

A New Method for evaluating and managing the complex risks
embedded in the life insurer's balance sheet:
basic ideas and preliminary results

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Abstract. We propose a new framework for evaluating and analyzing the complex risks embedded in a life insurance company's balance sheet, both on the asset and liability side. To this purpose, we apply an actuarial method not only to the insurance liabilities, but also to the corporate bonds and stocks as in the same manner as already developed in Kijima and Muromachi(1998a). And we introduce the VaRs and RAPMs, which are commonly used, to both asset and liability portfolios, and discuss their calculation methods with some preliminary numerical illustrations. The conclusion is that a life insurance company is recommended to segregate the total liabilities into several portions of the same nature and risk profile, and paste the appropriate hedging assets to them so as to maximize RAPM of each segment.

Key Words: HJM model, Amin-Jarrow model, hazard process, cohort model, default swap, VaR, RAPM, marginal VaR, Thiele's equation.

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1 Introduction

One of the most serious problems for the life insurance industry in Japan today is the continuing negative margin of the actual interest rate to the assumed rate. It would be the nationwide serious problem that the negative margin has aggravated the financial strength of the insurance companies, in particular, of middle-sized and old companies. The Japanese government and the Bank of Japan have adopted the super-low interest rate policy for 4 years since 1996. The official bank rate has kept 0.5% per annum, which is the lowest rate in the Japanese history, and we have once experienced that the 10-year yields of the Japanese Government Bond (hereafter, referred as JGB) become less than 1%.

Against the super-low interest rates, Japanese life insurance companies have taken actions to lower the assumed interest rate year by year, however, the assumed rates currently set on the existing business are now centered around 5.5%, so that the negative margins cannot easily vanish. Moreover, since Japanese life insurers have made the same interest rate assumption on both the policy reserve valuation and the premium calculations, they have compensated for the huge negative interest deficits by the expense and the mortality profits. Such a financial structure is abnormal for the life insurers, and the recognition of their true risk/return profiles is shadowed over. In order to solve these financial distresses by quantifying the performances and risks of products or business lines, and setting out the appropriate allocation of the company's capital or surplus, the new framework is anticipated to re-value the company's assets and liabilities on the fair market value basis, and to implement the precise risk management and ALM system.

In this paper, we propose a new framework to integrate the mortality and the withdrawal risks on the liability side, the credit and the equity risks on the asset side, and the interest rate risk on both sides, and show some simple numerical illustrations by this approach.

2 Previous studies

This chapter begins with the brief survey of the research on the asset/liability and risk management framework applied to the life insurance businesses, and proceeds to the theoretical setting of the pricing model of the risky assets and the quantification methods of the portfolio risk. Here, we deal with the individual stocks and the defaultable corporate bonds or loans as the risky assets.

One of the famous books introducing the risk management framework for the insurers is Daykin, Pentikäinen and Pesonen(1994); they dealt with mainly the non-life insurance businesses, but Chapter 15 went for the life insurance businesses, and Chapter 16 for the pension schemes. First, they introduced the classical actuarial approach to the mechanism of the profit generation in the life insurers operations, and the attribution analysis of the actuarial gains/losses. Second, they carried out the Monte-Carlo simulation of the actuarial gains/losses by randomizing the actuarial factors classically set as the fixed numbers, in which the concept of risk was incorporated. This type of approaches are well known and called the dynamic cashflow simulation method. You can find many researches on this method in the past AFIR articles ¹.

Moreover, you might find some limitations if you tried to measure or analyze the diversification effect on the asset side. In order to deal with this problem, you had better build a model describing the free surplus variations (= total assets minus total insurance liabilities), after re-valuing the insurance liabilities on the fair market value basis, as Taylor(1997) and Schnieper(1997) did. Both authors

¹See also Ziemba and Mulvey(1998).

dealt with the models focused mainly on the non-life insurance liabilities and the two asset categories, say, the risk-free assets and the volatile risky assets, to seek the optimal reinsurance cover, loadings, and capital allocation. Such models are appropriate to provide quantified informations for monitoring risk/return relationship on the periodical terms, and quite useful for planning tactical investment strategies. The capital allocation problem was originated from the banking risk management along with their risk-measurement concepts, VaR(Value-at-Risk), EaR(Earnings-at-Risk), and RAPM(Risk Adjusted Performance Measure), which were widely spread as the international standards for a short time. The banking risk management and RAPM are well explained in Matten(1996) and Jorion(1997). The insurance models seemed to follow or mimic these banking models, and examples of the insurance applications are Albrecht(1997), Albrecht, Bahrle and Konig(1996), and Albrecht, Bahrle, Mauerer and Schradin(1996).

This paper aims at constructing the mathematical framework of the risk management when considering risks except for the systematic risks, that is, the individual risks of assets and life insurance liabilities, such as credit risk, the equity unsystematic risk, the mortality risk, the withdrawal risk and the interest rate risk.

First, we should select an appropriate term structure model for our purpose. Heath, Jarrow and Morton model (1992), in which the forward rate process was modeled as a multi-factor stochastic process, is one of the best models because their model has such superior advantage that it holds no-arbitrage conditions as to the term structure of interest rates, and can involve all the available informations about the current yield curve. Amin and Jarrow (1992) applied HJM model to the ordinary stock price processes, and proposed a no-arbitrage model on stock and bond prices.

In recent years, there have been many outstanding contributions for pricing defaultable assets such as corporate bonds; for example, Jarrow and Turnbull(1995), Jarrow, Lando and Turnbull(1997), Duffie and Singleton(1999), and Logstaff and Schwartz(1995). However, their models are all applicable to only one risky asset, not to the portfolio of many assets, therefore, we cannot apply directly their models to the quantitative risk management tool for credit risk. On the other hand, some models for evaluating the credit risk of a portfolio have been proposed in the last two years. The most famous models are CreditMetricsTM of J.P.Morgan(1997), and CREDITRISK⁺ of Credit Suisse Financial Products(1997). CreditMetricsTM assumed to measure the obliger's future credit ability by its credit rating, and to assess the credit risk amount of the asset portfolio as the losses by default and the price variations by the change of the credit ratings on market value basis. CREDITRISK⁺ assumed that the occurrences of default events followed the Poisson processes, and calculated the loss distribution of the asset portfolio by analytical methods. These models have a lot of advantages, but some defects. In CreditMetricsTM, one of the disadvantages is that the spot rate curve used for the asset pricing is determined by its credit rating only. It means that CreditMetricsTM neglects the risk factors other than default risk, which would violate the consistency with the current asset prices, and ignores the interest rate risk that is the key factor to the total portfolio. It is also the shortcoming of CREDITRISK⁺ that the price variations are totally ignored; the risk amount is computed on a total loss basis, so that the amount is separated from the market risk valuation. It would cause a problem if you desire to integrate both credit and market risks.

In contrast to above models, Kijima and Muromachi(1998a, 1999) proposed a new risk measurement model to integrate both the interest rate risk and the credit risk simultaneously on fair market value basis. In their model, the default-free interest rate and the default rate are both stochastic processes, and the future price distribution of the given asset portfolio is computed by the Monte Carlo simulations. The risk-neutral measure and the real world measure are clearly distinguished,

and the current prices and future prices are obtained by the risk-neutral evaluation method, and the calculated current prices are consistent with the observed market prices.

In this paper, we propose a model for evaluating the synthetic risks of the asset portfolio and the liability portfolio simultaneously; the asset side model is the combination of Amin and Jarrow model (1992) and Kijima and Muromachi model (1998a), while the liability side model is based on the latter one. For pricing a life insurance liability, we adopt a default-swap pricing model proposed by Kijima and Muromachi(1998b) because the stochastic payoffs of such liabilities can be interpreted as those of the basket-type default swaps.

Since the HJM model has superior nature but not generally Markovian, it is difficult for practitioners to use the model widely. To the contrary, Inui and Kijima(1998) showed that the HJM model could induce the Markovian spot rate process by giving some restrictions to the volatility function, and Nakamura(1999) showed that many Markovian families could be generated by solving certain ODE's. In our numerical illustrations, the extended Vasicek model introduced by Hull and White(1990) is used because the model is one of such spot rate processes and is very tractable.

3 Modeling a life insurance company

In this section, following the framework of Daykin, Pentikäinen and Pesonen(1994)'s, we introduce a mathematical model to describe the management operation of the life insurance companies.

Suppose an insurer contracting a lot of life insurance policies, and that an individual life insurance policy is a contingent claim which pays to the insured some benefits assured at the insured person's death or at the maturity of the contract in compensation for the premium payments. At first, we describe a cohort model and discuss the evaluation of inflows and outflows in future, and then, by considering a lot of cohorts, we construct a simple stochastic model for a life insurance company.

3.1 Cohort model

Consider a cohort, the persons belonging to which are not identified individually so that they are homogeneous from the standpoint of the characteristics of risk such as his (her) death and the cancellation of the policy contract. Here, we consider the deterministic case, which implies that the mortality rate, the withdrawal rate, and the interest rate for discounting cash flows are all deterministic.

Defining $l(t)$ as the number of persons belonging to the cohort at time t under the discrete time economy $t = 0, 1, \dots, n$, $l(t)$ satisfies the following recursive equation

$$l(t) = l(t-1)[1 - q(t-1)],$$

where $q(t)$ denotes the one-year probability of death at time t of the cohort, that is, the conditional probability from t to $t+1$ given the survival at t . $q(t)$ of the cohort aged x_0 at $t=0$ is also written as q_{x_0+t} at t , which is estimated as the one-year probability of death at the age x_0+t from statistical data.

As a life insurance policy, suppose a contract with maturity $t=n$ so that the insured receives the following benefits:

- $S_d(t)$ at the time when the insured dies; or
- $S_e(t)$ at the maturity, $t=n$; or
- $S_w(t)$ at the time when the policy is withdrawn, where $S_w(t) = 0$ at $t < w - x_0$ and $S_w(t) > 0$ at $t \geq w - x_0$.

The policies with the benefits $S_d(t) + S_e(t)$ are called the endowment insurance policies.

In order to evaluate a life insurance policy, future cashflows generated by the policy are discounted to be the present value. Given the future interest rate $j(t)$ between t to $t + 1$, discount factors are given by

$$\begin{aligned} v(t) &= \frac{1}{1 + j(t)}, \\ v(t, s) &= v(t)v(t + 1) \cdots v(s - 1)v(s)^{1/2}, \\ v(t, t) &= v(t)^{1/2} = \frac{1}{(1 + j(t))^{1/2}}, \end{aligned}$$

where $v(t, s)$ is a discount factor from the start of the t financial year to the midyear of s financial year, assuming that the death of the insured occurs only at the midyear.

It is the basic principle in the insurance mathematics that the the amount of assets the insurer should reserve for the contract is equal to the difference between the present value of future benefit payments and that of future premium payments. This principle is written as

$$l(t)V(t) = \sum_{s=t+1}^{\infty} l(s-1)v(t, s)[S_e(s) + S_w(s) + q(s)S_d(s) - B(s) + E(s)], \quad (1)$$

where $V(t)$ denotes the *premium reserve* at time t , $B(s)$ is the *gross premium* paid in s financial year, and $E(s)$ is the *loading*. Especially, when $E(s) = 0$ in (1), $V(t)$ and $B(s)$ are called the *net premium reserve* and the *net premium*, respectively. Then, the gross premium is divided into the net premium and the loading. In countries where the institution is adopted such that the actuarial assumptions are set to be conservative, $V(t)$ is almost positive. The net premium is calculated from the *equivalent principle* at the time of policy issue

$$V(t) = 0,$$

while the loading is given exogenously. According to Schnieper(1997), for the loading, we assume that the loading for expenses and expenses cancel out, so that the loading in our paper implies the loading for profit only. And then, since the insurance policies are not traded in the market, we define the present value of a policy as the sum of the net premium reserve $V(t)$ and the discounted expectation of the loadings in future. Moreover, dividends to policyholders are ignored in this paper.

The next step to model the company's operation is to represent the external driving assumptions such as mortality rates $q(t)$, withdrawal rates $w(t)$, investment returns $j(t)$, expense rates $E(t)$ and discount rates as the stochastic processes. It is no common way to model these assumptions, but the following is one of the simplest models.

