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VALUATION OF ARITHMETIC AVERAGE CALLS:
FURTHER RESULTS[1]

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Abstract

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This paper extends the generating function approach to valuing arithmetic average fixed strike calls on a recombining binomial process. Neave (1997) and Neave-Stein (1998) define a forward induction procedure that gives both lower bounds on value and a means of finding exact solutions for European options. This paper introduces a new forward induction procedure that bounds the values of European and American options from both above and below. The new bounds, which can be tightened at the user's discretion, provide more nearly accurate valuations than other methods, especially when the underlying processes have relatively high volatilities. The solutions can be found with a relatively high degree of accuracy using just a PC and no more than a couple of minutes computing time. Since the main equipment limitation is available RAM, improved accuracy and even faster times can be obtained by installing our C++ programs on a workstation. In sum, the new methods provide practical solutions to both the European and the American valuation problems.

A Gaussian generating function helps to minimize calculations and to improve solution accuracy. First, the generating function suggests a natural way of organizing the data into sets called sub-bundles. Second, a sub-bundle may contain many paths, but all its paths are defined by a single geometric average, and attain a relatively small number of distinct arithmetic averages. Third, for valuations we only need the mean of the path arithmetic averages in a sub-bundle - unless the sub-bundle's arithmetic averages fall both above and below the strike price, in which case we say it is divided. To find either an exact solution or a refined approximation, we need to determine, or approximate as the case may be, the frequency distribution of the sub-bundle's distinct arithmetic averages. The sub-bundles which need this further investigation are easily identified and relatively few in number. Fourth, if we have enough RAM to store all the distinct arithmetic averages in the sub-bundles, our forward induction obtains an exact valuation. Fifth, if there are too many distinct averages to be stored individually, we classify them by intervals so as to create approximate frequency distributions from which upper and lower bounds on the solution can be calculated. If we need greater accuracy we introduce more classification intervals, although doing so increases computation time.

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1. INTRODUCTION

Neave (1997) uses a Gaussian generating function to structure the valuation of arithmetic average options defined on a recombining binomial asset process. Neave and Stein (1998) show that good approximations to the value of European options can be obtained.

This paper extends the Neave and Stein (1998). First, we introduce a new forward induction method[2] that both refines the earlier approximations and permits studying the tradeoff between approximation error and computation time. Second, we show how our methods extend to valuing American options. Unlike most other treatments, this paper controls any approximation error. Third we provide evidence that other methods of approximation are likely to be relatively accurate only if the underlying asset price process has low volatility.

Our methods apply to a variety of options, but this paper focuses on fixed strike average calls. The rest of this section specifies the asset price process, defines the options, and outlines the valuation methods. Section 2 elaborates some of the underlying theory. Section 3 reports computational results for European options, Section 4 for American options. Section 5 concludes.

1.2 The Process

We use a recombining binomial price process derived from an underlying Bernoulli process $\{X_t\}$, $t = 1, 2, ..., T$. Let $\{X_t\}$ be independent, identically distributed random variables:

$$X_t = \begin{cases} 0; & p \\ 1; & 1-p \end{cases} \tag{1.1}$$

Next, let

$$N_t \equiv \sum_{s=1}^{t} X_s; t = 1, 2, ..., T, \tag{1.2}$$

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[2] Hull and White (1993) and Chalasani et. al. (1998), who also use forward induction methods, reduce the numbers of state variable values using linear interpolation. We use properties revealed by the generating function to manage the number of state variable values while obtaining greater accuracy.

and

$$J_t = 2N_t - t; t = 1, 2, ..., T.$$ (1.3)

Henceforth the J_t are called *index variables*, and the realized values of the J_t, *node indices*.

The asset price process $\{S_t\}$, $t \, \varepsilon \, \{1, 2, ..., T\}$, is defined by:

$$S_t = u^{J_t}; t = 1, 2, ..., T$$ (1.4)

with $S_0 = 1$ and $u > 1$. Hereafter, the time t realized prices and their corresponding indices are called path *end points* and path *end indices* respectively.

