = 5$ percent and $g_2 = 3$ percent. That is, the customer who enters first is offered a higher minimum rate of return guarantee. The way of distributing the pooled bonus reserve is the same as in scenario 4. The only difference compared to scenario 4 is that the amount of the pooled bonus reserve evolves differently.

5.1. **Results.** In tables 3–12 the results of the simulations are given. Looking at the tables we have to consider the following issues in particular:

- Does a redistribution of bonus take place and who benefits (or is worse off) in the case of a redistribution?
- Are the contracts fair? Together as a whole as well as individually.
- The use of individual ξ's versus the use of one common ξ.

With respect to the last question, it is important to note that in scenarios 1, 2, and 3 the two contracts as a whole is fair if the sum of their values is equal to 2 (the amount deposited at date 0), whereas in scenarios 4 and 5 the fair value is only 1.6907. The reason is that in scenarios 4 and 5 customer 2 does not enter until date $T = 10$. Since the fair value is a "date 0" value, we have to discount customer 2's deposit (of 1) back to date 0. The discounting is done at the risk free rate, $r = 0.037$ which we have used in all the simulations, yielding a present value of, $1e^{-0.037\cdot 10} = 0.6907$.

First consider tables 3 and 4. They should in theory be identical because the two customers have identical contracts with respect to the minimum rate of return guarantee, entry date and exit date. Hence, all values in the two tables should be either 1 or 2. Because of simulation errors, however, there

Sc.1b	$\xi = 0.0099$	
	Individual bonus	Pooled bonus
Customer 1	1.0004	1.0008
Customer 2	0.9992	1.0008
Sum	1.9995	2.0015

TABLE 4. Common ξ, $g_1 = g_2 = 0.03$, and $T_1 = T_2 = 10$.

Sc.2a	$(\xi_1, \xi_2) = (0.0207, 0.0099)$	
	Individual bonus	Pooled bonus
Customer 1	0.9997	1.0288
Customer 2	0.9996	0.9602
Sum	1.9993	1.9889

TABLE 5. Ind. ξ's, $g_1 = 0.05$, $g_2 = 0.03$, and $T_1 = T_2 = 10$.

Sc.2b	$\xi = 0.0151$	
	Individual bonus	Pooled bonus
Customer 1	1.0545	1.0817
Customer 2	0.9550	0.9154
Sum	2.0095	1.9971

TABLE 6. Common ξ, $g_1 = 0.05$, $g_2 = 0.03$, and $T_1 = T_2 = 10$.

Sc.3a	$(\xi_1, \xi_2) = (0.0065, 0.0101)$	
	Individual bonus	Pooled bonus
Customer 1	1.0005	0.9860
Customer 2	0.9987	0.9993
Sum	1.9993	1.9853

TABLE 7. Ind. ξ, $g_1 = g_2 = 0.03$, $T_1 = 20$, and $T_2 = 10$.

are small deviations from these values. We use table 3 and 4 as indicators of how well the simulations perform. Since these deviations are so small (within 0.15% deviations), the simulation procedure seems to be working quite well.

Table 5 shows the effect of pooled bonus if the two customers have different minimum rate of return guarantees. In this table the contracts have different rate of payment fees, ξ, determined so that the contracts are individually fair. Thus, $\xi_1 > \xi_2$ reflects that customer 1 has a higher minimum rate of return guarantee than customer 2. Even though the customers are charged different rates of payment fees, we observe a significant redistribution of bonus from customer 2 to customer 1 when the bonus accounts are pooled. That is, the use of a pooled bonus account does (as we would expect, since customer 1 always receives at least as much as customer 2 from the bonus account in each period and the final bonus is shared equally at maturity) negatively affect the customer with the lowest minimum rate of return guarantee. Moreover, notice that the company also benefits from the use of pooled bonus (to a lesser extent), since the payout from the option that the company has issued to customer 1 is less when customer 2 is also contributing to the bonus reserve than when bonus is individual.

In table 6 we use a common rate of payment fee, ξ, determined such that the sum of the contracts is fair. In this case we see an even further redistribution from customer 2 to customer 1. Most of this

Sc.3b	$\xi = 0.0072$	
	Individual bonus	Pooled bonus
Customer 1	0.9856	0.9736
Customer 2	1.0254	1.0254
Sum	2.0110	1.9990

TABLE 8. Common ξ's, $g_1 = g_2 = 0.03$, $T_1 = 20$, and $T_2 = 10$.