1. Death of the j -th insured is judged by using a stochastic variable $\mathbf{q}_j(t)$ where $\mathbf{q}_j(t) = 1$ at probability $q_j(t)$ and $\mathbf{q}_j(t) = 0$ at probability $1 - q_j(t)$, and the total mortality number is $q(t) = \sum_j \mathbf{q}_j(t)$.
2. Withdrawal of the j -th policy is judged by using a stochastic variable $\mathbf{w}_j(t)$ where $\mathbf{w}_j(t) = 1$ at probability $w_j(t)$ and $\mathbf{w}_j(t) = 0$ at probability $1 - w_j(t)$, and the total withdrawal number is $w(t) = \sum_j \mathbf{w}_j(t)$.
3. Survival number $l(t)$ is determined by $l(t) = l(t - 1) - q(t) - w(t)$.
4. Investment return $j(t)$ is generated by an appropriate investment model such as Wilkie(1996).

5. Expense rates $E(t)$ are usually separated into the initial year costs and the following years' costs, the latter including inflationary elements, $E(1) = E_1$, $E(t) = E_2 \times \prod_{s=1}^t (1 + i(s))$, where $i(t)$ denotes the inflation rate.
6. Discount rates $v(t)$ are due to an appropriate term structure model of the interest rate.

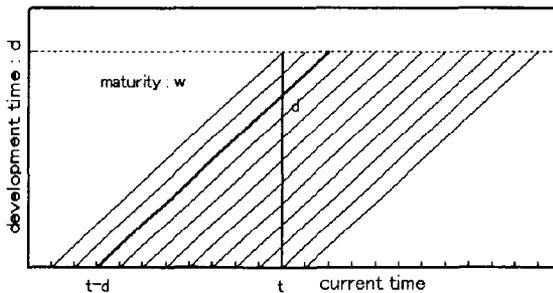
Our model is rather complex than this model in that both the mortality and the withdrawal rates follow the hazard processes, and the prices of assets and insurance liabilities are evaluated by the risk-neutral valuation method based on the hazard processes.

3.2 Firm-wide model of a life insurance company

The analysis of the total business of a life insurance company is based on breaking down the portfolio into cohorts of the same type. Each of the cohorts will be handled separately, applying the technique derived above, and the results for the whole business will be obtained by summing the outcomes of all cohorts. The cohort can be specified, for example, by the following characteristics;

- Type of policy; and
- Year in which the policy is written; and
- Gender of the insured; and
- Entry age of the insured.

Figure 1. Cohort Model



A sequence of cohorts with the same characteristics except entry years can be described as shown in Figure 1. The current time is denoted by t , and the time interval from the entry of the cohort up to the current year, say the development time, by d , hence, the entry year is $t - d$. Each type of cohort gives rise to a similar sequence of consecutive cohorts as is depicted in Figure 1. For example, the endowment and term insurance policies are specified by different cohorts according to entry age, maturity age, and so on.

One of the practical problems is that the number of cohorts can grow quite large if we distinguish cohorts by the types of policies, the entry ages, the maturity ages, and so on. Against this problem, it is an usual treatment to combine cohorts which have relatively similar relevant characteristics. The key variable in the stochastic analysis is the size of the dividend fund, now attributable to the total portfolio. However, in our framework, dividend matters will be neglected because of simplicity.

4 Stochastic modeling of a life insurance company

In this section, we propose a fundamental method for describing the price evolutions of assets and liabilities possessed by a life insurance company. For simplicity, suppose that the life insurance company has M stocks and N discount bonds issued by the government and the firms as assets, and that it has L life insurance policies as liabilities; no other kinds of assets and liabilities such as complicated derivatives or property assets are not considered in this paper. The bonds issued by the government imply the default-free bonds, and those issued by firms do the defaultable bonds.

Under the above conditions, risks included in the portfolios of asset and liability sides are the mortality risk, the withdrawal risk, the default risk, the default-free interest rate risk, and the stock price risk. Here, we introduce stochastic variables which express these risks respectively, and assume stochastic differential equations which describe the stochastic behaviors of the variables.

4.1 The mortality, withdrawal, and default processes

In our model, the mortality risk, the withdrawal risk, and the default risk are formulated in the same manner, therefore, we describe the formulation of the mortality risk as an example. You can see that the death of a policy means its withdrawal, and that the death of a firm does its bankruptcy, which is assumed to be equal to the default of the assets issued by the firm in this paper.

Suppose that there exist N persons, and let τ_j ($j = 1, 2, \dots, N$) denote the time of j -th person's death, and $h_j(t)$ be its mortality rate under the observed probability measure P . The mortality rate $h_j(t)$ is defined by

$$h_j(t) = \lim_{dt \rightarrow 0} \frac{P\{t < \tau_j \leq t + dt | \tau_j > t\}}{dt}, \quad t \geq 0,$$

and the cumulative mortality rate $H_j(u, v)$ is defined by

$$H_j(u, v) = \int_u^v h_j(s) ds, \quad 0 \leq u \leq v.$$

Since $h_j(t)$ is non-negative, $H_j(t, T)$ is non-decreasing in T , and $\lim_{T \rightarrow \infty} H(t, T) = \infty$. Hereafter, we refer to the mortality rate $h_j(t)$ as the hazard rate, in general.

It is assumed that the hazard rate processes $h_j(t)$ under P follow the system of stochastic differential equations (hereafter, abbreviated by SDEs)

$$dh_j(t) = \mu_j(\mathbf{h}(t), t)dt + \sigma_j(\mathbf{h}(t), t)dz_j(t), \quad j = 1, 2, \dots, N,$$

where $\mathbf{h}(t) = (h_1(t), h_2(t), \dots, h_N(t))$, μ_j and σ_j are the drift and volatility functions of $h_j(t)$, respectively, and $\mathbf{z}(t) = (z_1(t), z_2(t), \dots, z_N(t))$ is the n -dimensional standard Wiener process under P , assuming the following correlation structure

$$E[dz_j(t)dz_k(t)] = \rho_{jk}(t)dt, \quad j, k = 1, 2, \dots, N,$$

where $\rho_{jj}(t) = 1$. And, following Kijima(1998), we assume that there exist the *risk-premia adjustments* $\ell'_j(t)$ which are deterministic functions and satisfy

$$\tilde{h}_j(t) = h_j(t) + \ell'_j(t), \quad j = 1, 2, \dots, N, \quad (2)$$

where $\tilde{h}_j(t)$ are the hazard rates under the risk-neutral probability measure \tilde{P} ².

²If the defaultable bonds and policies are traded in the market, $\ell'_j(t)$ are derived from the market prices of them. However, since all of them are not traded in practice, we might assume $\ell'_j(t)$ based on other assumptions.

Next, we assume that τ_j are *conditionally independent* given the realization of the underlying stochastic processes ³. The *conditional independence* means that

$$P\{\tau_1 > t_1, \dots, \tau_N > t_N | \mathcal{F}_t\} = \prod_{j=1}^N P\{\tau_j > t_j | \mathcal{F}_t\}, \quad t \geq \max_j t_j.$$

By definition of the hazard rate, the conditional survival probabilities are given by

$$P_{t_0}\{\tau_j > t_j | \mathcal{F}_{t_0}\} = \exp\{-H_j(t_0, t_j)\}, \quad j = 1, 2, \dots, N,$$

therefore, by taking the expectation on the above equation at t_0 ($t_0 < \min_j t_j$), we obtain

$$P_{t_0}\{\tau_1 > t_1, \dots, \tau_N > t_N\} = E\left[\exp\left\{-\sum_{j=1}^N H_j(t_0, t_j)\right\}\right].$$

Notice that we impose no constraint on the correlation structures of hazard rates of different firms ⁴. Under \tilde{P} , we obtain the joint survival probability as

$$\tilde{P}_{t_0}\{\tau_1 > t_1, \dots, \tau_N > t_N\} = \left(\prod_{j=1}^N L'_j(t_0, t_j)\right) E\left[\exp\left\{-\sum_{j=1}^N H_j(t_0, t_j)\right\}\right]$$

where

$$L'_j(u, v) = \exp\left\{-\int_u^v \ell'_j(s) ds\right\}, \quad j = 1, 2, \dots, N.$$

4.2 Stochastic processes and pricing of assets and liabilities

In this subsection, stochastic price processes are summarized under P ⁵. As to the default-free interest rate process, we recommend the Heath-Jarrow-Morton (HJM) model (1992) because the model is one of the non-arbitrage term structure models, and it can incorporate all the current information in the yield curve. The proposed price processes of stocks are similar to the formulation for the defaultable forward rate proposed by Jarrow and Turnbull (1995), however, our model is very simple.

4.2.1 Basic stochastic processes

Let $S_j(t)$ denote the stock price of firm j ($j = 1, 2, \dots, M$), and $f(t, T)$ be the time t instantaneous default-free forward rate for date T , and $\rho(t)$ be the bank account

$$\rho(t) = \exp\left\{\int_0^t r(s) ds\right\},$$

where the instantaneous spot rate is $r(t) = f(t, t)$. We assume that a continuum of default-free discount bonds trade with differing maturities, and $p_0(t, T)$ denotes the time t price of the default-free discount bond with maturity T . And, for simplicity, we assume $S_j(t) = 0$ on $t \geq \tau_j$.

³This assumption makes it possible to construct the joint distribution of τ_j ($j = 1, 2, \dots, N$) from its marginal distributions which are derived from the hazard rates $h_j(t)$.

⁴If the *ordinary independence* is assumed, this is not derived.

⁵Price processes of assets and liabilities under \tilde{P} are necessary to evaluate the prices of derivatives written on them. They are summarized in Appendix A.

Let $h_j(t)$ ($j = 1, \dots, N$) denote the hazard rate for default of the j -th firm, $h_{N+k}(t) = h_{q_k}(t)$ ($k = 1, \dots, L$) be the mortality rate for the k -th person, and $h_{N+L+k}(t) = h_{w_k}(t)$ ($k = 1, \dots, L$) denote the withdrawal rate for the policy of the k -th person, respectively, and denote $\mathbf{h}(t) = (h_1(t), \dots, h_{N+2L})$. Under P , it is assumed that $h_j(t)$, $h_{q_j}(t)$, $h_{w_j}(t)$, $f(t, T)$ and $S_j(t)$ follow the SDEs

$$dh_j(t) = \mu_j(\mathbf{h}(t), t)dt + \sigma_j(\mathbf{h}(t), t)dz_j(t), \quad j = 1, 2, \dots, N + 2L, \quad (3)$$

$$df(t, T) = \alpha(t, T)dt + \sum_{i=N+2L+1}^{N+2L+K} \sigma_i(t, T)dz_i(t), \quad 0 \leq t \leq T,$$

$$\begin{aligned} \frac{dS_j(t)}{S_j(t)} &= [\mu_j(t) + X_j(t)h_j(t)]dt + \sum_{i=N+2L+1}^{N+2L+K+M} \delta_{ij}(t)dz_i(t) + dX_j(t) \\ &= \begin{cases} [\mu_j(t) + h_j(t)]dt + \sum_{i=N+2L+1}^{N+2L+K+M} \delta_{ij}(t)dz_i(t), & t < \tau_j, \\ -1, & t = \tau_j, \\ \mu_j(t)dt + \sum_{i=N+2L+1}^{N+2L+K+M} \delta_{ij}(t)dz_i(t), & t > \tau_j \end{cases}, \quad (4) \end{aligned}$$

$$E[dz_j(t)dz_k(t)] = \rho_{jk}(t)dt, \quad j, k = 1, 2, \dots, N + 2L + K + M,$$

where $X_j(t) = 1_{\{\tau_j > t\}}$, and $z_i(t)$ ($i = 1, 2, \dots, N + 2L + M + K$) are standard Brownian motions, and $\rho_{jk}(t) = 0$ ($j \neq k$; $j, k = N + 2L + 1, \dots, N + 2L + K + M$). (4) is similar to the formulation for describing the stochastic process of the defaultable forward rate used by Jarrow and Turnbull(1995), however, our case is very simple because of $S_j(t) = 0$ for $t \geq \tau_j$. Moreover, it is assumed that the unique solutions for (3) can be obtained, and that the solutions are denoted by $h_j(t)$. Under these assumptions, stochastic variables at the risk horizon, \bar{t} ($\bar{t} > 0$), are written by

$$f(\bar{t}, T) = f(0, T) + \int_0^{\bar{t}} \alpha(u, T)du + \sum_{i=N+2L+1}^{N+2L+K} \int_0^{\bar{t}} \sigma_i(u, T)dz_i(u),$$

$$\rho(\bar{t}) = \exp \left\{ \int_0^{\bar{t}} f(0, s)ds + \int_0^{\bar{t}} \int_u^{\bar{t}} \alpha(u, s)dsdu - \sum_{i=N+2L+1}^{N+2L+K} \int_0^{\bar{t}} a_i(u, \bar{t})dz_i(u) \right\},$$

$$p_0(\bar{t}, T) = \frac{p_0(0, T)}{p_0(0, \bar{t})} \exp \left\{ - \int_0^{\bar{t}} \int_{\bar{t}}^T \alpha(u, s)dsdu - \sum_{i=N+2L+1}^{N+2L+K} \int_0^{\bar{t}} [a_i(u, \bar{t}) - a_i(u, T)] dz_i(u) \right\},$$

$$\begin{aligned} S_j(\bar{t}) &= S_j(0) \exp \left\{ \int_0^{\bar{t}} \left(\mu_j(s) - \frac{1}{2} \sum_{i=N+2L+1}^{N+2L+K+M} \delta_{ij}^2(s) \right) ds + H_j(0, \bar{t}) \right. \\ &\quad \left. + \sum_{i=N+2L+1}^{N+2L+K+M} \int_0^{\bar{t}} \delta_{ij}(s)dz_i(s) \right\}, \quad \text{on } \bar{t} < \tau_j, \quad j = 1, 2, \dots, M, \end{aligned}$$

where

$$a_i(t, T) = - \int_t^T \sigma_i(t, u)du, \quad i = N + 2L + 1, N + 2L + 2, \dots, N + 2L + K.$$