We define the process' geometric averages using the entire price history:

$$C_t \equiv [\prod_{s=0}^{t} S_s]^{1/(t+1)}$$

$$= \prod_{s=0}^{t} [u^{J_s}]^{1/(t+1)}$$

$$= u^{V_t/(t+1)},$$
(1.5)

where

$$V_t = \sum_{s=0}^{t} J_s;$$ (1.6)

$t = 1, 2, ..., T$. The realized values of the V_t are called *index sums*. Values of the C_t can be determined from the V_t as indicated by (1.5). The arithmetic average is also defined over the entire price history:

$$A_t \equiv [\sum_{s=0}^{t} S_s]/(t+1) = [\sum_{s=0}^{t} u^{J_s}]/(t+1).$$ (1.7)

The realized values of the A_t can be determined from sequences of the realized values of J_s, $s = 0, \dots, t$.

1.3 Problem Formulation

The payoff to a European fixed strike average call with exercise date T is

$$(B_T - K)^+ \tag{1.8}$$

where B_T is a process average and $X^+ \equiv max \{ X, 0 \}$. Taking $B_T = C_T$ in (1.8) defines the call payoffs for the geometric average, taking $B_T = A_T$ defines them for the arithmetic average. Given an equivalent martingale measure Q, the time zero values of the options are

$$G_T{}^B(K) \equiv R^{-T} E_Q (B_T - K)^+ \tag{1.9}$$

where E_Q denotes expectation under Q and $R^t \equiv (1 + r)^t$ indicates the t - period accumulation of \$1 at the constant, single-period risk free interest rate r.

To see how computational difficulties arise, write (1.9) as

$$\Gamma_T^B(K) \equiv R^{-T} \sum_{j=0}^{T} p^j q^{T-j} \sum_{i=1}^{T!/(T-j)!j!} \{b_{2j-T}^i - K, 0\}^+ \tag{1.10}$$

where:

$B \; \varepsilon \; \{A_T, \; C_T\}$ indicates the type of average;
$R = 1 + r$ is a discount factor, determined by a constant interest rate r,
$p, \; q = 1 - p; \; 0 < p < 1$ indicate the martingale measure Q, and
$b_{2j-T}{}^i$ is a realized value of the average B.

Since each of the right-hand summations in (1.10) involves $T!/(T-j)!j!$ terms, the total number of paths is exponential:

$$\sum_{j=0}^{T} \binom{T}{j} = 2^T \tag{1.11}$$

Each realized average $b_{2j-T}{}^i$ may be attained by many more than one of the $T!/(T-j)!j!$ paths ending at $2j-T, \; j = 0, \dots, T$. To avoid calculating individual paths, we use a Gaussian generating function to help organize problem data; cf. Neave (1997) and Neave and Stein (1998). This paper develops a new forward induction for valuing European options, as well as a new backward induction for valuing American options. Following Neave (1997), we employ Markovian formulations whose numbers of state variable values are much smaller than the

number of individual paths.

The set of all possible paths with a given end point is called a bundle, denoted $B(t, j)$. The subset of all paths in $B(t, j)$ whose indices add to v is called a sub-bundle, and denoted $B(t, j, v)$. By definition, the paths in a sub-bundle all attain the same geometric average. Our valuations are based on organizing the 2^T paths of the problem into $T(T+1)/2 + 1$ sub-bundles, then rewriting (1.10) as

$$\Gamma_T^G(K) = R^{-T} \sum_{j=0}^{T} p^j q^{T-j} \sum_{v=v_{t,j}^-}^{v_{T,j}^+} \{c_{T,2j-T,v} - K, 0\}^+ g_{T,2j-T,v} ,$$

(1.12)

where in addition to previously used variables:

$c_{T,j,v}$ is a sub-bundle geometric average (a realized value of C_T), i.e.
$c_{T,j,v} \equiv u^{v/T}$; v a component value of $v_{T,j}$,
$v_{T,j}^-$ and $v_{T,j}^+$ are extremal values of $v_{T,j}$, and
$g_{T,j,v}$ is the number of paths in $B(T, j, v)$.

Formulation (1.12) repeats some of the component values of v_T, but even with this duplication the number of distinct records needed for exact valuation of the geometric average call is

$$\sum_{j=0}^{T} [j(T-j)+1](T+1) = (T^3 + 5T + 6)/6.$$

(1.13)

To help interpret the approach, Figure 1 shows the relations between a bundle, its sub-bundles, and the behaviour of path arithmetic averages within sub-bundles. Each cell represents a path, and each (horizontal) bar of cells represents a sub-bundle. The length of a bar gives the number of paths in the sub-bundle; the cell height indicates the arithmetic average of that path. The different cell heights within a bar indicate the distinct arithmetic averages attained by the paths in a sub-bundle.

