Optimal Dynamic Asset Allocation for Defined-Contribution Pension Plans[1] [2]

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Summary

We develop an optimal asset allocation model for the accumulation phase of a defined-contribution pension plan in the presence of non-hedgeable salary risk. The model considers a policyholder who each period, t, contributes a constant proportion, π, of his salary, $S(t)$, to a personal pension fund, $W(t)$. At the time of retirement, T, the fund is converted into an annuity paying $W(T)/a(T)$ each period, where $a(T)$ is the prevailing annuity rate at the time of retirement.

A general one-factor diffusion model is used for the risk-free rate of interest, and this underpins returns on n risky assets.

We focus attention on the replacement ratio achieved under a particular scheme: that is, $W(T)/(a(T)S(T))$, the pension at the time of retirement divided by the final salary at retirement. The success of any asset-allocation strategy is measured using a terminal utility function that depends on the replacement ratio, or, more generally, on $W(T)$ and the risk-free rate of interest, $r(T)$, at T. The problem considered is how should we invest in order to maximise expected terminal utility.

For the general model including non-hedgeable salary risk, we find that it is optimal to invest in a combination of three portfolios:

A: p_A : the minimum-risk portfolio measured relative to $S(t)$;

B: p_B : the minimum-risk portfolio measured relative to $S(t)a(t)$;

C: p_C : a more risky portfolio which is efficient when we measure risk and return relative to both $S(t)$ and $S(t)/a(t)$.

We show that these three portfolios do not depend upon the presence of non-hedgeable salary risk, although the precise mix of the three portfolios may do so.

We consider the power terminal utility function and find that expected future utility can only be of similar form if salary risk is entirely hedgeable using the available assets (that is, the minimum-risk portfolio A is, in fact, zero risk). In this event, we show that the proportion of the fund (counting the present value of unpaid future contributions as part of the fund's assets) invested in the risky portfolio C remains constant. The remaining assets shift gradually from portfolio A to portfolio B in an attempt to match partially the pension liability.

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1 Introduction

This paper examines optimal investment strategies for defined-contribution pension plans. We will focus on the replacement ratio as the central quantity of interest: that is, the pension at the time of retirement divided by the final salary at retirement. Related work (Blake, Cairns & Dowd, 1999) has concentrated on the evaluation of the value-at-risk for the replacement ratio with various confidence levels. Here we suppose that the policyholder has a specific terminal utility function which quantifies the value, to the policyholder, of different replacement ratios relative to one another. What, then, is the asset-allocation strategy over the accumulation phase of the plan that will maxmise the expected utility at the time of retirement?

A similar question has been posed recently by Boulier *et al.* (1999) and Deelstra *et al.* (1999). These authors considered a problem which involved:

- Vasicek and Cox-Ingersoll-Ross interest-rate models respectively;
- three assets (cash, a bond and equities);
- (ultimately) deterministic contribution rates;
- a guaranteed minimum benefit at retirement;
- terminal utility measured as power function of surplus *cash* over the guarantee.

Here we consider a problem without a guaranteed minimum benefit, but generalise the above models in other ways:

- a general one-factor interest rate model;
- cash plus n risky assets;
- random salary growth with a non-hedgeable element;
- utility as a function of the replacement ratio at retirement.

In particular, we have chosen the dependence on the replacement ratio to reflect the view that, in the normal course of events, the policyholder's personal pension fund, $W(t)$, will be converted into an annuity at retirement rather than taken in cash. In reality, the true picture may lie somewhere in between the two situations and this may depend upon national regulations. Thus policyholders may receive, at the time of retirement, a mixture of annuity and cash, with an element of discretion over the amount of cash and in the type of annuity purchased.

We find that the dependence of the terminal utility function upon the replacement ratio results in an optimal asset allocation strategy that is similar in some respects to previous work, but also qualitatively differs in other respects.

It is easiest to characterise the problem in discrete time. Time runs from $t = 0$ to $t = T$. Assume that all the dynamic processes in the model are Markov. Thus, if we wish to make forecasts about the future then it is sufficient that we know the current state of the world and not also its history. We will also consider asset-allocation strategies that are Markov. They may be stochastic or deterministic. Stochastic strategies may, for example, depend upon the current size of the policyholder's fund, his salary plus other economic variables.

For such a problem, stochastic dynamic programming provides the appropriate framework for finding an optimal solution (for example, see Whittle, 1982). This approach (based upon the Bellman principle) tells us that we can solve the optimisation problem by starting at time T and working backwards recursively. At each time, t, we consider expected utility one timestep ahead and choose the optimal asset allocation strategy that will apply from t to $t + 1$ by maximising this value. This gives us the expected utility at T starting from t. We then move back to $t - 1$ and repeat the process. In general, this is a very intensive computational problem. We therefore look for methods and results that help to reduce the amount of computing time required.

In Section 2 of the paper, we introduce a continuous-time version of the model that drives the dynamics of the personal pension fund and determines the amount of pension purchased at retirement. In Section 3 we specify the optimisation problem and derive some general results. Finally in Section 4 we consider the special case of a power utility function and derive some more specific solutions for the problem.

2 A continuous-time model for DC pension plans

In this section we develop a simple model in continuous time. This allows us to derive analytically certain results that could only be derived with difficulty in more complex models. Furthermore, numerical solutions to the problems in more complex models can be accelerated using the analytical solution derived here as a starting point.

2.1 Risk-free interest rate

We will consider relatively simple diffusion models for the risk-free interest rate, $r(t)$, which depend upon one factor or source of randomness. The stochastic differential equation for $r(t)$ is then:

$$dr(t) = \mu_r(r(t))dt + \sigma_r(r(t))dZ_1(t)$$

for suitable functions $\mu_r(r)$ and $\sigma_r(r)$ (Baxter & Rennie, 1996). $Z_1(t)$ is a standard Brownian motion under the real-world measure P.

In places we will consider the more specific Vasicek model:

$$dr(t) = \alpha_r(\mu_r - r(t))dt + \sigma_r dZ_1(t)$$

(Vasicek, 1977, or, for example, Duffie, 1996). In this case, $\mu_r(r) = \alpha_r(\mu_r - r)$ and $\sigma_r(r)^2 = \sigma_r^2$.

2.2 Asset returns

Assume we have $n+1$ assets. One unit of asset i with reinvestment of gross income has value $X_i(t)$ at time t.

Asset 0 is a risk-free account (or cash for short) satisfying:

$$dX_0(t) = r(t)X_0(t)dt$$

Assets 1 to n are risky and are governed by the following stochastic differential equations:

$$dX_i(t) = X_i(t)\left((r(t)+\lambda_i)dt + \sum_{j=1}^{n}\sigma_{ij}dZ_j(t)\right) \quad \text{for } i = 1,\ldots,n$$

where $Z(t) = (Z_1(t),\ldots,Z_n(t))'$ is standard n-dimensional Brownian motion (under the real-world probability measure P), λ_i is the risk premium on asset i and σ_{ij} is the volatility of asset i with respect to changes in $Z_j(t)$.

Let C be the matrix $(\sigma_{ij})_{i,j=1}^n$ (which we assume is non-singular) and $\lambda = (\lambda_1,\ldots,\lambda_n)'$, and define $\rho = C^{-1}\lambda$. Then ρ_j is the *market price of risk* associated with the source of risk $dZ_j(t)$: that is, ρ_j is the expected excess return per unit of risk from source j. More generally the σ_{ij} and the ρ_j could be functions of time and possibly also stochastic.