Sc.4a	$(\xi_1, \xi_2) = (0.0065, 0.0099)$	
	Individual bonus	Pooled bonus
Customer 1	0.9991	0.9876
Customer 2	0.6914	0.6871
Sum	1.6905	1.6747

TABLE 9. Ind. ξ, $g_1 = g_2 = 0.03$, entry date (1) $= 0$, $T_1 = 20$, entry date (2) $= 10$, and $T_2 = 10$.

Sc.4b	$\xi = 0.0070$	
	Individual bonus	Pooled bonus
Customer 1	0.9892	0.9825
Customer 2	0.7091	0.7067
Sum	1.6983	1.6892

TABLE 10. Common ξ, $g_1 = g_2 = 0.03$, entry date (1) $= 0$, $T_1 = 20$, entry date (2) $=$ 10, and $T_2 = 10$.

Sc.5a	$(\xi_1, \xi_2) = (0.0173, 0.0101)$	
	Individual bonus	Pooled bonus
Customer 1	1.0012	1.0106
Customer 2	0.6902	0.6446
Sum	1.6914	1.6553

TABLE 11. Ind. ξ's, $g_1 = 0.05$, $g_2 = 0.03$, entry date (1) $= 0$, $T_1 = 20$, entry date (2) $=$ 10, and $T_2 = 10$.

redistribution arises from the use of a common rate of payment fee as we see by comparing the contract values for the individual and common rate of payment fee both based on individual bonus calculations.

We consider the effect of different exit dates in tables 7 and 8. When the rates of payment fees are determined individually (table 7), we do not observe a very profound effect of pooled bonus. There is only a small redistribution to the company from the customer holding the long maturity contract. The value of the short maturity contract is unaffected by the introduction of pooled bonus. The two customers contribute equally to the bonus reserve until time customer 2 exits. If the bonus reserve at this time is negative, customer 1 carries the whole load, whereas if the bonus reserve is positive, customer 2 leaves with half of the bonus reserve. This means that the company, de facto, has transferred their liabilities with respect to customer 2 to customer 1.

When we change to a common rate of payment fee (table 8), we find the same kind of redistribution to the company from the customer with the long maturity contract. However, in this case this customer is also negatively affected by a redistribution to the customer with the short maturity contract following

Sc.5b	$\xi = 0.0142$	
	Individual bonus	Pooled bonus
Customer 1	1.0619	1.0711
Customer 2	0.6662	0.6210
Sum	1.7280	1.6921

TABLE 12. Common ξ, $g_1 = 0.05$, $g_2 = 0.03$, entry date (1) = 0, $T_1 = 20$, entry date (2) = 10, and $T_2 = 10$.

from the higher (common) rate of payment fee compared to his individually determined rate of payment fee.

Tables 9 and 10 illustrate the effect of different entry dates. In table 9 we see a redistribution to the company from customer 1 as well as customer 2. However, customer 1, who enters first, contributes more than customer 2 who enters when the first customer is halfway through his contract. This is due to the fact that when the last customer enters, the first customer has already built a bonus reserve (positive or negative). With a minimum rate of return guarantee of 3 percent the bonus reserve at the time customer 2 enters will on average (under the equivalent martingale measure) be positive, and therefore customer 2 will receive a higher return on his account (A_2) than he would have, had he entered when the bonus reserve was zero. The rates of payment fees in tables 9 are calculated based on an initial bonus reserve of zero for both customers. That is, if the bonus reserve is positive when the second customer enters, the company will have charged a rate of payment fee which is higher than the fair rate and hence leaves the company better off. In short, what happens in table 9 is that the company "cheats" customer 2 who "cheats" customer 1.

As usual we observe (table 10) a redistribution from the customer with the low individual rate of payment fee to the customer with the high individual rate of payment fee when we introduce a common rate of payment fee.

Finally, we consider the combined effect of different minimum rate of return guarantees and different entry dates in tables 11 and 12. This case illustrates the situation (at least as we see it) in Denmark today. In table 11 there is a redistribution from the customer with the short maturity contract to both the other customer and the company—the larger part goes to the company. The net redistribution stems from two separate effects working in the same direction. The first effect arises from customer 1 having a higher minimum rate of return guarantee than customer 2 as in scenario 2. The second effect is due to the different entry dates of the customers as in scenario 4. However, in this case (with customer 1's high minimum rate of return guarantee, i.e., 5 percent) the bonus reserve at date $T = 10$ is negative (on average) whereas it was positive in scenario 4. Therefore the direction of the redistribution between the customers is the other way around (i.e., from customer 2 to customer 1).