Apart from the values of the arithmetic averages, the remaining information conveyed by the graph can be obtained analytically, and all these features are invariant with respect to volatility.[3] Representing the fixed strike price by a plane parallel to x-y plane of Figure 1 suggests how valuation can be simplified. First, recall that each sub-bundle attains only a single geometric average, which shows why valuing the geometric average instrument is a relatively easy task. Building on this observation, the sub-bundle means of the arithmetic averages would appear to offer a good approximate solution for the arithmetic average instrument. Second, using

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[3] In Figure 1, the different averages are attributable to process volatility of $\sigma = 0.80$.

sub-bundle means would only introduce error if the strike price plane actually lay between different altitudes in a sub-bundle, and such conditions would appear to arise relatively infrequently. Third, the methods of linear interpolation used by Hull and White (1993) and more recently by Chalasani et. al. (1998) appear capable of introducing errors whose magnitude increases with volatility. All three of these conjectures are supported by the results we report below.

Since by definition all paths in $B(t, j, v)$ have the same probability of occurrence, the mean of the paths' arithmetic averages is

$$m_{t,j,v} / [n_1 m^1_{t,j,v} + \dots + n_g m^g_{t,j,v}] / n,$$

where n_i is the number of paths attaining the arithmetic average $m^i_{t,j,v}$, and

$$\sum_{i=1}^{g} n_i = g_{t,j,v} .$$

(1.14)

Section 2 shows that the value of the arithmetic average call is bounded from below by:

$$\hat{\Gamma}_T^A(K) = R^{-T} \sum_{j=0}^{T} p^j q^{T-j} \sum_{v=v_{T,j}^-}^{v_{T,j}^+} \{\mu_{T,2j-T,v} - K, 0\}^+ g_{T,2j-T,v} .$$

(1.15)

Approximation (1.15) can be interpreted in terms of Figure 1 as ignoring any height differences within the bar; i.e., making use of the sub-bundle means of the arithmetic averages.

To find an exact (or nearly exact) solution, we must find (or approximate) the frequency distributions of arithmetic averages within divided sub-bundles, a task that is much less burdensome than enumerating individual paths. For example, when $T = 20$, the problem's largest sub-bundle contains 5448 paths which attain 52 distinct arithmetic averages; cf. Neave and Stein (1998).

To see how frequency distributions are used, let $m^i_{T,j,v}$; $i = 1, 2, \dots, g$; $1 \leq g \leq g_{T,j,v}$, denote the distinct arithmetic averages attained by the paths in $B(T, j, v)$. Conditional on j and v, denote the ordered averages by:

$$m^1_{T,j,v} \leq \dots \leq m^g_{T,j,v.}$$

(1.16)

Let

$$k = \arg \min_n \{m^n_{T,j,v} : m^n_{T,j,v} > K\}$$

(1.17)

Determining the exact contribution to option value of the paths in a divided sub-bundle $B(T, j, v)$ then involves replacing the terms $\{\mu_{T,j,v} - K\}^+ g_{T,j,v}$ in (1.15) with the terms

$$(m^+_{T,j,v} - K)(n_k + ... + n_g),$$ (1.18)

where

$$m^+_{T,j,v} \equiv (n_k m^k_{T,j,v} + ... + n_g m^g_{T,j,v}) / (n_k + ... + n_g).$$ (1.19)

Finding the $m^+_{T,j,v}$ involves enumerating combinatorial possibilities, and recent computational experience indicates that it is no more time consuming to use our new recursion and compute upper and lower valuation bounds instead. Moreover, the new recursion adapts readily to valuing American options. We develop both these approaches below.

2. THEORETICAL RESULTS

We call the combination of an upper and lower valuation bound a set of bounds. The difference between an upper and a lower bound is called a gap, which measures the maximum approximation error created by the method of calculating the bounds. We describe the bounds themselves being loose or tight, and choosing between them involves tradeoffs. Tight bounds can be made progressively tighter, but loose bounds cannot. Loose bounds are computationally cheaper to generate, and a loose upper bound will be relatively less accurate than its tight counterpart. However, our loose lower bound is about as good its tighter counterparts, and in valuing European options it is computationally cheaper to employ a loose lower bound irrespective of whether we use a tight or a loose upper bound. On the other hand, we employ tight lower and upper bounds when valuing American options, because our backward recursion uses the data that generate those bounds.

2.1 Loose Lower Bound

Neave and Stein (1998) show that (1.15) defines the loose lower bound. For convenience, their lemma is reproduced here.