Consider the Vasicek model. Assume asset 1 is a bond account which rolls over zero-coupon bonds with τ years to maturity. The price at time t of such bonds per unit nominal is:

$$\begin{aligned} P(t,t+\tau) &= \exp(A(\tau) - B(\tau)r(t)) \qquad (2.2.1)\\ \text{where } B(\tau) &= \frac{1-e^{-\alpha_r\tau}}{\alpha_r}\\ A(\tau) &= (B(\tau)-\tau)\left(\tilde{\mu}_r - \frac{\sigma_r^2}{2\alpha_r^2}\right) - \frac{\sigma_r^2}{4\alpha_r}B(\tau)^2\\ \tilde{\mu}_r &= \mu_r + \frac{\lambda_1}{1-e^{-\alpha_r\tau}} = \mu_r - \frac{\sigma_r\rho_1}{\alpha_r} \end{aligned}$$

The volatility of asset 1 (in effect a tradeable index rather than a conventional asset) is $\sigma_{11} = -\sigma_r(1-\exp(-\alpha_r\tau))/\alpha_r$ with $\sigma_{1j} = 0$ for all $j > 1$.

It is well known that the prices of bonds rise when $r(t)$ falls. If we assume that $\sigma_r(r) < 0$ in the general model, it follows that excess expected returns on cash can only arise if ρ_1 is greater than than zero. (If $\sigma_r(r)$ was positive then ρ_1 would need to be negative in order to deliver a positive risk premium on bonds.) If we consider the Vasicek model, this means that $\tilde{\mu}_r$ (the mean reversion level under the risk-neutral measure Q) is greater than μ_r (the mean reversion level under the real-world measure P).

Where we are considering a more general model for $r(t)$ we can choose a portfolio which invests at time t in zero-coupon bonds with varying terms, τ, to maturity. The maturities chosen will depend upon $r(t)$. If the model permits, $\tau(r(t))$ can be chosen to ensure that the volatility of the bond portfolio remains constant over time. With a constant market price of risk, ρ_1, this ensures that the risk premium on (the synthesized) asset 1 remains constant.

2.3 Salaries

Let the policy holder's salary, $S(t)$, follow the simple model:

$$dS(t) = S(t)\left[(r(t)+\mu_s(t))dt + \nu_s dZ_s(t) + \sigma_s' dZ(t)\right]$$

where $\mu_s(t)$ is some deterministic function and σ_s allows us to model any links between salary growth and returns on some of the assets. In contrast, the term $\nu_s dZ_s(t)$ (where $Z_s(t)$ is a standard Brownian motion independent of $Z(t)$) allows us to incorporate non-hedgeable salary risks.

Consequently, we can write[3]:

$$S(t) = S(0)\exp\left[\int_0^t (r(s)+\mu_s(s))\,ds - \frac{1}{2}\nu_s^2 t - \frac{1}{2}\sigma_s'\sigma_s t + \nu_s Z_s(t) + \sigma_s' Z(t)\right] \tag{2.3.1}$$

Now $Z(t)$ is an n-dimensional Brownian motion under the real-world measure P. Let $\tilde{Z}_i(t) = Z_i(t) + \rho_i t$. Under the risk-neutral measure Q, where all risky assets have the same expected rate of return as cash, $\tilde{Z}(t)$ is an n-dimensional Brownian motion. Assume that $Z_s(t)$ is also a Brownian motion under Q. Then we can write:

$$S(t) = S(0)\exp\left[\int_0^t (r(s)+\mu_s(s))\,ds - \frac{1}{2}\nu_s^2 - \frac{1}{2}\sigma_s'\sigma_s t - \sigma_s'\rho t + \nu_s Z_s(t) + \sigma_s'\tilde{Z}(t)\right]$$

[3] An understanding of the solution to equation (2.3.1) can be gained from Øksendal (1998). In particular, the solution can be verified by applying Ito's formula.

$$\Rightarrow E_Q\left[\exp\left(-\int_0^t r(s)ds\right)S(t)\right] = S(0)\exp\left[\int_0^t \mu_s(s)ds - \sigma_s'\rho t\right] \tag{2.3.2}$$

We will refer back to this equation later in this section.

The inclusion of a deterministic, time-dependent adjustment to the growth rate, $\mu_s(t)$, allows us to incorporate age-dependent salary growth. For example, it is well known that salaries grow faster at younger ages, implying that $\mu_s(t)$ should be a decreasing function of time.

2.4 Pension fund

Let $W(t)$ be the value of the pension fund at time t. Then:

$$dW(t) = W(t)\left[\left(r(t) + p(t)'\lambda\right)dt + p(t)'CdZ(t)\right] + \pi S(t)dt$$

where $p(t) = (p_1(t), \ldots, p_n(t))'$ is the vector of proportions of the fund invested in each of the risky assets and π is the contribution rate (a fixed proportion of salary).

Let $Y(t) = W(t)/S(t)$. Then we can write:

$$\begin{aligned} dr(t) &= b_r(t,p,Y,r)dt + s_{rr}(t,p,Y,r)d\tilde{Z}^{(r)}(t) + s_{rv}(t,p,Y,r)d\tilde{Z}^{v}(t) \\ dY(t) &= b_y(t,p,Y,r)dt + s_{yr}(t,p,Y,r)d\tilde{Z}^{(r)}(t) + s_{yy}(t,p,Y,r)d\tilde{Z}^{v}(t) \end{aligned}$$

where $\tilde{Z}^{(r)}(t)$ and $\tilde{Z}^{(y)}(t)$ are independent and identically distributed, standard 1-dimensional Brownian motions. $\tilde{Z}^{(r)}(t) = Z_1(t)$ and $\tilde{Z}^{(y)}(t)$ depends upon $Z_2(t), \ldots, Z_n(t)$ and $Z_s(t)$,

$$\begin{aligned} b_r(t,p,y,r) &= \begin{cases} \mu_r(r) & \text{(in general)} \\ \alpha_r(\mu_r - r) & \text{(Vasicek model)} \end{cases} \\ b_y(t,p,y,r) &= \pi + \left(p'C(\rho - \sigma_s) - \mu_s(t) + \nu_s^2 + \sigma_s'\sigma_s\right)y \\ \sigma_{rr}(t,p,y,r)^2 &= \begin{cases} \sigma_r(r)^2 & \text{(in general)} \\ \sigma_r^2 & \text{(Vasicek model)} \end{cases} \\ \sigma_{rv}(t,p,y,r)^2 &= 0 \\ \sigma_{yr}(t,p,y,r)^2 &= \left(p'D_1p - 2p'C_1\sigma_{s1} + \sigma_{s1}^2\right)y^2 = (p'C_1 - \sigma_{s1})^2y^2 \\ \sigma_{yv}(t,p,y,r)^2 &= \left(\nu_s^2 + \sigma_{s2}'\sigma_{s2} + p'D_2p - 2p'\gamma_{s2}\right)y^2 \\ &= \left(\nu_s^2 + (C_2'\rho - \sigma_{s2})'(C_2'\rho - \sigma_{s2})\right)y^2 \end{aligned}$$

$$
\begin{aligned}
\text{where } C_1 &= (\sigma_{11},\ldots,\sigma_{n1})' \\
C_2 &= \begin{pmatrix} \sigma_{12} & \ldots & \sigma_{1n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \sigma_{n2} & \ldots & \sigma_{nn} \end{pmatrix} \\
\sigma_{s1} &= (\sigma_s)_1 \\
\sigma_{s2} &= ((\sigma_s)_2,\ldots,(\sigma_s)_n)' \\
D &= CC' \\
&= \text{instantaneous covariance matrix for assets} \\
D_1 &= C_1C_1' \\
D_2 &= C_2C_2' \\
\gamma_s &= C\sigma_s \\
&= \text{vector of covariances between assets and salary} \\
\gamma_{s1} &= C_1\sigma_{s1} \\
\gamma_{s2} &= C_2\sigma_{s2}, \quad \text{with } \gamma_s = \gamma_{s1} + \gamma_{s2}
\end{aligned}
$$

Let us, in addition, write $e_1 = (1,0,\ldots,0)'$. Then we have $C_1 = \hat{I}_1$ and $C_1'D^{-1}C_1 = e_1'C'D^{-1}Ce_1 = 1$.