These stochastic representations for time \bar{t} values are used in order to make future scenarios at \bar{t} , for example, by using the Monte Carlo simulation method.

4.2.2 Evaluation of defaultable discount bonds

The prices of defaultable bonds and life insurance policies are evaluated by the risk-neutral valuation framework ⁶. Assuming the followings

- The recovery rate δ_j ($0 \leq \delta_j \leq 1$) is constant; and
- If $\tau_j < T_j$, the investor receives the cash δ_j at the maturity T_j , regardless of the event $\{\tau_j \leq \bar{t}\}$ or $\{\tau_j > \bar{t}\}$; and
- $r(t)$ and $h_j(t)$ are independent,

the time \bar{t} price of the defaultable discount bond with maturity T_j issued by firm j , $p_j(\bar{t}, T_j)$, is given by

$$\begin{aligned} p_j(\bar{t}, T_j) &= \tilde{E}_{\bar{t}} \left[\exp \left\{ - \int_{\bar{t}}^{T_j} r(s) ds \right\} \left\{ \delta_j 1_{\{\tau_j \leq T_j\}} + 1_{\{\tau_j > T_j\}} \right\} \right] \\ &= \tilde{E}_{\bar{t}} \left[\exp \left\{ - \int_{\bar{t}}^{T_j} r(s) ds \right\} \right] \tilde{E}_{\bar{t}} \left[\delta_j 1_{\{\tau_j \leq T_j\}} + 1_{\{\tau_j > T_j\}} \right] \\ &= p_0(\bar{t}, T_j) \left[\delta_j + (1 - \delta_j) \tilde{P}_{\bar{t}} \{\tau_j > T_j\} \right] \quad \text{on } \tau_j > \bar{t}, \end{aligned}$$

where \tilde{E} denotes the conditional expectation operator under \tilde{P} . Since $p_j(\bar{t}, T_j) = \delta_j p_0(\bar{t}, T_j)$ on $\tau_j \leq \bar{t}$, we obtain

$$p_j(\bar{t}, T_j) = p_0(\bar{t}, T_j) \left[\delta_j + X_j(\bar{t})(1 - \delta_j) \tilde{P}_{\bar{t}} \{\tau_j > \bar{t}\} \right].$$

4.2.3 Evaluation of life insurance policies

Suppose a life insurance policy for the j -th insured person with the maturity T_j . Letting τ_q denote the time of the death of the insured, and τ_w denote the time of the withdrawal from the policy, we assume that the policyholder receives

- $S_{e,j}$ at time T_j on $\min\{\tau_q, \tau_w, T_j\} = T_j$; or
- $S_{w,j}(\tau_w)$ at time τ_w , on $\min\{\tau_q, \tau_w, T_j\} = \tau_w$; or
- $S_{d,j}$ at time τ_q , on $\min\{\tau_q, \tau_w, T_j\} = \tau_q$,

and that the policyholder pays the net premium $\pi_j(t_{j,i})$ and the loading $l_j(t_{j,i})$ at $t_{j,i}$ ($i = 1, 2, \dots, n$).

Such simple life insurance policies are similar to the basket-type default swaps with the first-to-default features. In order to explain their similarity, consider a default swap whose payoff is

- C_0 at time T if the maturity of this swap, T , comes first (that is, $\min\{\tau_1, \tau_2, T\} = T$); or
- C_1 at time τ_1 if the default of Firm 1 happens first (that is, $\min\{\tau_1, \tau_2, T\} = \tau_1$); or
- C_2 at time τ_2 if the default of Firm 2 happens first (that is, $\min\{\tau_1, \tau_2, T\} = \tau_2$),

⁶See Jarrow and Turnbull(1995) and Kijima and Muromachi(1998a, 1999) in detail.

and compare this with the above life insurance policy. If the default of Firm 1 is regarded as the withdrawal from the policy contract, and the default of Firm 2 as the death of the insured, then, the payoff of the above default swap is very similar to that of the above life insurance policy. Therefore, we can apply the default-swap pricing model to pricing the life insurance policy, and our pricing method is based on the default-swap pricing method proposed by Kijima and Muromachi(1998b). Here, we do not assume that $r(t)$ is independent of $h_{q_j}(t)$ and $h_{w_j}(t)$ necessarily ⁷.

The present value of the policy at time \bar{t} , $p_j^{LI}(\bar{t}, T_j)$, is given by

$$p_j^{LI}(\bar{t}, T_j) = PV_1^{out}(\bar{t}, T_j) + PV_2^{out}(\bar{t}, T_j) + PV_3^{out}(\bar{t}, T_j) - PV_{net}^{in}(\bar{t}, T_j) - PV_{load}^{in}(\bar{t}, T_j), \quad (5)$$

where

$$\begin{aligned} PV_1^{out}(\bar{t}, T_j) &= S_{e,j} \tilde{E}_{\bar{t}} \left[\exp \left\{ - \int_{\bar{t}}^{T_j} r(v) dv \right\} 1_{\{\min\{\tau_{q_j}, \tau_{w_j}, T_j\} = T_j\}} \right], \\ PV_2^{out}(\bar{t}, T_j) &= \tilde{E}_{\bar{t}} \left[S_{w,j}(\tau_{w_j}) \exp \left\{ - \int_{\bar{t}}^{\tau_{w_j}} r(v) dv \right\} 1_{\{\min\{\tau_{q_j}, \tau_{w_j}, T_j\} = \tau_{w_j}\}} \right], \\ PV_3^{out}(\bar{t}, T_j) &= S_{d,j} \tilde{E}_{\bar{t}} \left[\exp \left\{ - \int_{\bar{t}}^{\tau_{q_j}} r(v) dv \right\} 1_{\{\min\{\tau_{q_j}, \tau_{w_j}, T_j\} = \tau_{q_j}\}} \right], \\ PV_{net}^{in}(\bar{t}, T_j) &= \tilde{E}_{\bar{t}} \left[\sum_{\bar{t} < t_{ji} < T_j} \pi_j(t_{ji}) \exp \left\{ - \int_{\bar{t}}^{t_{ji}} r(v) dv \right\} 1_{\{\min\{\tau_{q_j}, \tau_{w_j}, T_j\} > t_{ji}\}} \right], \\ PV_{load}^{in}(\bar{t}, T_j) &= \tilde{E}_{\bar{t}} \left[\sum_{\bar{t} < t_{ji} < T_j} l_j(t_{ji}) \exp \left\{ - \int_{\bar{t}}^{t_{ji}} r(v) dv \right\} 1_{\{\min\{\tau_{q_j}, \tau_{w_j}, T_j\} > t_{ji}\}} \right]. \end{aligned}$$

Using the cumulative hazard rates, and defining $H_0(u, v) = \int_u^v r(s) ds$, we obtain

$$\begin{aligned} PV_1^{out}(\bar{t}, T_j) &= S_{e,j} \tilde{E}_{\bar{t}} \left[\exp \left\{ -H_0(\bar{t}, T_j) - H_{q_j}(\bar{t}, T_j) - H_{w_j}(\bar{t}, T_j) \right\} \right], \\ PV_2^{out}(\bar{t}, T_j) &= \tilde{E}_{\bar{t}} \left[\int_{\bar{t}}^{T_j} S_{w,j}(u) h_{w_j}(u) \exp \left\{ -H_0(\bar{t}, u) - H_{q_j}(\bar{t}, u) - H_{w_j}(\bar{t}, u) \right\} du \right], \\ PV_3^{out}(\bar{t}, T_j) &= S_{d,j} \tilde{E}_{\bar{t}} \left[\int_{\bar{t}}^{T_j} h_{q_j}(u) \exp \left\{ -H_0(\bar{t}, u) - H_{q_j}(\bar{t}, u) - H_{w_j}(\bar{t}, u) \right\} du \right], \\ PV_{net}^{in}(\bar{t}, T_j) &= \tilde{E}_{\bar{t}} \left[\sum_{\bar{t} < t_{ji} < T_j} \pi_j(t_{ji}) \exp \left\{ -H_0(\bar{t}, t_{ji}) - H_{q_j}(\bar{t}, t_{ji}) - H_{w_j}(\bar{t}, t_{ji}) \right\} \right], \\ PV_{load}^{in}(\bar{t}, T_j) &= \tilde{E}_{\bar{t}} \left[\sum_{\bar{t} < t_{ji} < T_j} l_j(t_{ji}) \exp \left\{ -H_0(\bar{t}, t_{ji}) - H_{q_j}(\bar{t}, t_{ji}) - H_{w_j}(\bar{t}, t_{ji}) \right\} \right]. \end{aligned}$$

Now, define

$$m_j(t, T_j, T_{q_j}, T_{w_j}) = \exp \left\{ -H_0(t, T_j) - H_{q_j}(t, T_{q_j}) - H_{w_j}(t, T_{w_j}) \right\},$$

then, it follows that

$$\begin{aligned} PV_1^{out}(\bar{t}, T_j) &= S_{e,j} \tilde{E} [m_j(\bar{t}, T_j, T_j, T_j)], \\ PV_2^{out}(\bar{t}, T_j) &= -\tilde{E} \left[\int_{\bar{t}}^{T_j} S_{w,j}(u) \frac{\partial}{\partial T_{w_j}} m_j(\bar{t}, T_j, T_{q_j}, T_{w_j}) \Big|_{T_j = T_{q_j} = T_{w_j} = u} du \right] \end{aligned} \quad (6)$$

⁷By Kijima and Muromachi's model, the effect of the correlation structures between the interest rate, the withdrawal rate and the mortality rate can be evaluated. This is one of the advantages of their model.