Lemma: Choose any sub-bundle $B(t, j, v)$. Then its contribution to approximation error, given by the difference between the exact solution and (1.15), is non-negative:

$$\sum_{i=1}^{g} n_i (m^i - K)^+ p^j q^{t-j} - n(\mu - K)^+ p^j q^{t-j} \geq 0.$$ (2.1)

Proof: i) If either $K \neq m^1_{t,j,v}$ or $m^g_{t,j,v} \neq K$, it follows immediately from the definition[*] of μ that (2.1) holds with equality.

ii) Now suppose $m^1_{t,j,v} < K < m^g_{t,j,v}$, and rewrite the contribution to approximation error as

$$\sum_{i=1}^{g} n_i (m^i - K)^+ p^j q^{t-j} - \{\sum_{i=1}^{g} n_i (m^i - K)\}^+ p^j q^{t-j} \geq 0.$$

(2.2)

Negative terms are valued as zero in the left-hand summation, but not in the right-hand summation. Hence the inequality follows.

2.2 Loose Upper Bound

Our loose upper bound, defined using a quadratic approximation, is developed in Ye (1998). Let $B(t,j,v)$ be a divided sub-bundle, and let m be a distinct arithmetic average in $B(T,j,v)$. That is,

$$m = \frac{1}{T+1} \sum_{t=0}^{T} S_t .$$

(2.3)

Define the squared average of m as

$$\|m\|^2 = \frac{1}{T+1} \sum_{t=0}^{T} S_t^2 .$$

(2.4)

Then define

$$f(m,K) = a(K)(\|m\|^2 - 2m^1_{t,j,v} + (m^1_{t,j,v})^2) + b(K)(m - m^1_{t,j,v}),$$

(2.5)

where,

$$a(K) = \min\{m^g_{t,j,v} - K, K - m^1_{t,j,v}\}/(\|m^g_{t,j,v}\|^2 - 2m^g_{t,j,v}m^1_{t,j,v} + (m^1_{t,j,v})^2)$$

(2.6)

$$b(K) = (m^g_{t,j,v} - K - a(K))/(m^g_{t,j,v} - m^1_{t,j,v}).$$

(2.7)

Lemma: Given any strike price K, then $f(m,K) \geq \max(m-K,0)$, for any m in a divided sub-bundle.

Proof: A proof can be obtained from Ye at the e-mail address given on the cover page.

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[*]Although (2.1) specifies a given sub-bundle, the subscripts t,j,v are suppressed to eliminate notational clutter.

We then define an upper bound by:

$$UB(T, j, v; K) = a(\upsilon_{T,j,v} - 2m^1_{t,j,v} M_{T,j,v} + (m^1_{t,j,v})^2) + b(M_{T,j,v} - m^1_{t,j,v}),\qquad(2.8)$$

where

$$\upsilon_{T,j,v} = E[\|m\|^2 \mid B(T,j,v)]$$
$$= \frac{1}{T+1}\sum_{t=0}^{T} E_{T,j,v}(S_t^2)\qquad(2.9)$$

and the expectation is taken over all paths in the sub-bundle. The $\upsilon_{T,j,v}$ can be calculated from the same data as the $\mu_{T,j,v}$. Section 3 shows the loose upper and lower bounds of this section provide a more nearly accurate estimate of option value than do the point estimates of Hull and White (1993), and without incurring additional computation costs.

2.3 Tight Upper And Lower Bounds

Our tight upper and lower bounds are calculated using a new forward recursion. First let

$$H_t \equiv (t+1)A_t,$$

Where A_t is defined in (1.7). Since

$$H_{t+1} = H_t + u^{J_t},$$

the values of H_t for all price paths ending at index J_t have the same time t increment. Examining the Gaussian generating function suggests calculating these values using the forward induction scheme illustrated next.

Forward induction scheme

$B(t-1,\ j+2,\ v)$	
	$B(t,\ j+1,\ v+j+1)$
$B(t-1,\ j,\ v)$	

The scheme displayed in the inset applies at time t-1 to values of j such that $-t+1 < j < t-1$. If $j = -t+1$, the lower branch of the figure is ignored in performing the forward induction, since the lowest permissible index value at time t will be $j = -t$. Correspondingly, if at time t-1 the index $j = t-1$, the upper branch of the figure is ignored in carrying out the induction forward to time t. If there are m distinct values of H_{t-1} in $B(t-1, j+2, v)$, and n distinct values of H_{t-1} in $B(t-1, j, v)$, then the maximal number of distinct values in $B(t, j+1, v)$, will be $m + n$. We need only tabulate the distinct averages and their frequencies.