The instantaneous covariance matrix for $(r(t), Y(t))'$ is

$$
\begin{pmatrix} a_{rr} & a_{ry} \\ a_{yr} & a_{yy} \end{pmatrix}
$$

where

$$
\begin{aligned}
a_{rr} &= \sigma_r(r)^2 \\
a_{ry} &= a_{yr} \\
&= (p'C_1 - \sigma_{s1}\sigma_r(r))y \\
a_{yy} &= (\mathrm{v}_s^2 + \sigma_s'\sigma_s + p'Dp - 2p'C\sigma_s)y^2 \\
&= \left(\mathrm{v}_s^2 + (C'\rho - \sigma_s)'(C'\rho - \sigma_s)\right)y^2
\end{aligned}
$$

2.5 Annuity purchase

We assume that at the time of retirement, T, the accumulated fund will be used to purchase an immediate level annuity at a price of $a(T, r(T))$ per unit of pension. If the retiree is aged x, then the fair value of $a(T, r(T))$ will be:

$$
a(T, r(T)) = \int_0^{\infty} P(T, T+s)\, {}_sp_x ds
$$

where ${}_sp_x$ is the survival function (under measure Q) of the retiree given survival to age x. (Thus $a(t, r(t))$ is a function of $r(t)$ only.)

As a proxy for this we will, in some cases (see Section 4), approximate $a(T, r(T))$ by:

$$e^{a-br(T)} \tag{2.5.1}$$

where the constants a and b are chosen to ensure the best approximation to $a(T, r(T))$ over the central range of $r(T)$ (say, the mean of $r(T)$ plus or minus two standard deviations). The constants may be determined more precisely by, for example, minimising $E\left[\left(a(T, r(T)) - e^{a-br(T)}\right)^2\right]$ over a and b. This is similar to duration matching although here it is done in a less precise manner (that is, we find a constant match which is approximately correct most of the time rather than a varying match which is correct all of the time). Affine term-structure models (for example, the Vasicek and Cox-Ingersoll-Ross models) have zero-coupon prices which are equal to $\exp[A(d) - B(d)r(T)]$ for some functions $A(d)$ and $B(d)$ (for example, see equation 2.2.1). This means that the approximation (2.5.1) is equivalent to having a constant holding in zero-coupon bonds with a particular fixed term d to maturity: d is chosen to ensure that $B(d) = b$.

3 Terminal utility and optimal asset allocation

3.1 Terminal utility

Suppose that we have some terminal utility $K(Y(T), r(T))$. Let:

$$\begin{aligned} J(t, y, r)(p) &= E\left[K(Y(T), r(T)) \mid Y(t) = y, r(t) = r, p(s, y(s), r(s))\right] \\ \text{and let } \phi(t, y, r) &= \sup_p J(t, y, r)(p) \end{aligned}$$

be the optimal value function.

3.2 Optimal asset allocation

The Hamilton-Jacobi-Bellman equation (hereafter referred to as the Bellman equation) for this problem (for example, see Fleming & Rishel, 1975, Merton, 1992, Øksendal, 1998) is:

$$\sup_p \left[\phi_t + b_y\phi_y + b_r\phi_r + \frac{1}{2}a_{yy}\phi_{yy} + a_{yr}\phi_{yr} + \frac{1}{2}a_{rr}\phi_{rr}\right] = 0$$

where $b_y = b_y(t, y, r, p)$ *etc.* and ϕ_t, ϕ_{yr} are first and second partial derivatives of $\phi(t, y, r)$ with respect to t and to y and r, respectively *etc.*

Thus we have:

$$\sup_p \Bigg\{ \phi_t + \Big(\pi + \Big[p'C(\rho - \sigma_s) - \tilde{\mu}_s(t)\Big] y\Big) \phi_y + \mu_r(r)\phi_r + \frac{1}{2}\Big(\nu_s^2 + \sigma_s'\sigma_s + p'Dp - 2p'C\sigma_s\Big) y^2 \phi_{yy} + (p'C_1 - \sigma_{s1})\sigma_r(r) y \phi_{yr} + \frac{1}{2}\sigma_r(r)^2 \phi_{rr} \Bigg\} = 0 \tag{3.2.1}$$

where $\tilde{\mu}_s(t) = \mu_s(t) - \nu_s^2 - \sigma_s'\sigma_s$.

Proposition 3.2.1

The optimal asset allocation strategy takes the form:

$$\begin{aligned} p^*(t,y,r) &= D^{-1}\left(C\sigma_s - C(\rho - \sigma_s)\frac{\phi_y}{y\phi_{yy}} - C_1\sigma_r(r)\frac{\phi_{yr}}{y\phi_{yy}}\right) \\ &= C'^{-1}\left(\sigma_s - (\rho - \sigma_s)\frac{\phi_y}{y\phi_{yy}} - e_1\sigma_r(r)\frac{\phi_{yr}}{y\phi_{yy}}\right) \end{aligned} \tag{3.2.2}$$

Proof

We differentiate the expression $\{\cdot\}$ in (3.2.1) with respect to p and equate to zero. Thus:

$$yC(\rho - \sigma_s)\phi_y + (Dp - C\sigma_s)y^2\phi_{yy} + Ce_1\sigma_r(r)y\phi_{yr} = 0$$

Solving for p the result follows.

□

Let us consider the interpretation of the three portfolios, $C'^{-1}\sigma_s$, $C'^{-1}(\rho - \sigma_s)$ and $C'^{-1}e_1\sigma_r(r)$, in equation (3.2.2).

Suppose that $\pi = 0$. The instantaneous variance of $dY(t)/Y(t)$ is $v(p) = \nu_s^2 + \sigma_s'\sigma_s + p'Dp - 2p'C\sigma_s$ and the expected rate of growth is $m(p) = p'C(\rho - \sigma_s) - \tilde{\mu}_s(t)$.

First, let us minimise $v(p)$ over p.

$$\begin{aligned} \Rightarrow 2Dp - 2C\sigma_s &= 0 \\ \Rightarrow p^*(0) &= D^{-1}C\sigma_s = C'^{-1}\sigma_s = p_A \\ \Rightarrow m(p_A) &= m_A \\ &= \sigma_s'C^{-1}D^{-1}C'^{-1}(\rho - \sigma_s) - \tilde{\mu}_s(t) \\ &= \sigma_s'(\rho - \sigma_s) - \tilde{\mu}_s(t) \end{aligned}$$

Now minimise $v(p)$ over p subject to $m(p) = m$.