$$= - \int_{\bar{t}}^{T_j} S_{w,j}(u) \frac{\partial}{\partial T_{w_j}} \tilde{E} \left[m_j(\bar{t}, T_j, T_{q_j}, T_{w_j}) \right] \Big|_{T_j=T_{q_j}=T_{w_j}=u} du, \quad (7)$$

$$\begin{aligned} PV_3^{out}(\bar{t}, T_j) &= -S_{d,j} \tilde{E} \left[\int_{\bar{t}}^{T_j} \frac{\partial}{\partial T_{q_j}} m_j(\bar{t}, T_j, T_{q_j}, T_{w_j}) \Big|_{T_j=T_{q_j}=T_{w_j}=u} du \right] \\ &= -S_{d,j} \int_{\bar{t}}^{T_j} \frac{\partial}{\partial T_{q_j}} \tilde{E} \left[m_j(\bar{t}, T_j, T_{q_j}, T_{w_j}) \right] \Big|_{T_j=T_{q_j}=T_{w_j}=u} du, \end{aligned} \quad (8)$$

$$PV_{net}^{in}(\bar{t}, T_j) = \sum_{\bar{t} < t_{ji} < T_j} \pi_j(t_{ji}) \tilde{E} [m_j(\bar{t}, t_{ji}, t_{ji}, t_{ji})], \quad (9)$$

$$PV_{load}^{in}(\bar{t}, T_j) = \sum_{\bar{t} < t_{ji} < T_j} l_j(t_{ji}) \tilde{E} [m_j(\bar{t}, t_{ji}, t_{ji}, t_{ji})], \quad (10)$$

assuming that the differentiations and the integrations are exchangeable. From (5), (6), (7), (8), (9) and (10), we obtain $p_j^{LI}(\bar{t}, T_j)$ if $\tilde{E} [m_j(t, T_j, T_{q_j}, T_{w_j})]$ are given definitely. For example, if $r(t)$ is independent of $h_{q_j}(t)$ and $h_{w_j}(t)$, and if (2) is satisfied and $l_j^*(t)$ are set to be zero for the mortality rate and the withdrawal rate, $\tilde{E} [m_j(t, T_j, T_{q_j}, T_{w_j})]$ are reduced to the following ⁸:

$$\tilde{E} [m_j(t, T_j, T_{q_j}, T_{w_j})] = p_0(t, T_j) E [m_j(t, T_{q_j}, T_{w_j})],$$

where

$$m_j(t, T_{q_j}, T_{w_j}) = \exp \{ -H_{q_j}(t, T_{q_j}) - H_{w_j}(t, T_{w_j}) \}.$$

Given the stochastic characteristics of $r(t)$, $h_{q_j}(t)$ and $h_{w_j}(t)$, the net premium $\pi_j(t_{ji})$ is calculated from the equivalent principle. For simplicity, we assume that $\pi_j(t_{ji})$ is constant, π_j , and that π_j is determined by the equivalent principle at the time of policy issue, $t = 0$. Then, π_j is given by

$$\pi_j = \frac{PV_1^{out}(0, T_j) + PV_2^{out}(0, T_j) + PV_3^{out}(0, T_j)}{\sum_{0 < t_{ji} < T_j} \tilde{E} [m_j(\bar{t}, t_{ji}, t_{ji}, t_{ji})]}.$$

By definition of the net premium reserve for the j -th policy, $V_j(t)$ is given by

$$V_j(t) = PV_1^{out}(t, T_j) + PV_2^{out}(t, T_j) + PV_3^{out}(t, T_j) - PV_{net}^{in}(t, T_j),$$

which is equal to $p_j^{LI}(t, T_j) + PV_{load}^{in}(t, T_j)$. Notice that the stochastic spot rate is used to calculate the net premium reserve in our framework, while the assumed interest rate is used from a regulatory point of view. There exist some differences between the standard actuarial framework and ours, however, the basic concepts of the former remains in the latter ⁹.

If the insured dies, or cancels the contract, or the maturity arrives, the insurance policy disappears from the liability and the insurer must pay the benefits, so that the total amount of assets of the insurer is reduced. As a liability, the time t_0 value of the insurance policy is $p_j^{LI}(t_0, T_j)$, and the value at the risk horizon \bar{t} ($\bar{t} > t_0$) is $p_j^{LI}(\bar{t}, T_j) 1_{\{\min(\tau_{q_j}, \tau_{w_j}, T_j) > \bar{t}\}}$. For simplicity, assuming that the benefits which the insurer must pay to the insured are borrowed from the money market at the time when one

⁸Since the risk-premia adjustments for the mortality rate and the withdrawal rate are assumed to be zero, the expectation operator \tilde{E} changes to E .

⁹For example, *Thiele's differential equation* is derived in our framework. See in Appendix B.

of the contracted event occurs, and that they are discharged at time \bar{t} , then, the time \bar{t} value of the stochastic benefits, $S_j(t_0, \bar{t})$, is written as

$$\begin{aligned}
 S_j(t_0, \bar{t}) &= S_{e,j} \exp \left\{ \int_{T_j}^{\bar{t}} r(v) dv \right\} \mathbf{1}_{\{\min\{\tau_{q_j}, \tau_{w_j}, T_j\} = T_j \leq \bar{t}\}} \\
 &+ S_{w,j}(\tau_{w_j}) \exp \left\{ \int_{\tau_{w_j}}^{\bar{t}} r(v) dv \right\} \mathbf{1}_{\{\min\{\tau_{q_j}, \tau_{w_j}, T_j\} = \tau_{w_j} \leq \bar{t}\}} \\
 &+ S_{d,j} \exp \left\{ \int_{\tau_{q_j}}^{\bar{t}} r(v) dv \right\} \mathbf{1}_{\{\min\{\tau_{q_j}, \tau_{w_j}, T_j\} = \tau_{q_j} \leq \bar{t}\}} \\
 &- \sum_{t_0 < t_{j_i} < \bar{t}} \{\pi_j + l_j(t_{j_i})\} \exp \left\{ \int_{t_{j_i}}^{\bar{t}} r(v) dv \right\} \mathbf{1}_{\{\min\{\tau_{q_j}, \tau_{w_j}, T_j\} > t_{j_i}\}}. \quad (11)
 \end{aligned}$$

We assume that $S_j(t_0, \bar{t})$ is subtracted from the value of assets at \bar{t} .

Under the future scenarios made by the basic stochastic processes shown in 4.2.1, the future prices of assets and liabilities at the risk horizon \bar{t} are evaluated by using the non-arbitrage price of defaultable discount bonds described in 4.2.2 and of life insurance policies derived here.

4.2.4 Market yield adjusted reserve and premium

In the classical actuarial modeling, the yield curve for discounting is constant, while it changes with time in our approach, therefore, the concepts of the net reserve in these cases are quite different for the policies issued before. Here, we introduce a new concept for the net reserve called the *market yield adjusted reserve*, which takes a middle position between the two reserves¹⁰.

For the endowment policies, as an example, the market yield adjusted reserve $V_{x+t-0:\overline{n-t}|}^*$ (before) and $V_{x+t+0:\overline{n-t}|}^*$ (after), and the market yield adjusted premium $P_{x+t:\overline{n-t}|}^*$ are defined recursively by¹¹

$$\begin{aligned}
 V_{x+t-0:\overline{n-t}|}^* &= A_{x+t:n-t|}^{(t)} - P_{x+t:n-t|}^* \ddot{a}_{x+t:n-t|}^{(t)} = A_{x+t:n-t|}^{(t-1)} - P_{x+t-1:n-t+1|}^* \ddot{a}_{x+t-1:n-t+1|}^{(t-1)}, \\
 V_{x+t+0:\overline{n-t}|}^* &= V_{x+t-0:\overline{n-t}|}^* + P_{x+t:n-t|}^*, \\
 P_{x+t:n-t|}^* &= \frac{A_{x+t:n-t|}^{(t)} - A_{x+t:n-t|}^{(t-1)} + P_{x+t-1:n-t+1|}^* \ddot{a}_{x+t:n-t|}^{(t-1)}}{\ddot{a}_{x+t:n-t|}^{(t)}},
 \end{aligned}$$

where $^{(t)}$ denotes that the yield curve at time t is used for discounting, and the initial values for $t = 0$ are given by $P_{x:n|}^* = A_{x:n|}^{(0)} / \ddot{a}_{x:n|}^{(0)}$ and $V_{x-0:\overline{n}|}^* = 0$. These variables reflect the change of the yield curve with time. If the yield curve used for discounting does not change, these variables are identical to the values in the normal actuarial approach.

4.3 Simulation-based VaRs

It is likely that we may use the Monte Carlo simulation method in order to evaluate the future characteristics, such as the distribution of future price, of the existing portfolio¹². In this subsection,

¹⁰Notice that the *market yield adjusted reserve* is not equal to the market yield reserve defined in 4.2.3. In the former, the net premium changes with time recursively, therefore, the information up to date is always necessary. However, not in the latter.

¹¹In these equations, normal actuarial representations are used.

¹²Some easier calculation methods than the Monte Carlo simulation method are proposed. One of them is the use of the Cornish-Fisher expansion, which is discussed in Kijima and Muromachi(1999).

we summarize an important notice for the simulation, and derive an approximate confidence interval of the calculated VaRs in brief.

4.3.1 A notice for the simulation

In this paper, we have distinguished definitely the observed probability measure P from the risk-neutral probability measure \tilde{P} , while in evaluating the market risk, these two measures are not distinguished clearly, and are treated as the same measure. This is because the risk horizon for evaluating the market risk is quite short (about few days), therefore, the difference of the two measures is small so that the one can be used as a good approximation for another. However, in evaluating the credit risk, the risk horizon is so long (usually one year or more) that the approximation cannot be accepted.

We must use the governing equations under P in the simulation so that we obtain a distribution of future scenarios¹³ at the risk horizon \tilde{t} . Notice that the risk-neutral measure \tilde{P} must be used if it is necessary to evaluate the future values of contingent claims written on the scenarios. In our case, future values of default-free bonds and stocks are given by the simulation under P , and the future values of defaultable bonds and insurance policies must be evaluated by the risk-neutral valuation framework. However, since the future values of defaultable bonds and policies are calculated analytically in our case, the risk-neutral measure \tilde{P} is not used definitely in this paper.

4.3.2 Approximate confidence interval of VaRs

VaRs calculated by simulations are approximate values because the calculated future price distributions are approximate. Here, the analysis of calculated VaRs is discussed briefly.

Defining $f(x)$ as the density function of a continuous stochastic variable X , and ξ_α as the $100(1-\alpha)$ -percentile, then, according to Cramer(1946), the calculated $100(1-\alpha)$ -percentile ζ_α , from large size of independent data X_i ($i = 1, 2, \dots, n$), is asymptotically normally distributed with mean ξ_α and variance $\frac{\alpha(1-\alpha)}{n f^2(\xi_\alpha)}$. For simplicity, after rearranging the data such that $X_1 \leq X_2 \leq \dots \leq X_n$, we assume that there exists X_m such that $\zeta_\alpha = X_m$ ($2 \leq m \leq n-1$). Substituting ζ_α and $f(\zeta_\alpha)$ for the unknown values ξ_α and $f(\xi_\alpha)$, and using the central difference approximation

$$f(\zeta_\alpha) = \frac{F_D(X_{m+1}) - F_D(X_{m-1})}{(X_{m+1} - X_{m-1})} = \frac{2}{n(X_{m+1} - X_{m-1})},$$

where $F_D(x)$ denotes the empirical distribution function of data X_i , we obtain that ζ_α becomes asymptotically normally distributed with mean $\mu_\alpha = X_m$ and variance $\sigma_\alpha^2 = n\alpha(1-\alpha)(X_{m+1} - X_{m-1})^2/4$. Therefore, ξ_α lies in $(\mu_\alpha - z_\beta\sigma_\alpha, \mu_\alpha + z_\beta\sigma_\alpha)$ with probability β where z_β is the $(1+\beta)/2$ -percentile of the standard normal distribution¹⁴.

5 Firm-wide model of a life insurance company

In this section, we describe the return and the risk analyses, and propose an evaluation method of individual assets and liabilities based on the risk-adjusted performance measure (RAPM). Figure 2 shows the illustration of our overview of the firm-wide model.

¹³In this context, a future scenario does not mean a future price directly. A scenario means a set of future values of basic stochastic variables whose SDEs are defined definitely, that is, $h_j(t)$, $f(t, T)$ and $S_j(t)$ in this case.

¹⁴Morokoff, Lagnado and Owen(1998) discussed more practical evaluation methods.