The tabulation procedure is organized to combine ordered vectors of distinct averages and their frequencies. This procedure means that existing distinct averages need only be scanned once in each forward inductive step, improving computation time. When we calculate approximate solutions, we limit the number of distinct averages stored. (This limit is subsquently indicated by the value of the parameter L.) Rather than forcing the averages into a grid, we use existing distinct values and combine frequencies in an approach that minimizes the approximation errors created thereby. Suppose, for example, we have a vector of distinct averages $\mathbf{H}$ ($\equiv \mathbf{H}_{t,j,v}$) and its frequencies $\mathbf{F}$ ($\equiv \mathbf{F}_{t,j,v}$). To reduce the number of stored values by one[5], we first find the smallest frequency[6]. We then combine the smallest frequency with the adjacent[7] larger frequency if defining an upper bound, the adjacent smaller frequency if defining a lower bound.

The scheme is shown in the following inset. Let h_{n-1}, h_n and h_{n+1} be components of $\mathbf{H}$, arranged in increasing order, while f_{n-1}, f_n and f_{n+1} are the corresponding (positive) frequencies in $\mathbf{F}$. If we do not wish to carry the value h_n forward to the next iteration, the upper bound calculation proceed as shown next. The analogue for the lower bound calculation is obvious. Note that the manner in which the scheme is defined gives an immediate proof that with enough storage space, the upper and lower limits will both converge to the exact solution. With enough storage space, the exact frequency distribution of the distinct averages is stored for each sub-bundle[8]

Original stored values	h_{n-1}	h_n	h_{n+1}
Original frequencies	f_{n-1}	f_n	f_{n+1}
New frequencies for upper bound calculation	f_{n-1}	0	$f_n + f_{n+1}$

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[5] The amendments if we wish to make more reductions are obvious.

[6] In the case of a tie, we take the frequency associated with the smallest average.

[7] If the chosen frequency is a maximum or a minimum we select the second smallest frequency instead, thus ensuring that we always have the value of the maximal and minimal averages in s asub-bundle.

[8] Unlike our calculations using loose bounds, we do not here distinguish between divided sub-bundles and those which are not divided. Keeping track of the difference requires as much computation time as generating the frequency distributions for all sub-bundles.

We use the above recursion and the time *t-1* frequency distributions of distinct arithmetic averages inductively to generate the time *t* frequency distributions. By design, the recursions are efficient with respect to calculating and storing the frequency distributions of distinct arithmetic averages. However, when T is sufficiently large the number of distinct arithmetic averages can become too great to store in random access memory. Moreover, writing values to disc and then reading them back increases computation time considerably. Computation times can be improved by approximating the frequency distributions with upper (and lower) distributions. We generate the upper (and lower) distributions by respectively increasing (decreasing) some of the averages so they equal other observed values. This strategy keeps the computation times reasonable, keeps the number of recorded distinct averages below a specified maximum, and bounds the true value of the option from above (below). We define the maximal number of distinct averages stored to be L. Increasing values of L imply both increased computational accuracy and increased computation time.

3. COMPUTATIONAL RESULTS: EUROPEAN OPTIONS

This section examines the comparative performance of our methods. First, we consider tradeoffs between accuracy and computation time. Second, we compare the present method's accuracy with that of Hull and White (1993). Third, we examine the effect of volatility on both option values and the accuracy of the approximations.

3.1 Accuracy and Computation Time

Since our methods exploit the structure of the data, they are both computationally more efficient and more nearly accurate than competing methods.[9] The speed and accuracy of our two sets of bounds are compared in Table 1. If the user can tolerate errors in the fourth significant digit, the loose bound approach is particularly suitable. For example, it might be employed when many options needed to be valued quickly. On the other hand, the tight bound approach might be preferred in the case of a single option with a relatively large value. Figure 3 shows how increasing L tightens the bounds.

The increase in computation time needed to obtain tight bounds is not inconsequential, both because calculating tighter bounds involves more floating point calculations and because we are only using a PC with a Pentium chip rather than a work station. Using a work station with say, 200MB of RAM can improve computation times substantially over those reported here. Regardless, by changing L we can find a desired trade-off between reasonable computation times and reasonable approximation errors for whatever equipment is used.