Let $L(p, \psi) = v(p) + 2\theta(m(p) - m)$.

$$\begin{aligned}
\frac{\partial L}{\partial p} &= 2Dp - 2C\sigma_s + 2\psi C(\rho - \sigma_s) \\
&= 0 \\
\Rightarrow p^*(m - m_A) &= C'^{-1}[\sigma_s - \psi(\rho - \sigma_s)] \\
\frac{\partial L}{\partial \psi} &= 2(m(p) - m) \\
&= 0 \\
\Rightarrow [C\sigma_s - \psi C(\rho - \sigma_s)]' D^{-1} C(\rho - \sigma_s) - \tilde{\mu}_s(t) - m &= 0 \\
\Rightarrow \psi &= \psi(m) \\
&= \frac{m_A - m}{(\rho - \sigma_s)'(\rho - \sigma_s)}
\end{aligned}$$

The important point to note is that the optimal portfolio $p^*(m - m_A)$ is a weighted average of the minimum-variance portfolio $p_A = C'^{-1}\sigma_s$ (with weight $1 + \psi$) and another efficient portfolio $p_C = C'^{-1}\rho$ (with weight $-\psi$).

Now consider $V(t) = Y(t)/a(t, r(t))$ where $a(t, r(t))$ is the price at t of an immediate level annuity. Since $a(t, r(t))$ is a function of $r(t)$ only, we can write $da(t, r(t))$ as $a(t, r(t))\left[-d_a(r)dr(t) + \frac{1}{2}c_a(r)(dr(t))^2\right]$, where $d_a(r)$ is the duration of the annuity function:

$$\begin{aligned}
d_a(r) &= -\frac{1}{a(t,r)}\frac{\partial a(t,r)}{\partial r} \\
\text{and } c_a(r) &= \frac{1}{a(t,r)}\frac{\partial^2 a(t,r)}{\partial r^2}
\end{aligned}$$

Then:

$$\begin{aligned}
dV(t) &= V(t)\Big[\left(p'C(\rho - \sigma_s) - \tilde{\mu}_s(t)\right)dt - \nu_s dZ_s(t) + (p'C - \sigma_s')dZ(t) \\
&\quad + d_a(r)\left(\mu_r(r)dt + \sigma_r(r)dZ_1(t)\right) \\
&\quad + \left(d_a(r)^2 - \frac{1}{2}c_a(r)\right)\sigma_r(r)^2 dt + d_a(r)\sigma_r(r)\left(p'Ce_1 - \sigma_{s1}\right)dt\Big] \\
&\stackrel{\mathcal{D}}{=} V(t)\left(m(p,r)dt + \sqrt{v(p,r)}d\tilde{Z}\right)
\end{aligned}$$

where $\tilde{Z}(t)$ is a Brownian motion, $\stackrel{\mathcal{D}}{=}$ means '*equivalent in distribution*',

$$\begin{aligned} m(p,r) &= p'C(\rho-\sigma_s)-\tilde{\mu}_s(t)+d_a(r)\mu_r(r)+d_a(r)(p'Ce_1-\sigma_{s1})\sigma_r(r) \\ &\quad +\left(d_a(r)^2-\frac{1}{2}c_a(r)\right)\sigma_r(r)^2 \end{aligned}$$

$$\begin{aligned} \text{and } v(p,r) &= \nu_s^2+\sigma_s'\sigma_s+p'Dp-2p'C\sigma_s+2d_a(r)p'Ce_1\sigma_r(r) \\ &\quad -2d_a(r)\sigma_{s1}\sigma_r(r)+d_a(r)^2\sigma_r(r)^2 \end{aligned}$$

First, minimise $v(p,r)$ over p:

$$\Rightarrow \quad 2Dp-2C\sigma_s+2d_a(r)\sigma_r(r)Ce_1=0$$

$$\begin{aligned} \Rightarrow p &= D^{-1}(C\sigma_s-d_a(r)\sigma_r(r)Ce_1) \\ &= C'^{-1}\left(\sigma_s-d_a(r)\sigma_r(r)e_1\right) \\ &= p_B \end{aligned}$$

Note that $p_B=p_A$ if $d_a(r)=0$ (that is, if we are funding for cash). Let $m_B=m(p_B,r)$. Next, minimise $v(p,r)$ over p subject to $m(p,r)=m$. Let $p_B(m-m_B)$ be the optimal p for this problem. Let $L(p,\psi)=v(p,r)+2\psi(m(p,r)-m)$.

Then:

$$\begin{aligned} \frac{\partial L}{\partial p} &= 2Dp-2C\sigma_s+2d_a(r)\sigma_r(r)Ce_1+2\psi C(\rho-\sigma_s+d_a(r)\sigma_r(r)Ce_1)=0 \\ \Rightarrow p &= C'^{-1}(\sigma_s-d_a(r)\sigma_r(r)e_1-\psi(\rho-\sigma_s+d_a(r)\sigma_r(r)e_1)) \\ &= (1+\psi)C'^{-1}(\sigma_s-d_a(r)\sigma_r(r)e_1)-\psi C'^{-1}\rho \\ &= (1+\psi)p_B-\theta p_C \\ \frac{\partial L}{\partial \psi} &= 0 \\ \Rightarrow \psi &= \frac{m_B-m}{(\rho-\sigma_s+d_a(r)\sigma_r(r)e_1)'(\rho-\sigma_s+d_a(r)\sigma_r(r)e_1)} \end{aligned}$$

As before we see that the optimal asset allocation strategy, $p_B(m_B-m,r)$ is a weighted average of the minimum variance portfolio p_B and the more risky, but efficient, portfolio p_C.

Let us refer back to equation (3.2.2) which gave the optimal asset allocation strategy for the dynamic optimisation problem:

$$p(t,y,r)=C'^{-1}\left(\sigma_s-(\rho-\sigma_s)\frac{\phi_y}{y\phi_{yy}}-e_1\sigma_r(r)\frac{\phi_{yr}}{y\phi_{yy}}\right)$$

We now see that this can be written in the form:

$$
\begin{aligned}
p(t,y,r) &= \theta_A p_A + \theta_B p_B + \theta_C p_C \\
\text{where } \theta_A &= \theta_A(t,y,r) \\
&= 1 - \frac{\phi_{yr} - d_a(r)\phi_y}{d_a(r) y \phi_{yy}} \\
\theta_B &= \theta_B(t,y,r) \\
&= \frac{\phi_{yr}}{d_a(r) y \phi_{yy}} \\
\theta_C &= \theta_C(t,y,r) \\
&= 1 - \theta_A(t,y,r) - \theta_B(t,y,r) \\
&= -\frac{\phi_y}{y\phi_{yy}} \\
p_A &= C'^{-1}\sigma_s \\
p_B &= C'^{-1}\left(\sigma_s - d_a(r)\sigma_r(r)e_1\right) \\
\text{and } p_C &= C'^{-1}\rho
\end{aligned}
\tag{3.2.3}
$$

The three portfolios p_A, p_B and p_C correspond to the three portfolios described below:

A: p_A (with $p_{A0} = 1 - \sum_{i=1}^{n} p_{Ai}$), the minimum-risk portfolio measured relative to $S(t)$;

B: p_B (with $p_{B0} = 1 - \sum_{i=1}^{n} p_{Bi}$), the minimum-risk portfolio measured relative to $S(t)a(t)$;

C: p_C (with $p_{C0} = 1 - \sum_{i=1}^{n} p_{Ci}$), a more risky portfolio which is efficient when we measure risk and return relative to both $S(t)$ and $S(t)/a(t)$.