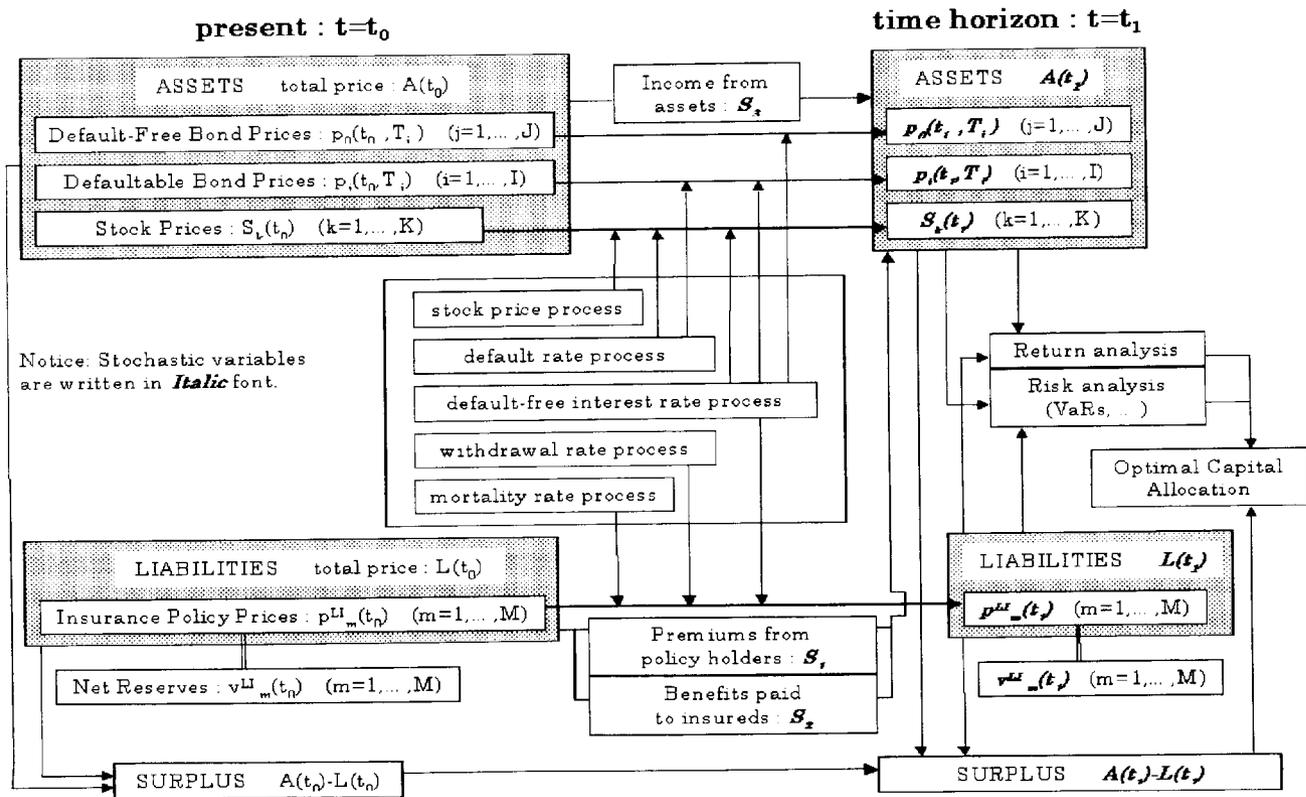


Figure 2. Overview of the firm-wide model

Suppose a portfolio consisting of N_A assets and N_L liabilities, and let $u(\bar{t})$ denote the value of the portfolio at the risk horizon \bar{t} ,

$$u(\bar{t}) = \sum_{j=1}^{N_A} w_j A_j(\bar{t}) - \sum_{j=N_A+1}^{N_A+N_L} w_j L_j(\bar{t}),$$

where w_j is the weight of asset or liability j in the portfolio, and $A_j(\bar{t})$ and $L_j(\bar{t})$ are present values of asset and liability per unit volume at \bar{t} , respectively. Additionally, in order to discuss the VaR of the portfolio, we define $G(x)$ as the distribution function of $u(\bar{t})$, and $G^{-1}(\alpha)$ denotes the $100(1 - \alpha)$ -percentile of $u(\bar{t})$.

5.1 Return analysis

The expected increase of the price of the portfolio from t_0 to \bar{t} is given by

$$E[\Delta u(\bar{t})] = \sum_{j=1}^{N_A} w_j E[\Delta A_j(\bar{t})] - \sum_{j=N_A+1}^{N_A+N_L} w_j E[\Delta L_j(\bar{t})], \quad (12)$$

where $\Delta u(\bar{t}) = u(\bar{t}) - u(t_0)$, $\Delta A_j(\bar{t}) = A_j(\bar{t}) - A_j(t_0)$ and $\Delta L_j(\bar{t}) = L_j(\bar{t}) - L_j(t_0)$. Divided by $u(t)$, (12) means that the expected return of the portfolio is the weighted sum of the returns of individual assets and liabilities.

Assuming that the time changes of the risk factors Y_k , such as $h_j(t)$, $f(t, T)$, and $S_j(t)$, are small enough to neglect higher orders of the changes, and that Y_k is subject to $N(0, \sigma_{Y_k}^2)$, we have the following relation approximately (the Delta-Gamma method):

$$\Delta A_j \simeq \frac{\partial A_j}{\partial t} \Delta t + \sum_k \frac{\partial A_j}{\partial Y_k} \Delta Y_k + \frac{1}{2} \sum_k \sum_l \frac{\partial^2 A_j}{\partial Y_k \partial Y_l} \Delta Y_k \Delta Y_l, \quad (13)$$

where $\Delta t = \bar{t} - t_0$ and $\Delta Y_k = Y_k(\bar{t}) - Y_k(t_0)$, respectively. Using (13), it follows that

$$\begin{aligned} \Delta u \simeq & \left(\sum_{j=1}^{N_A} w_j \frac{\partial A_j}{\partial t} - \sum_{j=N_A+1}^{N_A+N_L} w_j \frac{\partial L_j}{\partial t} \right) \Delta t + \sum_k \left(\sum_{j=1}^{N_A} w_j \frac{\partial A_j}{\partial Y_k} - \sum_{j=N_A+1}^{N_A+N_L} w_j \frac{\partial L_j}{\partial Y_k} \right) \Delta Y_k \\ & + \frac{1}{2} \sum_k \sum_l \left(\sum_{j=1}^{N_A} w_j \frac{\partial^2 A_j}{\partial Y_k \partial Y_l} - \sum_{j=N_A+1}^{N_A+N_L} w_j \frac{\partial^2 L_j}{\partial Y_k \partial Y_l} \right) \Delta Y_k \Delta Y_l. \end{aligned} \quad (14)$$

In the case where the risk factor is the flat interest rate only, (14) becomes simple, and the necessary and sufficient condition that the price of the portfolio always increases against the change of the interest rate is the famous *Redington's immunization* (1952). However, in order to evaluate the contributions to the return of the individual risk factors during a finite period, we must use the numerical simulations because simple and useful relations such as (14) do not exist in general.

Ignoring the third term in the right hand side in (14), we get the equation

$$\Delta u - E[\Delta u] \simeq \sum_k \left(\sum_{j=1}^{N_A} w_j \frac{\partial A_j}{\partial Y_k} - \sum_{j=N_A+1}^{N_A+N_L} w_j \frac{\partial L_j}{\partial Y_k} \right) \Delta Y_k.$$

On the first order approximation, the gain of the capital during a period, Δu , is divided into $w_j \frac{\partial A_j(\text{or } L_j)}{\partial Y_k} \Delta Y_k$ of all the assets and liabilities. The analysis of profit used practically in the life insurance companies, such as Schnieper(1997), is generalized in this framework¹⁵.

¹⁵See Appendix B and C. In Appendix C, partial derivatives necessary to do this analysis are given, and the Thiele's differential equation is derived in Appendix B.

5.2 Risk analysis

For the risk analysis, we introduce a method by which the total risk of a portfolio is divided into the risk of individual assets. This idea is proposed by Litterman(1997).

We assume that the amount of risk of the portfolio, R_u , is a homogeneous function with the order one¹⁶. Then, it follows that

$$R_u = \sum_{j=1}^{N_A+N_L} w_j \frac{\partial R_u}{\partial w_j},$$

which implies that the total risk R_u can be divided into $w_j \partial R_u / \partial w_j$ for the j -th asset or liability.

In general, R_u and the marginal amount of risk $\partial R_u / \partial w_j$ are not expressed as an analytical formula, therefore, they are evaluated by the numerical calculations. However, the appropriate numerical calculation of them might be difficult because of the discontinuity of the percentile calculated by the monte carlo simulations¹⁷.

And, unfortunately, it is not the similar problem to divide the total risk into the risk of individual risk factors, because R_u is not homogeneous on the risk factors. So, in evaluating the contribution of each risk factor on the total risk, we will compare many calculated risks R_u with different artificial conditions each other. For example, the interest rate risk is estimated from the R_u s by comparing some numerical results in different σ_0 cases.

5.3 RAPM

Suppose that the contribution of risk with regard to asset or liability j is given by $w_j \partial R_u / \partial w_j$, then, we can propose a candidate for RAPM¹⁸ as the following

$$RAPM_j = \frac{w_j (E[\Delta p_j(\bar{t})] - p_j(t_0)r_{rf})}{w_j \partial R_u / \partial w_j} = \frac{E[\Delta p_j(\bar{t})] - p_j(t_0)r_{rf}}{\partial R_u / \partial w_j},$$

where $r_{rf} = -\ln[p_0(t_0, \bar{t})] / (\bar{t} - t_0)$ and $p_j(t_0)$ denotes the price per unit volume. Notice that this RAPM depends on the construction of the portfolio. In general, assets with lower RAPM should be excluded from our portfolio, and assets with higher RAPM be incorporated.

The RAPM can be used for optimizing the asset and liability allocation for the insurer. The weight vector $\mathbf{w} = (w_1, w_2, \dots, w_{N_A}, w_{N_A+1}, \dots, w_{N_A+N_L})$ of the optimal portfolio satisfies

$$\frac{\sum_{j=1}^{N_A} w_j (E[\Delta A_j(\bar{t})] - A_j(t_0)r_{rf}) - \sum_{j=N_A+1}^{N_A+N_L} w_j (E[\Delta L_j(\bar{t})] - L_j(t_0)r_{rf})}{VaR_u} \rightarrow \text{maximum},$$

subject to

$$\sum_{j=1}^{N_A} w_j A_j(t_0) - \sum_{j=N_A+1}^{N_A+N_L} w_j L_j(t_0) = u(t_0) \quad (\text{initial surplus}).$$

The above equation can be solved by the normal Lagrange multiplier method.

¹⁶For example, $E[u(\bar{t})] - G^{-1}(\alpha)$, $u(t_0) - G^{-1}(\alpha)$, and the standard deviation of $u(\bar{t})$ satisfy this assumption.

¹⁷Although we can easily calculate the partial derivatives literally, the calculated values reflect only on one or two scenarios which are selected accidentally. I'm afraid that such values are used for the following fine calculations. One of the promising methods for calculation is the non-parametric approach, however, this method might not be practical if the portfolio consists of lots of assets and liabilities, which implies that the number of independent variables are large for the non-parametric estimation. Another method called "nVaR" is proposed by Mausser and Rosen(1998).

¹⁸This RAPM belongs to RARORAC (risk-adjusted return on risk-adjusted capital).

6 Numerical examples

In this section, a very simplified model satisfying the stochastic models discussed in section 4 is proposed, and some results of numerical experiments are shown.

6.1 A simplified model

Assume that $h_j(t)$, $r(t)$ and $S_j(t)$ under P satisfy the SDEs ¹⁹

$$\begin{aligned} dh_j(t) &= b_j(t)dt + \sigma_j dz_j(t), \quad j = 1, 2, \dots, N + 2L, \\ dr(t) &= a_0(m_0 - r(t))dt + \sigma_0 dz_0(t), \\ \frac{dS_j(t)}{S_j(t)} &= [\mu_j + X_j(t)h_j(t)]dt + \delta_{0j}dz_0(t) + \delta_{N+2L+j}dz_{N+2L+j}(t) + dX_j(t), \quad j = 1, 2, \dots, M, \end{aligned}$$

where σ_j , a_0 , m_0 , and δ_{ij} are nonnegative constants, especially, $\sigma_j = 0$ ($j = N + 1, \dots, N + L$) (the mortality rate is a deterministic function of time), and μ_j is constant, $b_j(t)$ is a deterministic function of time, the correlations are constant, $\rho_{jk}(t) = \rho_{jk}$, and $\rho_{0k} = 0$ ($r(t)$ is independent of other stochastic variables). Then, $h_j(\bar{t})$ is given by

$$h_j(\bar{t}) = h_j(0) + \int_0^{\bar{t}} b_j(s)ds + \sigma_j z_j(\bar{t}).$$

Now, we assume that $E[h_j(t)]$ for the default are the hazard rate of the delayed Weibull distributions,

$$E[h_j(t)] = h_j(0) + \int_0^t b_j(s)ds = \lambda_j \gamma_j (t + \eta_j)^{\gamma_j - 1}, \quad j = 1, 2, \dots, N,$$

where λ_j , γ_j , and η_j are nonnegative constants, and assume that $E[h_j(t)]$ for the mortality and the withdrawal are given numerically.