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[9] We can only offer direct comparisons for the volatilities reported in the literature, and these are relatively low in relation to the ones we can and do study.

Table 1 European Asian Call Option Prices and Computing Times

T	Loose Lower Bound	Loose Upper Bound	Gap	CT	Loose Lower Bound	Tight Upper Bound	Gap	CT
6	0.136519	0.136519	0.000000	0	0.136519	0.136519	0.000000	0
12	0.137026	0.137048	0.000022	0	0.137026	0.137036	0.000010	0
18	0.137213	0.137265	0.000052	0	0.137213	0.137231	0.000018	1
24	0.137322	0.137422	0.000100	0	0.137322	0.137355	0.000033	4
30	0.137391	0.137543	0.000152	1	0.137391	0.137442	0.000051	14
36	0.137439	0.137645	0.000206	1	0.137439	0.137502	0.000063	31
42	0.137474	0.137735	0.000261	2	0.137474	0.137560	0.000086	65
48	0.137501	0.137816	0.000315	3	0.137501	0.137605	0.000104	118
54	0.137522	0.137894	0.000372	6				
60	0.137538	0.137965	0.000427	10				
66	0.137552	0.138035	0.000473	18				
72	0.137564	0.138104	0.000540	28				
78	0.137574	0.138170	0.000596	44				
84	0.137582	0.138235	0.000653	68				
90	0.137590	0.138300	0.000710	95				
100	0.137600	0.138405	0.000805	170				

Notes:
1) Initial stock price $1.00, strike price $1.00, risk-free interest rate 0.10 per annum, stock price volatility 0.40 annually, time to maturity 1.5 years. CT is computation time in seconds. In calculations of the tight bounds, L = 100.
2) Valuation bounds are calculated using a C++ program in a Pentium 200 computer. The gaps are the differences between the upper and lower bounds.

3.2 Comparisons of Accuracy

Table 2 compares the accuracy of our methods to those of Hull and White (1993). Not only can our loose bounds be calculated very quickly, but for low volatilities they produce roughly the values as the methods of Hull and White. However, Hull-White use linear interpolation of averages' value, and our investigations show that the distribution of averages is both nonlinear and relatively irregular when compared, say, to the lognormal[10] (cf. Figures 1 and 2). Thus for higher volatilities we would expect our methods to give greater accuracy than can linear

[10] Of course, the sum of lognormally distributed variables is not itself lognormal. Our comparison is intended only to show how very irregular the discrete distribution can be.

Table 2 Comparisons of Accuracy

Years to Maturity	Strike price	40	45	50	55	60
0.5	H&W upper bound	10.755	6.363	3.012	1.108	0.317
						0.3164
	loose upper bound	10.75480	6.36145	3.00839	1.10557	7
		10.75	6.360	3.007	1.104	
	tight upper bound	455	78	62	56	0.31552
		10.75	6.360	3.007	1.104	0.314
	loose lower bound	439	44	33	13	99
1.0	H&W upper bound	11.545	7.616	4.522	2.420	1.176
						1.177
	loose upper bound	11.54539	7.61578	4.52185	2.42008	70
		11.54	7.613	4.519	2.41	1.1747
	tight upper bound	415	98	84	750	1
		11.54	7.613	4.519		1.173
	loose lower bound	351	15	00	2.41632	09
1.5	H&W upper bound	12.285	8.670	5.743	3.585	2.124
		12.28	8.671	5.7	3.588	2.128
	loose upper bound	678	73	4489	69	60
			8.668	5.741	3.584	2.123
	tight upper bound	12.28435	75	41	36	40
			8.667	5.739	3.582	2.12
	loose lower bound	12.28320	37	89	21	063
2.0	H&W upper bound	12.953	9.582	6.792	4.633	3.057
				6.7	4.638	3.064
	loose upper bound	12.95718	9.58589	9627	77	87
		12.95	9.581	6.791	4.632	3.0575
	Tight upper bound	358	56	28	51	6
		12.95	9.57	6.789	4.62	3.053
	Loose lower bound	186	961	05	958	38

Notes: The initial stock price is $50, the risk-free interest rate is 10% per year, the stock price volatility is 30% per year. The 40-time-steps binomial tree model is used in computations. In calculations of the tight bounds, $L = 100$. H&W upper bounds are quoted from Hull and White(1993). This Table uses loose rather than tight lower bounds, since they are very nearly the same. The interested reader can compare the loose lower bounds in this Table with the corresponding tight lower bounds reported in the next Table.

interpolations, and the subsequent data show this generally to be true, especially for deep in the money options where the Hull-White (1993) method creates relatively large errors. Hull-White offer no means of controlling approximation error, although the extensions of Chalasani et. al. (1988) do.