We observe the same kind of redistribution in table 12 where we use a common rate of payment fee as we observed in table 11. In addition, we see the usual redistribution as a result of switching from the individual rate of payment fee to a common rate of payment fee. That is, a redistribution from customer 2 to customer 1. In this combined case the effect is quite profound.

In addition, we recalculated the different scenarios using different volatilities, σ. However, this did not alter the redistribution effects significantly, and therefore, we have not reported the result of these calculations.

6. Concluding Remarks

In this paper we have set up a model which we think is fairly close to the institutional setup that prevails within the life insurance and pension fund industry in Denmark. The model prices contracts with minimum rate of return guarantees using the principle of fair valuation. The minimum rate of return guarantees that we consider are equipped with an option on the final bonus reserve. We use a smooth bonus distribution mechanism in order to even out the annual returns on the customers' accounts. Since we have used the principle of fair valuation to find the terms of the contracts, there is no need for an up-front premium. The customers simply pay for the contracts by paying an annual fee. Of course they also have to provide an initial deposit.

We have looked at two different ways that the company can collect payment for issuing the contracts. The direct method, where the company collects payment by charging a rate of payment fee (i.e., a certain percentage of the amount in the customer's account), and the indirect method, where the company receives a fraction of the excess bonus. We have found that the direct method allows for a greater variety of contract specifications, that is, different minimum rate of return guarantees and αs. In particular, the rate of payment fee is more or less independent of α. The direct method is, in addition, much easier for the company to implement.

The current market practise in Denmark is to charge a rate of payment fee of 0.5 percent and to offer a minimum rate of return guarantee of 3 percent.[12] We have shown that under the current market conditions (i.e., an (after-tax) short term interest rate of 3.7 percent and a volatility of 10 percent on the reference portfolio) the offered contracts are fair if their maturity is 30 years (see table 3). This illustrates, according to our model, that the companies have charged a correct premium for the minimum rate of return guarantees issued. The companies in Denmark today claim that with the current low interest rate level it is difficult to construct investment portfolios which yield a return distribution sufficient to cover the issued guarantees, hence they indirectly claim that the contracts are not fair (indeed favorable to the customers). Therefore they wish to lower the minimum rate of return on the already established contracts. This is, however, regulated by legislation and it is therefore a decision to be made by the politicians. Since the current contracts are fair according to our model, reasons for the inadequate investment opportunities must be found outside our model. This could be related to e.g., incomplete markets, transactions costs, or stochastic interest rates.

Moreover, our model has shown that the practise of pooling the inhomogeneous customers' bonus reserves makes the company better off leaving at least one of the customers worse off. This weakens the companies' claim even further, since a 30 year contract with a minimum rate of return guarantee of 3 percent and a 0.5 percent rate of payment fee must be a favorable contract for the companies if the customers are entering or leaving at different dates.

Lately in Denmark there has been a lot of discussion whether old customers with a 5 percent minimum rate of return guarantee "cheat" new(er) customers with a minimum rate of return guarantee of 3 percent. In our model we have shown that this is, in fact, the case. More precisely, figures from scenario 5, cf. table 12 show a redistribution in the area of 10 percent of the initial deposit from new(er) customers to old customers. This last observation indicates that the companies should keep track of each customer's bonus reserve separately and not pool them, implying that they should also calculate an individual rate of payment fee for each customer.

[12] Up until spring 1999 companies have offered contracts with minimum rate of return guarantees of 3 percent. However, recently some companies have lowered the minimum rate of return guarantee offered to new customers, after having experienced difficulties finding investment opportunities with returns high enough to cover the guarantee.

It is important to remember that the findings of our model are limited by the Black-Scholes assumptions of constant short term interest rate, log-normally distributed asset returns, and complete markets. In addition, we have only considered the case of one initial deposit by the customer. Interesting extensions of our model are stochastic interest rates, annual deposits, mortality risk, hedging aspects in markets with some degree of incompleteness, and the incentive issues of premature surrender of contracts.

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