Under \tilde{P} , we assume that $r(t)$ satisfy the SDE

$$dr(t) = (\phi_0(t) - a_0 r(t))dt + \sigma_0 d\tilde{z}_0(t), \quad t \geq 0,$$

where

$$\phi_0(t) = a_0 f(0, t) + \frac{\partial f(0, t)}{\partial t} + \frac{\sigma_0^2}{2a_0} (1 - e^{-2a_0 t}),$$

which is the extended Vasicek model proposed by Hull and White(1990) ²⁰.

Under these assumptions, the stochastic variables at the risk horizon \bar{t} are given by

$$\begin{aligned} S_j(\bar{t}) &= S_j(0)X_j(\bar{t}) \exp \left\{ \left(\mu_j - \frac{1}{2}(\delta_{0j}^2 + \delta_{N+2L+j}^2) \right) \bar{t} + H_j(0, \bar{t}) \right. \\ &\quad \left. + \delta_{0j}z_0(\bar{t}) + \delta_{N+2L+j}z_{N+2L+j}(\bar{t}) \right\}, \\ p_0(\bar{t}, T) &= R_1(\bar{t}, T) \exp \{-R_2(\bar{t}, T)r(\bar{t})\}, \\ p_j(\bar{t}, T_j) &= p_0(\bar{t}, T_j) [\delta_j + X_j(\bar{t})(1 - \delta_j)L_j(\bar{t}, T_j)P_{\bar{t}}\{\tau_j > T_j\}], \quad j = 1, 2, \dots, N, \end{aligned}$$

¹⁹ $h_j(t)$ are subject to the random walk model, $r(t)$ is to the Vasicek(1977) model, and $S_j(t)$ are to the Black-Scholes model.

²⁰According to Inui and Kijima(1998), this model is consistent with the Markovian HJM model. In this case, the market price of risk is written as $\beta(t) = (a_0 m - \phi_0(t))/\sigma_0$.

where

$$\begin{aligned}
R_1(t, T) &= \exp \left\{ \frac{\sigma_0^2}{2} \int_t^T R_2^2(u, T) du - \int_t^T \phi_0(u) R_2(u, T) du \right\}, \\
R_2(t, T) &= \frac{1 - e^{-a_0(T-t)}}{a_0}, \\
P_t\{\tau_j > T\} &= \exp \left\{ -B_j(t, T) + \frac{1}{2} S_j^2(t, T) \right\}, \\
B_j(t, T) &= E[H_j(t, T)] = (T-t)h_j(t) + \int_t^T (T-s)b_j(s) ds. \\
S_j^2(t, T) &= V[H_j(t, T)] = \frac{1}{3} \sigma_j^2 (T-t)^3.
\end{aligned}$$

Since $r(\bar{t})$, $h_j(\bar{t})$ and $H_j(0, \bar{t})$ are normally distributed on $\tau_j > \bar{t}$, $p_0(\bar{t}, T)$ and $S_j(\bar{t})$ are lognormally distributed, while $p_j(\bar{t}, T_j)$ is subject to the mixture of the lognormal distributions. And, under the same assumptions, we obtain

$$\bar{E}[m_j(t, T_j, T_q, T_{w_j})] = p_0(t, T_j) e^{-B_{q_j}(t, T_{q_j}) - B_{w_j}(t, T_{w_j}) + \frac{1}{2} S_{w_j}^2(t, T_{w_j})}. \quad (15)$$

Substituting (15) into (6), (7), (8), (9) and (10), we obtain

$$\begin{aligned}
p_j^{LJ}(t, T_j) &= S_{c,j} p_0(t, T_j) Q_j(t, T_j) + \int_t^{T_j} S_{w,j}(u) \left\{ E[h_{w_j}(u)] - \frac{1}{2} \sigma_{w_j}^2 (u-t)^2 \right\} p_0(t, u) Q_j(t, u) du \\
&\quad + S_{d,j} \int_t^{T_j} E[h_{d_j}(u)] p_0(t, u) Q_j(t, u) du - \sum_{t < t_{j_i} < T_j} (\pi_j(t_{j_i}) + l_j(t_{j_i})) p_0(t, t_{j_i}) Q_j(t, t_{j_i}),
\end{aligned}$$

where

$$Q_j(u, v) = \exp \left\{ -B_{q_j}(u, v) - B_{w_j}(u, v) + \frac{1}{2} S_{w_j}^2(u, v) \right\}.$$

The stochastic benefits $S_j(t_0, \bar{t})$ is given by (11), however, the equation is valid for a single life insurance policy. Since we consider the life insurance policy as a group of the policies which have the same characteristics of risks in this calculation, we adopt another simpler treatment; we assume that the benefits of the policies are paid from the insurer's assets at the time horizon \bar{t} only, and that the benefits for the death and for the cancellation are $S_{d,j}$ and $S_{w,j}(\bar{t})$, respectively.

6.2 Numerical results

As a preliminary results, we show some outputs such as price distributions of assets and liabilities and VaRs. Main data used for this calculation are summarized in Appendix D, although some detailed data, which are necessary for calculating the reserves and net premiums of the policies, are omitted ²¹. We hope that you will regard our numerical results as illustrations of our frameworks. Here, the interval from the present t_0 to the time horizon \bar{t} is set to be one year, and the number of simulation runs is 10,000 ²². The following five cases are mainly discussed ²³:

²¹These omitted data are modeled or calculated based on some recent Japanese data such as the mortality rate, the withdrawal rate, and so on.

²²As a random number generation tool, we use the java program written by Shoji Yumae at NLI Research Institute. The program returns the random numbers generated by the Mersenne Twister method.

²³As long as the Vasicek model is used, the larger σ_0 cannot be selected in the super-low interest rate economy in order to avoid the frequent occurrence of the negative interest rate, therefore, our numerical results are not appropriate in order to evaluate the effect of the interest rate risk.

Table 1. 5 simulation cases.

	Case A	Case B	Case C	Case D	Case E
σ_0	0	0.001	0.002	0.001	0.001
σ_w	0.0025	0.0025	0.0025	0	0.005

Figure 3. Distributions of asset, liability and surplus in Case B.

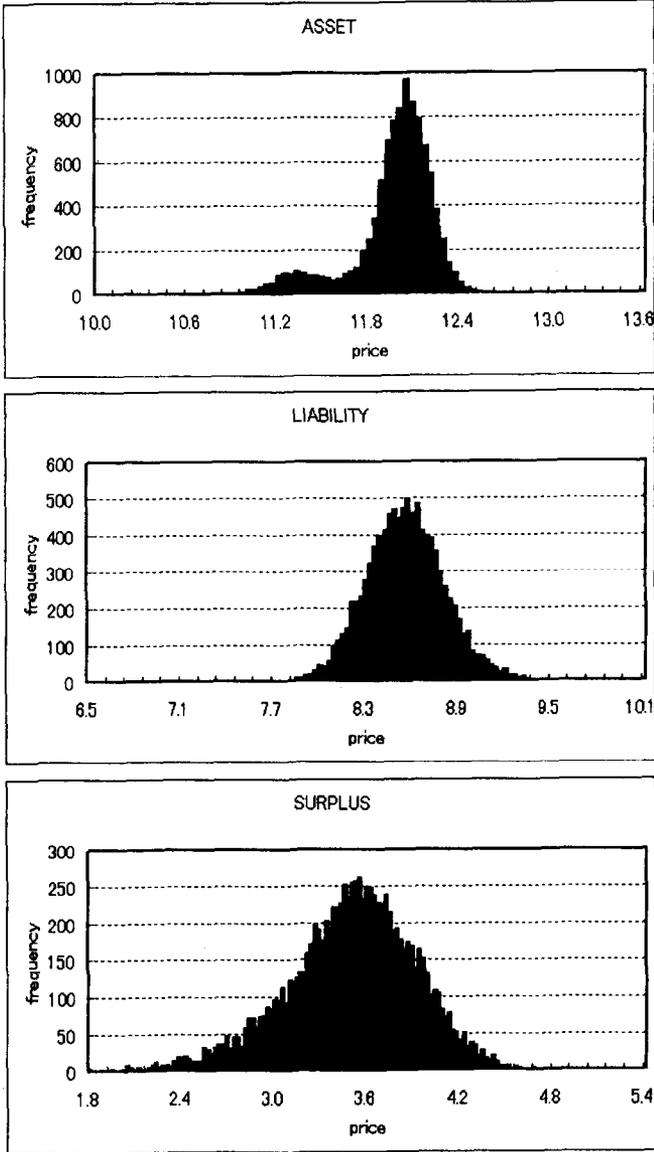


Figure 3 depicts the future price distributions of the prices of assets,²⁴ liabilities, and the surplus (assets-liabilities) in Case B. The distribution of the asset price has two peaks; the right large one corresponds roughly to the non-default scenarios, and the small left to some default scenarios. Such a bimodal distribution appears in the price distributions of defaultable bonds and the stocks, and, roughly speaking, the shape of the distribution of the asset price reflects on those of the stock price and the defaultable bond price in our five cases. This is because the amount of the stocks is small in the asset (hereafter, called small holding of stocks cases). In the larger holding of stocks cases, the left small peak would be hidden by the right large one because of the effect of the non-defaultable fluctuations of the stock price, and the shape of the distribution of the asset would become unimodal. Therefore, the dominant risk-factors for the asset price distribution are the fluctuation of the stock price including the occurrence of default, and, in particular, the occurrence of default is important for evaluation of the VaRs. On the other hand, the distributions of liabilities in five cases are all unimodal, and it does not change so much with σ_0 ; which implies that its shape is roughly determined by the fluctuation of the withdrawal rate, which we call the "withdrawal rate risk". Reflecting mainly on the distribution of the liability price, that of the surplus is also unimodal with a more gentle slope than the liability price, and it does not change so much with σ_0 . Therefore, the most dominant factor for the surplus distribution in our five cases is the withdrawal rate risk, and the next is the stock price risk including the occurrence of default²⁵.

The effect of σ_w on the distributions of the liability price and the surplus is shown in Figure 4. Only the interest rate risk is considered in Case D, while both the interest rate risk and the withdrawal rate risk are considered in Case B and E. The distribution of the liability price changes drastically with the σ_w . On the distribution of the surplus, the effect of the default becomes remarkable only in small σ_w case, while the effect of the withdrawal rate risk becomes dominant in large σ_w cases.