In addition, the Hull-White methods create systematic biases, as indicated in Table 2a. In contrast, our methods define intervals within which the option value is guaranteed to lie, and while these intervals increase with time and volatility, there does not appear to be any other systematic bias in them. Moreover, our interval estimates can be sharpened by using the tight bound approach and high values of the parameter L. Finally, the cost of obtaining the better estimates is a modest an increase in computation time, as Table 1 already showed.

Table 2a: Hull-White Upper Bound Relative to Our Tight Upper Bound

Years/Strike	40	45	50	55	60
0.5	1.000042	1.000349	1.001456	1.003114	1.004691
1.0	1.000074	1.000265	1.000476	1.001034	1.001618
1.5	1.000053	1.000144	1.000277	1.000179	1.000283
2.0	0.999955	1.000046	1.000106	1.000106	0.999817

3.3 Accuracy and Volatility

Table 3 shows how both the option values and the accuracy of our bounds are affected by increases in volatility. There are no similar published results concerning the effects of higher volatility on the valuations obtained by competing methods. Figure 4 shows how our methods' approximation errors compare to the differences in value from modest changes in volatility. These results suggest to us that greater payoffs to future research are more likely to come from improving volatility estimates rather than from further attempts to improve the accuracy of our bounds.

Table 3 Accuracy and Volatility for Tight Bounds

T	Volatility 0.4			Volatility 0.8		
	Tight Lower Bound	Tight Upper Bound	Gap	Tight Lower Bound	Tight Upper Bound	Gap
6	0.136519	0.136519	0.000000	0.231945	0.231945	0.000000
12	0.137036	0.137036	0.000000	0.232966	0.232966	0.000000
18	0.137231	0.137231	0.000000	0.233474	0.233474	0.000000
24	0.137339	0.137355	0.000016	0.233682	0.233829	0.000147
30	0.137399	0.137442	0.000043	0.233796	0.234085	0.000289
36	0.137441	0.137502	0.000061	0.233822	0.234327	0.000510
42	0.137476	0.137560	0.000084	0.233901	0.234570	0.000669
48	0.137502	0.137605	0.000103	0.233963	0.234853	0.000890

Notes: The initial stock price is $1, the strike price is $1, the risk-free interest rate is 10% per year, time to maturity is 1.5 years, and L = 100.

3.4 Stochastic Volatility Options

Since our methods can generate both upper and lower bounds for given volatility, we can provide a range within which values would fall if volatility were stochastic. Since option values increase with volatility, all we need do is take the upper bound for the maximum volatility, and the minimum bound for the minimum attainable volatility, to obtain a quick estimate of the values that could be attained by a stochastic volatility option. While these values might not be very precise, they are computationally much cheaper than recognizing the effects of stochastic volatility more systematically. Moreover as we have already pointed out, getting the volatility process correct is highly important, and yet that problem has probably received less attention than that of calculating valuations.

4. COMPUTATIONAL RESULTS: AMERICAN OPTIONS

Since our methods for calculating tight bounds use forward recursion to generate the necessary data, we can readily adapt them to develop a backward recursion to value American options. Table 4 reports computing accuracy and speed are reported. The time steps are from 6 up to 30, and L=100 and 200. For time steps not greater than 18, with L = 100, we get exact values, which we specify to the fifth significant digit. For time steps equal to 24 or 30, with L=200, we get fourth significant digit accuracy. Computing times are small, even on a PC. For example, when 24 time steps are used and L=100, we value the American call in one minute, and our solution is accurate to the fourth significant digit.

Table 4 American Asian Call Option Prices and Computing Times

T	L=100				L=200			
	Lower bound	Upper bound	Gap	CT	Lower bound	Upper bound	Gap	CT
6	0.141268	0.141268	0.	0				
12	0.146479	0.146479	0.	1				
18	0.148574	0.148633	0.000059	5				
24	0.149445	0.150294	0.000849	60	0.149670	0.149981	0.000311	165
30	0.150001	0.151745	0.001744	160	0.150395	0.151168	0.000773	980

Notes: Initial stock price $1.00, strike price $1.00, risk-free interest rate 0.10 per annum, stock price volatility 0.40 annually, time to maturity 1.5 years.