Within each of the three portfolios the proportion invested in each of the $n+1$ assets remains constant over time. However, the proportion of the fund as a whole invested in each of the three portfolios (that is, $(\theta_A(t,y,r), \theta_B(t,y,r), \theta_C(t,y,r))$) does vary over time and depending upon $r(t)$ according to the equations (3.2.3).

Remark 3.2.2

We can see that the term θ_C is equal to the reciprocal of the degree of relative risk aversion.

As a consequence we note two points. First, since relative risk aversion is positive (but possibly dependent upon t and y) the investment in portfolio C is necessarily positive. Second, if relative risk aversion is constant, it will be optimal to invest a constant proportion in the risky portfolio $C'^{-1}\rho$ over time.

Remark 3.2.3

Note that the three portfolios $p_A = C'^{-1}\sigma_s$, $p_B = C'^{-1}(\sigma_s - d_a(r)\sigma_r(r)e_1)$ *and* $p_C = C'^{-1}\rho$, *do not depend upon the level of non-hedgeable salary risk,* ν_s. *However, the precise mix may depend upon* ν_s *through its effect on* $\phi(t,r)$.

Section 4 below develops this remark further.

Corollary 3.2.4

Suppose that $\phi(T,y,r) = K(y/a(T,r))$: *that is, the terminal utility is a function of the pension as a proportion of final salary (replacement ratio) achieved at time* T.

Then $\theta_A(T,y,r) = 0$ *for all* y, r.

Proof:

At $t = T$, we find that:

$$\begin{aligned}
\phi_y &= \frac{1}{a(T,r)}K'(y/a(T,r)) \\
\phi_{yr} &= \frac{d_a(r)}{a(T,r)}K'(y/a(T,r)) + \frac{yd_a(r)}{a(T,r)^2}K''(y/a(T,r)) \\
\phi_{yy} &= \frac{1}{a(T,r)^2}K''(y/a(T,r))
\end{aligned}$$

It is then straightforward to confirm that $\theta_A = 0$.

□

This means that, as we approach retirement, we reduce to zero the proportion of the personal pension fund invested in portfolio A. In general, though, before the retirement date part of the fund will be invested in portfolio A. The exception to this result occurs if we are funding for cash. Then $d_a(r) = 0$ for all r and portfolios A and B are identical.

Conjecture 3.2.5

As $T - t$ *tends to infinity* $\theta_B(t,y,r)$ *tends to zero.*

We make this conjecture on the following basis. The further we are from retirement the less able are we to predict what interest rates will be at the time of retirement. This means that ϕ_r and ϕ_{yr} are likely to tend to 0 as $T - t$ increases. This result is illustrated in the case of the Vasicek model developed in Section 4.3.

Let us return now to the solution of the Bellman equation (3.2.1). Inserting the solution

for p^* given in equation (3.2.2) we get:

$$\begin{aligned}
\phi_t + (\pi - \tilde{\mu}_s(t)y)\phi_y + (\rho - \sigma_s)' \left(\sigma_s - (\rho - \sigma_s)\frac{\phi_y}{y\phi_{yy}} - e_1\sigma_r(r)\frac{\phi_{yr}}{y\phi_{yy}} \right) y\phi_y \\
+\mu_r(r)\phi_r + \frac{1}{2}(\sigma_s^2 + \sigma_s'\sigma_s)y^2\phi_{yy} \\
+\frac{1}{2}\left(\sigma_s - (\rho - \sigma_s)\frac{\phi_y}{y\phi_{yy}} - e_1\sigma_r(r)\frac{\phi_{yr}}{y\phi_{yy}} \right)' \left(\sigma_s - (\rho - \sigma_s)\frac{\phi_y}{y\phi_{yy}} - e_1\sigma_r(r)\frac{\phi_{yr}}{y\phi_{yy}} \right) y^2\phi_{yy} \\
-\sigma_s' \left(\sigma_s - (\rho - \sigma_s)\frac{\phi_y}{y\phi_{yy}} - e_1\sigma_r(r)\frac{\phi_{yr}}{y\phi_{yy}} \right) y^2\phi_{yy} \\
+\left(\sigma_s - (\rho - \sigma_s)\frac{\phi_y}{y\phi_{yy}} - e_1\sigma_r(r)\frac{\phi_{yr}}{y\phi_{yy}} \right)' e_1\sigma_r(r)y\phi_{yr} + \frac{1}{2}\sigma_r(r)^2\phi_{rr} = 0
\end{aligned} \tag{3.2.4}$$

Corollary 3.2.6

The optimal value function is fully determined by ρ, σ_s, π, $\tilde{\mu}_s(t)$, $\mu_r(r)$, $\sigma_r(r)$ *and* $K(y, r)$.

Remark 3.2.7

In particular, this means that once the market price of risk, ρ, *has been specified, the optimal value function,* ϕ, *is not affected by the choice of assets used (provided* C *is non-singular).*

For example, there is no advantage to using bonds of one duration over another. Similarly, there would be no advantage to investment in equity derivatives over the underlying stocks. However, consider equation (3.2.2) for

$$\begin{aligned}
p^*(t, y, r) &= \theta_A p_A + \theta_B p_B + \theta_C p_C \\
&= C'^{-1}(\theta_A\sigma_s + \theta_B(\sigma_s - d_a(r)\sigma_r(r)e_1 + \theta_C\rho)
\end{aligned}$$

Now θ_A, θ_B and θ_C are functions only of ϕ and its various partial derivatives and of the form of the annuity function $a(T, r(T))$. It follows that $p^*(t, y, r)$ will depend upon the individual assets available through the volatility matrix C (since the dynamics of these assets are governed by $r(t)$, ρ and C only).

Let us now introduce an arbitrary deterministic function $\varepsilon(t)$ which will be defined later. For the sake of brevity, in the equations which follow, we will take ε to mean $\varepsilon(t)$, $\tilde{\mu}_s$ to mean $\tilde{\mu}_s(t)$ and so on wherever this is convenient to do so. Expanding equation (3.2.4)

and dividing this by $(y+\varepsilon)^2\phi_{yy}$ gives:

$$
\begin{aligned}
&\left\{\frac{\phi_t}{(y+\varepsilon)^2\phi_{yy}}\right\}+\left(\frac{\pi+\varepsilon\tilde{\mu}_s}{y+\varepsilon}\right)\left\{\frac{\phi_y}{(y+\varepsilon)\phi_{yy}}\right\}\\
&-\tilde{\mu}_s\left\{\frac{\phi_y}{(y+\varepsilon)\phi_{yy}}\right\}-\frac{1}{2}(\rho-\sigma_s)'(\rho-\sigma_s)\left\{\frac{\phi_y}{(y+\varepsilon)\phi_{yy}}\right\}^2\\
&+\frac{1}{2}\nu_s^2\left(1-\frac{2\varepsilon}{y+\varepsilon}+\frac{\varepsilon^2}{(y+\varepsilon)^2}\right)\left\{1\right\}+\mu_r(r)\left\{\frac{\phi_r}{(y+\varepsilon)^2\phi_{yy}}\right\}\\
&+\sigma_s'(\rho-\sigma_s)\left(1-\frac{\varepsilon}{y+\varepsilon}\right)\left\{\frac{\phi_y}{(y+\varepsilon)\phi_{yy}}\right\}\\
&+\sigma_s'e_1\sigma_r(r)\left(1-\frac{\varepsilon}{y+\varepsilon}\right)\left\{\frac{\phi_{yr}}{(y+\varepsilon)\phi_{yy}}\right\}\\
&-(\rho-\sigma_s)'e_1\sigma_r(r)\left\{\frac{\phi_y}{(y+\varepsilon)\phi_{yy}}\right\}\left\{\frac{\phi_{yr}}{(y+\varepsilon)\phi_{yy}}\right\}\\
&-\frac{1}{2}\sigma_r(r)^2\left\{\frac{\phi_{yr}}{(y+\varepsilon)\phi_{yy}}\right\}^2+\frac{1}{2}\sigma_r(r)^2\left\{\frac{\phi_{rr}}{(y+\varepsilon)^2\phi_{yy}}\right\}= \quad 0
\end{aligned}
\tag{3.2.5}
$$

4 Special case: power utility

Let us consider now the special case of the power utility function (which assumes constant relative risk aversion):

$$\phi(T,y,r)=\frac{1}{\beta}y^{\beta}e^{\gamma r}$$

for $\beta,\gamma<0$. If we assume that the annuity price $a(r)$ can be well approximated by ke^{-dr} for some constant $d>0$, then it is natural to take $\gamma=\beta d$.