The return and the risk characteristics in five cases are summarized in Table 2²⁶. Notice that the benefits for the policyholders and the premiums from them between t_0 and \bar{t} , $S_j(t_0, \bar{t})$, are included in the asset price, however, they have little influence on the distribution of the asset price. Some comments described above can be confirmed in this table; for example, the standard deviation due to the interest rate risk on the liabilities is 0.0863 (Case D), while that due to the withdrawal risk is 0.2388 (Case A); the latter is three times as large as the former. Hereafter, we describe some other noticeable features only. First, see the VaRs of the assets. It seems natural that the Risk indices, such as the standard deviation and the VaRs, become large with the increase of the risk, however, the 90% VaR for the assets gives one of the counterexamples. As discussed in Kijima and Muromachi(1999), such a result can happen depending on the shape of the distribution function, therefore, we should take care of some VaRs with different confidence levels simultaneously. Second, the interest rate risk is not so remarkable on not only the assets but also the surplus. The former is because the dominant risk factor is the stock price risk, and the latter is because the interest rate risk for the liabilities is partly cancelled out by that for the assets. This cancellation effect is suggested in Case D; the risk

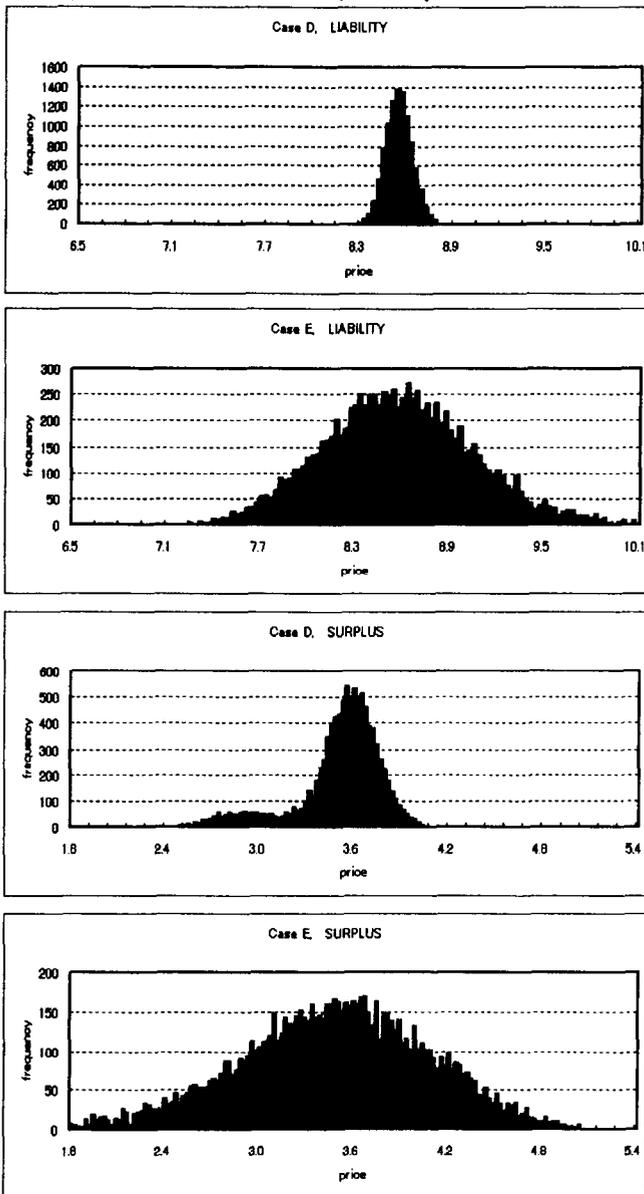
²⁴This "asset" price includes the contribution of the benefits and the premiums from t_0 to \bar{t} .

²⁵Notice that this result depends strongly on the composition of the assets. In reality, since the volume of stocks in the assets is much larger in the Japanese life insurance companies, the stock price risk would become more dominant than in our cases.

²⁶In these simulations, the VaR at a certain confidence level α is defined as the absolute difference between the mean value of the future price and the percentile. For the assets and the surplus, the percentile is $100(1 - \alpha)$ -percentile, while it is 100α -percentile for the liabilities. And, in Table 2 and Table 3, "return" means the expected return, and "s.d." means the standard deviation.

indices of the surplus are close to those of the assets²⁷.

Figure 4. Distributions of liability and surplus in Case D and E.



²⁷In spite of this cancellation effect, all the risk indices of the surplus are not smaller than those of the assets. This is because we assume the positive correlation between the interest rate and the stock prices.

Table 2. Return and risk profiles of five cases

(unit:trillion yen)

ASSET		Case A	Case B	Case C	Case D	Case E
initial price		11,7844				
future price	mean	11.9687	11.9680	11.9673	11.9680	11.9680
	return	1.56%	1.56%	1.55%	1.56%	1.56%
	s. d.	0.2853	0.2874	0.3008	0.2893	0.2856
VaR	90%	0.4411	0.4409	0.4358	0.4418	0.4385
	95%	0.6546	0.6567	0.6644	0.6588	0.6554
	99%	0.8953	0.9057	0.9362	0.9062	0.8995
LIABILITY		Case A	Case B	Case C	Case D	Case E
initial price		8.4236				
future price	mean	8.4914	8.4903	8.4892	8.4828	8.5134
	return	0.80%	0.79%	0.78%	0.73%	0.97%
	s. d.	0.2388	0.2531	0.2934	0.0863	0.4889
VaR	90%	0.3054	0.3220	0.3766	0.1105	0.6285
	95%	0.4001	0.4229	0.4991	0.1438	0.8304
	99%	0.5905	0.6350	0.7336	0.2089	1.2526
SURPLUS		Case A	Case B	Case C	Case D	Case E
initial price		3.3607				
future price	mean	3.4773	3.4777	3.4781	3.4852	3.4546
	return	3.47%	3.48%	3.49%	3.62%	3.04%
	s. d.	0.4230	0.4250	0.4290	0.2905	0.6247
VaR	90%	0.5547	0.5601	0.5670	0.4362	0.8081
	95%	0.7673	0.7639	0.7684	0.6567	1.0787
	99%	1.1484	1.1536	1.1576	0.9116	1.6307

Table 3. Returns and risks of group of assets and liabilities in Case B.

(unit:trillion yen)

		corporate bonds	JGB	stocks		ASSETS
initial price		7.2517	2.7327	1.8000		11.7844
future price	mean	7.3664	2.7796	1.9907		12.1368
	return	1.58%	1.72%	10.60%		2.99%
	s. d.	0.1377	0.0166	0.2017		0.2893
VaR	90%	0.2322	0.0211	0.2694		0.4419
	95%	0.3104	0.0267	0.4276		0.6591
	99%	0.4462	0.0386	0.5696		0.9076
		business 1	business 2	business 3	business 4	LIABILITIES
initial price		0.2027	4.8220	2.5021	0.8968	8.4236
future price	mean	0.2220	4.9819	2.5496	0.9056	8.6590
	return	9.52%	3.32%	1.90%	0.97%	2.79%
	s. d.	0.0241	0.1938	0.0478	0.0043	0.2477
VaR	90%	0.0315	0.2466	0.0610	0.0056	0.3152
	95%	0.0412	0.3302	0.0796	0.0071	0.4138
	99%	0.0614	0.4863	0.1151	0.0100	0.6234

As a reference for evaluating each asset and liability, the more detailed return/risk data in Case B are listed in Table 3²⁸. The total risk of the portfolio is much smaller than the sum of the risks of all the assets and liabilities. In this case, the stocks make a great contribution to the expected return of the assets in spite of the small fraction, and the corporate bonds are bad assets because their expected return is smaller and their risk indices are larger than the default-free bonds (JGB). In the liabilities, the main contribution to the return and the risk is made by the life business 2; the life business 1 has the highest return of them, but the contribution is small because of its small fraction.

The confidence intervals of the VaRs with 90% confidence level, $z_{0.9}\sigma_\alpha$, are roughly estimated in Case B. They are 0.0592, 0.0179, 0.0209 for $\alpha = 90\%, 95\%, 99\%$ VaRs in the asset, 0.0032, 0.0125, 0.0080 in the liability, and 0.0180, 0.0042, 0.0793 in the surplus. Some of the estimated intervals are relatively small, but some are large. The large interval appears in the range where the simulation runs are sparsely distributed.

7 Concluding remarks

In this paper we propose a new integrated risk management framework for managing the complex risks including the mortality, withdrawal, interest rate, default and equity market risks, and shows simple preliminary numerical examples as illustrations. Our simulation results imply that, in the small holding of stocks cases, the most important factors for calculating the VaRs are the default risk and the withdrawal risk; the former is dominant for the asset, the latter is for the liabilities. We cannot evaluate the interest rate risk enough, however, it might be possible that our results are regarded as the results of the portfolio in which the interest rate risk is almost hedged by some techniques such as the duration matching.

In order to manage the risks embedded in the insurance liabilities, the firm should segregate the total liabilities into several portions of the same nature and risk profile, and paste the appropriate hedging assets to them. For this purpose, it is necessary to combine the established firm's segregated accounting system with its risk management system. For the insurance blocks with huge negative margins and the thin economic loadings, there is little room to take any risk so that the classical Redington's immunization (1952) might be the best strategy. However, in this deflationary economy, the life insurance companies cannot but seek the opportunity to exploit the yield-gap gains in the corporate debt markets to some extent. It may be difficult to manage corporate credit risks which cannot be well diversifiable because the corporate itself is a customer of the company. Since Japanese life insurers have held the huge amount of many companies' stocks, the collapse of the companies means not only to abandon irrecoverable loans and defaulted bonds but also to vanish the values of the holding stocks. Under these situations, it will be an emergent task to build an integrated risk management system workable.

Also in order to embody this framework, the next important step will be to establish an accounting system to value the insurance liabilities on market value basis. Especially, the deferred acquisition expenses treatment will be of vital importance as in US GAAP accounting. Moreover, it will be a big challenge to develop a numerical technical methods to calculate VaRs and the relating Greeks efficiently and effectively. For the management purpose, how to use this framework for the performance measurement on annual- or quarterly-base will be a problem. We hope that this article will contribute

²⁸In Table 3, (life) "business" means a group of the life insurance policies, all the insured of which belong to a certain cohort. See Appendix D. In contrast to Table 2, the contribution of the benefits and the premiums are included in the corresponding life business.

to the corporate actuaries or financial officers that consider the firm-wide integrated risk management of the life insurance companies.

A Basic stochastic processes under the risk-neutral measure

In section 4, we discuss the stochastic structures under the observed probability measure P , which are necessary to make future scenarios. In this appendix, we consider the stochastic structures under the risk-neutral probability measure \tilde{P} , because it is necessary to evaluate the non-arbitrage prices of derivatives written on $S_j(t)$ and $r(t)$, if they are included in the portfolio.

Suppose that the corresponding SDEs under P are all described in section 4. According to Amin and Jarrow(1992), assuming some regularity conditions, there exists a unique equivalent martingale measure \tilde{P} under which $p_0^*(t, T) = p_0(t, T)/\rho(t)$ and $S_j^*(t) = S_j(t)/\rho(t)$ are martingale, and their prices are uniquely determined. Then, under \tilde{P} , the stochastic variables are followed by the SDEs²⁹:

$$\begin{aligned} df(t, T) &= - \sum_{i=N+2L+1}^{N+2L+K} \sigma_i(t, T) a_i(t, T) dt + \sum_{i=N+2L+1}^{N+2L+K} \sigma_i(t, T) d\tilde{z}_i(t), \\ \frac{dp_0^*(t, T)}{p_0^*(t, T)} &= \sum_{i=N+2L+1}^{N+2L+K} a_i(t, T) d\tilde{z}_i(t), \\ \frac{dS_j^*(t)}{S_j^*(t)} &= X_j(t) \tilde{h}_j(t) dt + dX_j(t) + \sum_{i=N+2L+1}^{N+2L+K+M} \delta_{ij}(t) d\tilde{z}_i(t), \quad j = 1, 2, \dots, M, \end{aligned} \quad (\text{A.1})$$

where $\tilde{z}_i(t)$ ($i = N + 2L + 1, \dots, N + 2L + K + M$) are the independent Brownian motions under \tilde{P} . Then, the stochastic variables under \tilde{P} at the risk horizon \bar{t} are given by

$$\begin{aligned} f(\bar{t}, T) &= f(0, T) - \sum_{i=N+2L+1}^{N+2L+K} \int_0^{\bar{t}} \sigma_i(u, T) a_i(u, T) du + \sum_{i=N+2L+1}^{N+2L+K} \int_0^{\bar{t}} \sigma_i(u, T) d\tilde{z}_i(u), \\ \rho(\bar{t}) &= \frac{1}{p_0(0, \bar{t})} \exp \left\{ \frac{1}{2} \sum_{i=N+2L+1}^{N+2L+K} \int_0^{\bar{t}} a_i^2(u, \bar{t}) du - \sum_{i=N+2L+1}^{N+2L+K} \int_0^{\bar{t}} a_i(u, \bar{t}) d\tilde{z}_i(u) \right\}, \\ p_0(\bar{t}, T) &= \frac{p_0(0, T)}{p_0(0, \bar{t})} \exp \left\{ \sum_{i=N+2L+1}^{N+2L+K} \int_{\bar{t}}^T \int_0^{\bar{t}} \sigma_i(u, s) a_i(u, s) du ds \right. \\ &\quad \left. - \sum_{i=N+2L+1}^{N+2L+K} \int_0^{\bar{t}} \int_{\bar{t}}^T \sigma_i(u, s) ds d\tilde{z}_i(u) \right\}, \\ S_j(\bar{t}) &= S_j(0) \rho(\bar{t}) \exp \left\{ L_j(0, \bar{t}) H_j(0, \bar{t}) - \frac{1}{2} \sum_{i=N+2L+1}^{N+2L+K+M} \int_0^{\bar{t}} \delta_{ij}^2(u) du \right. \\ &\quad \left. + \sum_{i=N+2L+1}^{N+2L+K+M} \int_0^{\bar{t}} \delta_{ij}(u) d\tilde{z}_i(u) \right\}, \quad \text{on } \bar{t} < \tau_j, \quad j = 1, 2, \dots, M. \end{aligned}$$

Some of the parameters included in these equations are determined so that the calculated present values are consistent with the current observed market prices, some are derived from the statistical analysis of the time series data, and others are assumed. For example, $\ell_j(t)$ is included in the first category. See Kijima and Muromachi(1998a, 1999) in detail.