4.1 A possible solution

Let us investigate the possibility of a solution of the form:

$$\phi(t,y,r) \quad = \quad \delta(t)\,(y+\varepsilon(t))^{\beta(t)}\,e^{\gamma(t)r} \tag{4.1.1}$$

for some deterministic functions $\delta(t)$, $\varepsilon(t)$, $\beta(t)$ and $\gamma(t)$.

In the following equations we use the abbreviated forms ε for $\varepsilon(t)$, ε' for $\partial\varepsilon(t)/\partial t$ and so on, where appropriate, for compactness. Differentiating $\phi(t, y, r)$ we get:

$$
\begin{aligned}
\phi_y &= \delta\beta(y+\varepsilon)^{\beta-1}e^{\gamma r} \\
\phi_{yy} &= \delta\beta(\beta-1)(y+\varepsilon)^{\beta-2}e^{\gamma r} \\
\phi_r &= \delta\gamma(y+\varepsilon)^{\beta}e^{\gamma r} \\
\phi_{yr} &= \delta\beta\gamma(y+\varepsilon)^{\beta-1}e^{\gamma r} \\
\phi_{rr} &= \delta\gamma^2(y+\varepsilon)^{\beta}e^{\gamma r} \\
\phi_t &= \delta'(y+\varepsilon)^{\beta}e^{\gamma r} + \delta\gamma' r(y+\varepsilon)^{\beta}e^{\gamma r} + \delta\beta\varepsilon'(y+\varepsilon)^{\beta-1}e^{\gamma r} \\
&\qquad + \beta'\log(y+\varepsilon)\delta(y+\varepsilon)^{\beta}e^{\gamma r} \\
\Rightarrow \frac{\phi_y}{(y+\varepsilon)\phi_{yy}} &= \frac{1}{\beta-1} \\
\frac{\phi_{yr}}{(y+\varepsilon)\phi_{yy}} &= \frac{\gamma}{\beta-1} \\
\frac{\phi_r}{(y+\varepsilon)^2\phi_{yy}} &= \frac{\gamma}{\beta(\beta-1)} \\
\frac{\phi_{rr}}{(y+\varepsilon)^2\phi_{yy}} &= \frac{\gamma^2}{\beta(\beta-1)} \\
\frac{\phi_t}{(y+\varepsilon)^2\phi_{yy}} &= \frac{1}{\delta\beta(\beta-1)}\left(\delta' + \delta\gamma' r + \frac{\delta\beta\varepsilon'}{y+\varepsilon} + \delta\beta'\log(y+\varepsilon)\right)
\end{aligned}
$$

We insert these expressions into equation (3.2.5) and equate the resulting terms in $\log(y+\varepsilon)$, $1/(y+\varepsilon)$ and so on to zero.

Equating terms in $\log(y+\varepsilon)$ to zero:

$$
\begin{aligned}
\Rightarrow \beta'(t) &= 0 \\
\Rightarrow \beta(t) &\equiv \beta \text{ for all } t
\end{aligned}
$$

Equating terms in $1/(y+\varepsilon)^2$ to zero:

$$
\begin{aligned}
\Rightarrow \frac{1}{2}\nu_s^2 &= 0 && (4.1.2) \\
\text{or } \varepsilon(t) &= 0 \text{ for all } t && (4.1.3)
\end{aligned}
$$

Equating terms in $1/(y+\varepsilon)$ to zero:

$$
\Rightarrow \varepsilon'(t) + \pi + \tilde{\mu}_s(t)\varepsilon(t) - \sigma_s'(\rho-\sigma_s)\varepsilon(t) - \sigma_s' e_1\sigma_r(r)\gamma(t)\varepsilon(t) = 0 \text{ for all } t, r \qquad (4.1.4)
$$

Equating terms independent of y to zero:

$$\begin{aligned}\Rightarrow \frac{\delta'(t)}{\delta(t)} + \gamma'(t)r - \beta\tilde{\mu}_s(t) - \frac{1}{2}(\rho-\sigma_s)'(\rho-\sigma_s)\frac{\beta}{\beta-1} + \frac{1}{2}\nu_s^2 \\ +\mu_r(r)\gamma(t) + \beta\sigma_s'(\rho-\sigma_s) + \beta\sigma_s' e_1\sigma_r(r)\gamma(t) \\ -(\rho-\sigma_s)' e_1\sigma_r(r)\gamma(t)\frac{\beta}{\beta-1} - \frac{1}{2}\sigma_r(r)^2\gamma(t)^2\frac{1}{\beta-1} &= 0\end{aligned} \tag{4.1.5}$$

As with equation (4.1.4), equation (4.1.5) must hold for all t and r.

We see from equation (4.1.4) that the proposed solution $\varepsilon(t) \equiv 0$ in equation (4.1.3) can only be achieved if $\pi = 0$. Since we assume $\pi > 0$, a solution compatible with power utility requires salary risk to be perfectly hedgeable, that is, $\nu_s = 0$.

The requirement that $\pi = 0$ or $\nu_s = 0$ is apparently quite restrictive. In reality ν_s will be non-zero. However, typically it will be quite small meaning that the solution which may have to be found using numerical methods ought to be similar in form to the case where $\nu_s = 0$ (the exception being close to the boundary $y = -\varepsilon(t)$). We have not attempted here to consider how accurate this approximation might be.

We now present two examples where equations (4.1.4) and (4.1.5) can be satisfied for certain functions $\gamma(t)$, $\varepsilon(t)$ and $\delta(t)$.

4.2 Funding for cash

Assume that $\gamma(t) \equiv 0$ and $\nu_s = 0$ for all t: that is, we are funding for cash as a proportion of final salary at time T rather than an annuity, and salary risk is hedgeable.

Then equations (4.1.4) and (4.1.5) become:

$$\varepsilon'(t) + \pi + \tilde{\mu}_s(t)\varepsilon(t) - \sigma_s'(\rho-\sigma_s)\varepsilon(t) = 0 \tag{4.2.1}$$

$$\frac{\delta'(t)}{\delta(t)} - \beta\tilde{\mu}_s(t) - \frac{1}{2}(\rho-\sigma_s)'(\rho-\sigma_s)\frac{\beta}{\beta-1} + \beta\sigma_s'(\rho-\sigma_s) = 0 \tag{4.2.2}$$

with boundary conditions:

$$\varepsilon(T) = 0 \tag{4.2.3}$$

$$\delta(T) = 1/\beta \tag{4.2.4}$$

Define

$$\begin{aligned} M_s(t) &= \int_t^T \tilde{\mu}_s(u)du = \int_t^T \mu_s(u)du - \sigma_s'\sigma_s(T-t) \\ \Delta_0 &= -\frac{1}{2}(\rho-\sigma_s)'(\rho-\sigma_s)\beta/(\beta-1) + \beta\sigma_s'(\rho-\sigma_s) \\ \text{and } \Delta_1 &= \sigma_s'(\rho-\sigma_s)\end{aligned}$$

The solutions to equations (4.2.1) to (4.2.4) are then:

$$\varepsilon(t) = \pi e^{M_s(t)-(T-t)\Delta_1} \int_t^T e^{-M_s(\tau)+(T-\tau)\Delta_1} d\tau \tag{4.2.5}$$

$$\delta(t) = \frac{1}{\beta} e^{-\beta M_s(t)+(T-t)\Delta_0} \tag{4.2.6}$$

Referring back to equation (2.3.2) and recalling that $\tilde{\mu}_s(t) = \mu_s(t) - \nu_s^2 - \sigma_s'\sigma_s = \mu_s(t) - \sigma_s'\sigma_s$, it is straightforward to see that:

$$\varepsilon(t) = \pi \int_t^T E_Q \left[\exp\left(-\int_t^\tau r(u)du \right) \frac{S(\tau)}{S(t)} \,\middle|\, \mathcal{F}_t \right] d\tau \tag{4.2.7}$$

where Q is the risk-neutral measure. Thus, $\varepsilon(t)$ is equal to the economic value (or present value) of the future contributions as a proportion of today's salary[4]. Economic value here has a precise meaning, since we asume that the market is complete and that future salaries are hedgeable (that is, $\nu_s = 0$ and Q is unique).

We note the following:

- The optimal value function is independent of the form of the model for the risk-free rate of interest: that is, of $\mu_r(r)$ and $\sigma_r(r)$. (This, though, depends upon the particular model for $S(t)$ used here.)
- The optimal allocation is:

$$\begin{aligned} p^*(t,y,r).y &= C'^{-1}(\sigma_s y - (\rho - \sigma_s)(y + \varepsilon(t))/(\beta - 1)) \\ &= -\varepsilon(t)C'^{-1}\sigma_s + \Big(y + \varepsilon(t)\Big)C'^{-1}\left[\frac{\rho - \beta\sigma_s}{1-\beta}\right] \\ &= -\varepsilon(t)p_A + \Big(y + \varepsilon(t)\Big)\left[\frac{1}{1-\beta}p_C + \frac{(-\beta)}{1-\beta}p_A\right] \end{aligned} \tag{4.2.8}$$

  Thus we undertake the following investment strategy:

- First, we short sell portfolio A up to the value $\varepsilon(t)$, as indicated in equation (4.2.8). This part of the fund therefore has value $-\varepsilon(t)$ at time t.
- This leaves us with an amount equal to $y + \varepsilon(t)$ for investment (which we will call the *surplus*). Again by reference to equation (4.2.8) we see that a proportion $1/(1-\beta)$ of this surplus is invested in the risky portfolio C, while the remainder (a proportion $(-\beta)/(1-\beta)$) is invested in portfolio A.

  Note that the proportions of the surplus invested in each portfolio do not depend upon the amount of surplus, nor do they vary over time or depend upon $r(t)$.

[4]This is related to some results by Karatzas, Lehoczky & Shreve (1987), Boulier *et al.* (1999) and Deelstra *et al.* (1999).

The third portfolio p_B plays no part in this strategy as we are funding for cash rather than an annuity.

This optimal asset allocation strategy generalises, in some respects, that derived by Deelstra *et al.* (1999), who considered a specific model for $r(t)$, the Cox, Ingersoll & Ross (1985) model (that is, $\mu_r(r) = \alpha_r(\mu_r - r)$ and $\sigma_r(r) = \sigma_r\sqrt{r}$). With three assets (cash, bond and stock), they find that the proportion invested in stock remains constant over time, while the proportions invested in the bond and cash vary over time. This result is not directly comparable with the present result for two reasons. First, Deelstra *et al.* (1999) use a bond with a fixed maturity date whereas here asset 1 is a bond index with constant volatility. Second, they consider a different (deterministic) model for salary growth.

The combination here of the form of the salary model, the asset model and with funding for cash means that the problem essentially reverts to that examined in Merton (1990).

4.3 Special case: Vasicek model for $r(t)$

Let us now assume that we are using the Vasicek model for interest rates and that $\gamma(T) = \gamma < 0$. Thus $\mu_r(r) = \alpha_r(\mu_r - r)$ and $\sigma_r(r) = \sigma_r$. We also assume that $v_s^2 = 0$ as in equation (4.1.2).

Let us now solve equations (4.1.4) and (4.1.5). Equation (4.1.5) becomes:

$$\begin{aligned}
\frac{\delta'(t)}{\delta(t)} - \beta\tilde{\mu}_s(t) - \frac{1}{2}(\rho - \sigma_s)'(\rho - \sigma_s)\frac{\beta}{\beta - 1} \\
+\alpha_r\mu_r\gamma(t) + \beta\sigma_s'(\rho - \sigma_s) + \beta\sigma_s' e_1\sigma_r\gamma(t) \\
-(\rho - \sigma_s)' e_1\sigma_r\frac{\beta}{\beta - 1}\gamma(t) - \frac{1}{2}\sigma_r^2\frac{1}{\beta - 1}\gamma(t)^2 \\
+\left[\gamma'(t) - \alpha_r\gamma(t)\right] r &= 0
\end{aligned} \tag{4.3.1}$$

Equating terms in r to zero:

$$\begin{aligned}
\gamma'(t) - \alpha_r\gamma(t) &= 0 \\
\Rightarrow \gamma(t) &= \gamma e^{-\alpha_r(T-t)}
\end{aligned}$$

Equating the remaining terms to zero:

$$\begin{aligned}
0 &= \frac{\delta'(t)}{\delta(t)} - \beta\tilde{\mu}_s(t) - \beta\sigma_s'\sigma_s + \xi_0 + \xi_1 e^{-\alpha_r(T-t)} + \xi_2 e^{-2\alpha_r(T-t)} \\
&= \frac{\delta'(t)}{\delta(t)} - \beta\mu_s(t) + \xi_0 + \xi_1 e^{-\alpha_r(T-t)} + \xi_2 e^{-2\alpha_r(T-t)} \\
\text{where } \xi_0 &= -\frac{1}{2}(\rho-\sigma_s)'(\rho-\sigma_s)\frac{\beta}{\beta-1} + \beta\sigma_s'\rho \\
\xi_1 &= \alpha_r\mu_r\gamma + \sigma_s e_1\sigma_r\gamma\frac{\beta^2}{\beta-1} - \rho e_1\sigma_r\gamma\frac{\beta}{\beta-1} \\
\xi_2 &= -\frac{1}{2}\frac{\sigma_r^2\gamma^2}{(\beta-1)}
\end{aligned}$$

Let $K_s(t) = \int_t^T \left(\beta\mu_s(u) + \xi_0 + \xi_1 e^{-\alpha_r(T-u)} + \xi_2 e^{-2\alpha_r(T-u)}\right) du$. Combining this with $\delta(T) = 1/\beta$ gives us the solution:

$$\delta(t) = \frac{1}{\beta} e^{K_s(t)}$$

Now let us consider equation (4.1.4). With the known solution for $\gamma(t)$ this gives us:

$$\varepsilon'(t) + \pi + \varepsilon(t)\left(\tilde{\mu}_s(t) - \sigma_s'(\rho-\sigma_s) - \sigma_s' e_1\sigma_r\gamma e^{-\alpha_r(T-t)}\right) = 0$$