²⁹(A.1) implies that the $S_j^*(t)$ is martingale under \tilde{P} because there exists a hazard process $\tilde{h}_j(t)$ such that $X_j(t) + \int_0^t X_j(s) \tilde{h}_j(s) ds$ is martingale.

B Net premium reserve and Thiele's differential equation

Here, we derive *Thiele's differential equation* from the viewpoint of our model.

Consider a group in which all the insured and the contracts have the same properties with the same maturity $T_j(> \bar{l})$, and define $N(t)$ as the conditional expectation of the number of contracts at time t , then, it follows that

$$\begin{aligned} d[N(t)V_j(t)] &= N(t)[dPV_1^{out}(t, T_j) + dPV_2^{out}(t, T_j) + dPV_3^{out}(t, T_j) - dPV_{net}^{in}] \\ &= r(t)N(t)V_j(t)dt - N(t)[\mu_{w_j}(t)S_{w_j}(t) + \mu_{q_j}(t)S_{d,j}]dt + \sum_{t_{ji}} N(t)\pi_j 1_{\{t < t_{ji} \leq t+dt\}}. \end{aligned}$$

Using the following relation

$$\frac{dN(t)}{dt} = -[\mu_{q_j}(t) + \mu_{w_j}(t)]N(t),$$

we obtain ³⁰

$$\frac{dV_j(t)}{dt} = \mu_{q_j}(t)(V_j(t) - S_{d,j}) + \mu_{w_j}(t)(V_j(t) - S_{w_j}(t)) + \sum_{t_{ji}} \pi_j \delta(t - t_{ji}) + r(t)V(t),$$

which means *Thiele's differential equation*.

C Partial derivatives of assets and liabilities

Under the assumptions adopted in 6.2, the partial derivatives of $p_j(t, T)$ and $p_j^{LI}(t, T)$ are given by

$$\begin{aligned} \frac{\partial p_j(t, T)}{\partial r(t)} &= -R_2(t, T)p_j(t, T), \\ \frac{\partial p_j(t, T)}{\partial h_j(t)} &= -X_j(t)(1 - \delta_j)(T - t)p_0(t, T)L'_j(t, T) \exp \left\{ -B_j(t, T) + \frac{1}{2}S_j^2(t, T) \right\}, \\ \frac{\partial p_j^{LI}(t, T)}{\partial r(t)} &= -S_{e,j}R_2(t, T)p_0(t, T)Q_j(t, T) \\ &\quad - \int_t^T S_{w,j}(u) \left\{ E[h_{w_j}(u)] - \frac{1}{2}\sigma_{w_j}^2(u - t)^2 \right\} R_2(t, u)p_0(t, u)Q_j(t, u)du \\ &\quad - S_{d,j} \int_t^T E[h_{q_j}(u)]R_2(t, u)p_0(t, u)Q_j(t, u)du \\ &\quad + \sum_{t < t_{ji} < T} (\pi_j(t_{ji}) + l_j(t_{ji}))R_2(t, t_{ji})p_0(t, t_{ji})Q_j(t, t_{ji}), \\ \frac{\partial p_j^{LI}(t, T)}{\partial h_{q_j}(t)} &= -S_{e,j}(T - t)p_0(t, T)Q_j(t, T) \\ &\quad - \int_t^T S_{w,j}(u)(u - t) \left\{ E[h_{w_j}(u)] - \frac{1}{2}\sigma_{w_j}^2(u - t)^2 \right\} p_0(t, u)Q_j(t, u)du \\ &\quad + S_{d,j} \int_t^T (1 - (u - t)E[h_{q_j}(u)]) p_0(t, u)Q_j(t, u)du \\ &\quad + \sum_{t < t_{ji} < T} (\pi_j(t_{ji}) + l_j(t_{ji}))(t_{ji} - t)p_0(t, t_{ji})Q_j(t, t_{ji}), \end{aligned}$$

³⁰ $\delta(x)$ denotes the Dirac's delta function.

$$\begin{aligned} \frac{\partial p_j^{LI}(t, T)}{\partial h_w(t)} &= -S_{e,j}(T-t)p_0(t, T)Q_j(t, T) \\ &+ \int_t^T S_{w,j}(u) \left(1 - (u-t) \left\{ E[h_{w_j}(u)] - \frac{1}{2}\sigma_{w_j}^2(u-t)^2 \right\} \right) p_0(t, u)Q_j(t, u)du \\ &- S_{d,j} \int_t^T (u-t)E[h_{q_j}(u)]p_0(t, u)Q_j(t, u)du \\ &+ \sum_{t < t_{ji} < T} (\pi_j(t_{ji}) + l_j(t_{ji}))(t_{ji} - t)p_0(t, t_{ji})Q_j(t, t_{ji}). \end{aligned}$$

In the last two equations, notice the following equation ³¹

$$\frac{\partial E[h_j(u)]}{\partial h_j(t)} = 1, \quad u \geq t, \quad j = 1, 2, \dots, N + 2L.$$

D Main data for simulation

1. Balance sheet at the start time t_0 .

(unit:trillion yen)

ASSET		LIABILITY & SURPLUS	
corporate bonds	7.2517	life business 1	0.2027
corporate stocks	1.8000	life business 2	4.8220
JGB	2.7327	life business 3	2.5021
		life business 4	0.8968
		LIABILITY	8.4236
		SURPLUS	3.3607
ASSET	11.7844	LIABILITY+SURPLUS	11.7844

These values are written in market values. More detailed compositions of assets and liabilities are shown below.

2. Composition of the holding asset.

The composition of the asset portfolio is summarized in the following table, and the unit is one trillion. The term "single" means a single asset, and "group" means an assemblage of indistinguishable assets. Therefore, the price of the former asset jumps down suddenly if the default occurs, while the price of the latter decreases gradually with the rate of default.

³¹This is obtained from $E[h_j(u)|h_j(t)] = h_j(t) + \int_t^u b_j(s)ds$ where $b_j(s)$ is given as a deterministic function.

(unit:trillion yen)

	rating	5-year bond		10-year bond		stock price	default	single/group
		face	price	face	price			
corporate A	Aaa	0.4	0.3772	0.4	0.3409	0.2	defaultable	single
corporate B	Aaa	0.2	0.1886	0.2	0.1704	0.2	defaultable	single
corporate C	Baa	0.4	0.3716	0.4	0.3308	0.2	defaultable	single
corporate D	Baa	0.2	0.1858	0.2	0.1654	0.2	defaultable	single
corporate E	B	0.4	0.3447	0.4	0.2847	0.4	defaultable	single
corporate F	B	0.2	0.1724	0.2	0.1424	0.4	defaultable	single
corporate G	Aaa	0.4	0.3772	0.4	0.3409		defaultable	single
corporate H	Aaa	0.2	0.1886	0.2	0.1704		defaultable	single
corporate I	Baa	0.4	0.3716	0.4	0.3308		defaultable	single
corporate J	Baa	0.2	0.1858	0.2	0.1654		defaultable	single
corporate K	B	0.4	0.3447	0.4	0.2847		defaultable	single
corporate L	B	0.2	0.1724	0.2	0.1424		defaultable	single
bond index 1	Aaa	0.2	0.1886				defaultable	group
bond index 2	Aa	0.2	0.1876				defaultable	group
bond index 3	A	0.2	0.1867				defaultable	group
bond index 4	Baa	0.2	0.1858				defaultable	group
bond index 5	Ba	0.2	0.1812				defaultable	group
bond index 6	B	0.2	0.1724				defaultable	group
JGB		1.5	1.4286	1.5	1.3040		non	
stock index						0.2	non	

3. Composition of the holding liabilities and premiums.

As policy liabilities, we consider the following four assemblages of life insurance policies. $S_{e,j}$, $S_{d,j}$ and the number of holders are in million, net premiums are in thousand. The ratio of economic loading means a percentage against the net premium.

	age	T_j	$S_{e,j}$	$S_{d,j}$	number of holders	net premium	ratio of loading
life business 1	32	28	2	20	1	87.64	10%
life business 2	40	20	2.5	15	4	57.53	5%
life business 3	50	10	1.25	7.5	2	4.72	15%
life business 4	57	3	0.833	5	1	5.29	15%

The net premiums and the cash surrender values $S_{w,j}(t)$ are calculated based on the current and past Japanese data, respectively. $S_{w,j}(t)$ and basic data for calculating them are omitted.

4. Hazard rate parameters for default and the recovery rates.

	Aaa	Aa	A	Baa	Ba	B
λ_j	0.00005116	0.00023357	0.00028899	0.00153925	0.01249443	2.164202
γ_j	2.0142	1.5656	1.6963	1.4221	1.1998	0.1725
m_j	0	0	0	0	0	9.721
σ_j	0.00088458	0.00099323	0.00117139	0.00205425	0.00550477	0.00800367

The correlation matrix between the credit ratings from Aaa to B is

$$(\rho_{jk}) = \begin{pmatrix} 0.8 & 0.7 & 0.5 & 0.3 & 0.2 & 0.1 \\ 0.7 & 0.8 & 0.6 & 0.4 & 0.3 & 0.2 \\ 0.5 & 0.6 & 0.8 & 0.5 & 0.4 & 0.3 \\ 0.3 & 0.4 & 0.5 & 0.8 & 0.5 & 0.4 \\ 0.2 & 0.3 & 0.4 & 0.5 & 0.8 & 0.5 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.8 \end{pmatrix}.$$

For example, the correlation coefficient of Aaa and Aa is 0.7, and that of Aaa and B is 0.1. Notice that the correlation coefficient of the companies belonging to the same rating is 0.8, but that of the same company is 1, of course. And the recovery rates δ_j ($j = 1, \dots, N$) are assumed to be 0.4.

5. Interest rate parameters and initial forward rates.

Parameters for the Vasicek model are assumed to be $a_0 = 0.02$, $m_0 = 0.01$, and $\sigma_0 = 0, 0.001, 0.002$. We call $\sigma_0 = 0$, $\sigma_0 = 0.001$, $\sigma_0 = 0.002$ as Case A, CaseB, and Case C in this paper. It is assumed that the initial forward rates $f_j(0, t)$ are given by $f_j(0, t) = f_j^0 + f_j^1 t + f_j^2 t^2$, and the parameters are given in the following table.

	default-free(JGB)	Aaa	Aa	A	Baa	Ba	B
f_j^0	0.005	0.007	0.008	0.009	0.010	0.015	0.025
f_j^1	0.002	0.002	0.002	0.002	0.002	0.002	0.002
f_j^2	-0.00003	-0.00003	-0.00003	-0.00003	-0.00003	-0.00003	-0.00003

6. Parameters for stocks.

Parameters for stocks are assumed to be $\mu_j = 0.1$, $\delta_{N+2L+j} = 0.1$, and $\delta_{0j} = 0.01$ ($j = 1, \dots, M$).

7. Mortality rate and withdrawal rate.

The mean of the mortality rate at each age used in the simulation is based on the recent Japanese Experience Table 1996 (Male), but the values of them are omitted. The mean of the withdrawal rate at each year of policy, $E[h_w(s)]$, is given as a function of the time from the policy issue, s ,

$$E[h_w(s)] = 0.09 \exp\left\{-\frac{s}{30}\right\}.$$

In the standard cases σ_q is set to be zero, while σ_w is chosen to be 0.0025 or 0.005.

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