Define

$$\begin{aligned}
N_s(t) &= \int_t^T \tilde{\mu}_s(u)du - \sigma_s'(\rho-\sigma_s)(T-t) - \gamma\sigma_s' e_1\sigma_r\left(1-e^{-\alpha_r(T-t)}\right)/\alpha_r \\
&= \int_t^T \mu_s(u)du - \sigma_s'\rho - \gamma\sigma_s' e_1\sigma_r\left(1-e^{-\alpha_r(T-t)}\right)/\alpha_r \\
\text{Then } \varepsilon(t) &= \pi e^{N_s(t)}\int_t^T e^{-N_s(\tau)}d\tau
\end{aligned}$$

As in equation (4.2.7) we find that this can also be written as:

$$\begin{aligned}
\varepsilon(t) = \quad & \pi\int_t^T \exp\left\{-\frac{\gamma\sigma_s' e_1\sigma_r}{\alpha_r}e^{-\alpha_r(T-\tau)}\left(1-e^{-\alpha_r(\tau-t)}\right)\right\} \\
& \times E_Q\left[\exp\left(-\int_t^\tau r(u)du\right)\frac{S(\tau)}{S(t)}\,\middle|\,\mathcal{F}_t\right]d\tau
\end{aligned}$$

If $\gamma < 0$ and $\sigma_s' e_1 > 0$ then $\varepsilon(t)$ is greater than the expression in equation (4.2.7). The role of the term $\exp\left\{-\gamma\sigma_s' e_1\sigma_r e^{-\alpha_r(T-\tau)}\left(1-e^{-\alpha_r(\tau-t)}\right)/\alpha_r\right\}$ is not entirely clear.

Previously we considered the case where $\gamma = 0$. It followed that $\varepsilon(t)$ could be interpreted as the market value of the future contributions. If $\gamma \neq 0$ we lose this simple interpretation.

Recall that:

$$\begin{aligned}
p(t,y,r) &= \theta_B C'^{-1}(\sigma_s - d_a(r)\sigma_r e_1) + \theta_A C'^{-1}\sigma_s + (1-\theta_B-\theta_A)C'^{-1}\rho \\
\text{where } \theta_B &= \theta_B(t,y,r) \\
&= \frac{\phi_{yr}}{d_a(r)y\phi_{yy}} \\
\text{and } \theta_A &= \theta_A(t,y,r) \\
&= 1 - \frac{\phi_{yr} - d_a(r)\phi_y}{d_a(r)y\phi_{yy}}
\end{aligned}$$

If we assume that $d_a(r)$ is constant and equal to γ/β, then we have:

$$\begin{aligned}
\theta_B &= \frac{\beta\gamma(t)}{\gamma(\beta-1)}\frac{(y+\varepsilon)}{y} \\
\theta_A &= -\frac{\varepsilon(t)}{y} + \left(1 - \frac{\beta\gamma(t)}{\gamma(\beta-1)} + \frac{1}{\beta-1}\right)\frac{y+\varepsilon(t)}{y} \\
1-\theta_B-\theta_A &= -\frac{1}{\beta-1}\frac{(y+\varepsilon(t))}{y}
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{p(t,y,r)y + \varepsilon(t)C'^{-1}\sigma_s}{y+\varepsilon(t)} \\
&= \frac{\beta\gamma(t)}{\gamma(\beta-1)}(p_B - p_A) + \frac{\beta}{\beta-1}p_A - \frac{1}{\beta-1}C'^{-1}\rho \\
\text{where } p_B &= C'^{-1}(\sigma_s - d_a(r)\sigma_r e_1) \\
p_A &= C'^{-1}\sigma_s
\end{aligned}$$

Now $\gamma(t)$ increases to γ as t increases to T. Thus, lower risk assets are shifted from portfolio A ($p_A = C'^{-1}\sigma_s$) into portfolio B ($p_B = C'^{-1}(\sigma_s - \gamma\sigma_r e_1/\beta)$). By time T, there are no investments remaining in portfolio A (as proved in Corollary 3.2.4). This gives the most tangible difference between the present case and the case where we are funding for cash ($\gamma = 0$). On the other hand, the proportion of the excess over $-\varepsilon(t)$ invested in the risky portfolio C ($p_C = C'^{-1}\rho$) does not depend on either time or y. None of the investment proportions depend upon r. This arises, in the present work, for two reasons. First, the $r(T)$ components in the salary and asset parts of the model cancel out. Second, the choice of a power utility function produces a similar lack of dependence upon t and r as arises in simpler optimisation problems (for example, see Merton, 1990).

4.4 Further comment

If $\nu_s > 0$, then there is non-hedgeable salary risk. With a power utility function, which takes the value minus infinity for negative values of y,we need to ensure that the fund size stays positive at all times. To do this, we need to ensure that the *proportions* of the fund held in each asset are bounded in some neighbourhood of zero.

Further investigation shows that, with $\pi = 0$ and $\nu_s > 0$, we still retain the power form of the solution (equation 4.1.1) when $\varepsilon(t) = 0$. This suggests the following outcome in problems with non-zero payment streams. Suppose we replace the salary-related and non-hedgeable contribution stream $\pi S(t)$ by a related, possibly stochastic, contribution stream $C(t)$ which is hedgeable (for example, if assets include index-linked bonds and if $C(t)$ is linked to the same index). Suppose also that final salary is the product of a hedgeable component and a component which is independent of asset returns. Then the independent component will separate out in a power utility problem and have no effect on the optimisation procedure. In combination with the hedegable contribution stream $C(t)$ the observation for $\pi = 0$ suggests that this new model will retain the power form of the solution (equation 4.1.1).

5 Conclusions

In this paper we have extended the recent work of Boulier *et al.* (1999) and Deelstra *et al.* (1999) in a number of ways:

- We consider a general one-factor diffusion model for interest rates rather than the Vasicek or Cox-Ingersoll-Ross models.
- We place no restriction on the number of risky assets.
- Salary growth is random and contains a non-hedgeable element.
- The terminal utility function can be a function of the terminal fund value in salary units and of the risk-free interest rate at that time. In particular, we took it to be a function of the pension purchased at retirement as a proportion of final salary.

With this structure we are able to prove some general results using the Bellman equation. In particular:

- the optimal portfolio is composed of a time-dependent mixture of just three portfolios (p_A, p_B and p_C) which are themselves constant over time;
- the optimal utility depends only upon the market prices of risk and not, in addition, on the particular range of assets available;

- the low risk component of the total fund gradually shifts from low-risk cash investments into assets that match the future pension liability. (This result is similar in some respects to a result derived in Blake (1998).)

With more restrictive assumptions and the special case of power utility, we were able to establish more specific results: in particular, circumstances under which the power-utility structure is preserved through time. We found that the dependence of the utility function upon the replacement ratio rather than the cash value of the fund at retirement leads to an optimal asset-allocation strategy that is similar in some respects to that derived by Boulier *et al.* (1999) and Deelstra *et al.* (1999) (in particular, the stable investment proportion in risky assets) but qualitatively different in other respects (e.g., a shift from low-risk cash assets towards liability-matching assets).

The model considered here did not include any form of minimum guaranteed benefits. However, the main results in this paper should extend to problems involving guarantees with utility measured as a function of the excess pension benefit as a proportion of final salary.

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