Table 5 compares the accuracy of our method[11] with Hull and White (1993) and Chalasani, et al. (1998). Table 5 shows that our upper bound calculated with L=100 is similar to H&W with h=0.005; and our upper bounds calculated with L ≥ 150 are tighter than H&W with h=0.003. Next, we get both the upper bounds and lower bounds, while Hull and White only get upper bounds. Furthermore, when L=300 is used, the bounds converge to the exact value, as by design they must. Since Hull and White do not indicate computing times, we cannot offer this type of performance comparison. In comparison to those obtained in Chalasani, et al. , our upper bound with L ≥ 150 and lower bound with L ≥ 200 are tighter. In other words, we get a more nearly accurate solution for the T=20 problem in only 15 seconds on a PC.

Table 5 Comparisons of Accuracy: American Asian Call

Tight Bounds					
L	100	150	200	250	300
Upper bound	4.81552	4.81343	4.81273	4.81258	4.81248
Lower bound	4.80988	4.81161	4.81225	4.81240	4.81248
Gap	0.00564	0.00182	0.00048	0.00018	0.00000
CT	9	12	15	20	47
Chalasani, et al. Bounds					
Upper bound	4.814				
Lower bound	4.812				
Gap	0.002				
H&W Upper Bounds					
H	0.1	0.05	0.01	0.005	0.003
Upper bound	5.197	4.971	4.823	4.815	4.814

Notes: The initial stock price is $50, the strike price is $50, the risk-free interest rate is 0.10 per annum, $\sigma = 0.30$, T = 20. H&W upper bounds are quoted from Hull and White(1993). Chalasani, et al. bounds are quoted from Chalasani, et al. (1998).

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[11] For reasons of space, we only compare our method with the two indicated. Many other solutions have also been proposed, as our references show.

5. CONCLUSION

This paper provided upper and lower bounds on the values of both European and American arithmetic average fixed strike calls. For European options, our loose bounds are better than those in the literature and as quick or quicker to calculate. Our tight bounds, calculated using forward induction, take longer to calculate but can obtain as close an approximation to the true value as desired. Our values for American options are more nearly accurate than those of competing methods and are also obtained with only small amounts of computing time.

References

Chalasani, Prasad, Feyzullah Egriboyun, Somesh Jha and Ashok Varikooty, "A Refined Binomial Lattice for Pricing Asian Options," Eighth Annual Derivative Securities Conference, Boston University, 1998.

Curran, M., "Beyond Average Intelligence," Risk, November 1992, p. 60.

Geman, Helyette, and Marc Yor, "Bessel Processes, Asian Options, and Perpetuities," Mathematical Finance 3, 1993, 349 - 375.

Hull, John, and Alan White, "Efficient Procedures for Valuing European and American Path-Dependent Options," The Journal of Derivatives, Fall 1993, 21-31.

Levy, Edmond, "A Note on Pricing European Average Options," Nomura Bank International Working Paper, 1991.

Levy, Edmond, and Stuart M. Turnbull, "Average Intelligence," Risk, 1992.

Neave, Edwin H., "A Frequency Distribution Method for Valuing Average Options," ASTIN Bulletin 27, 1997, 173 - 205.

Neave, Edwin H., and Christopher A. Stein, "Exact and Approximate Solutions for Arithmetic Average Fixed Strike Calls," Presented to Eighth Derivative Securities Conference, Boston University, 1998.

Ritchken, Peter, L. Sankarasubramanian, and A. M. Vijh, "The Valuation of Path Dependent Contracts on the Average," Management Science 39 (10), 1993, 1202-1213.

Rogers, L. C. G. and Shi, Z., "The Value of an Asian Option," Queen Mary and Westfield College, University of London, 1993.

Tilley, James A., "Valuing American Options in a Path Simulation Model," Actuarial Approach For Financial Risks, vol. 1, pp. 249 - 266, Third AFIR International Colloquium, Rome, 1993.

Turnbull, Stuart M. and Lee Macdonald Wakeman, "A Quick Algorithm for Pricing European Average Options," Journal of Financial and Quantitative Analysis 26, 1991, 377 - 389.

Ye, George L., "Pricing Asian Options Using A Path Bundling Technique," PhD Dissertation, Queen's School of Business, Queen's University, 1998 (in progress).

Yor, M., "On Some Exponential Functions of Brownian Motion," Advances in Applied Probability 24, 1992, 509-531.

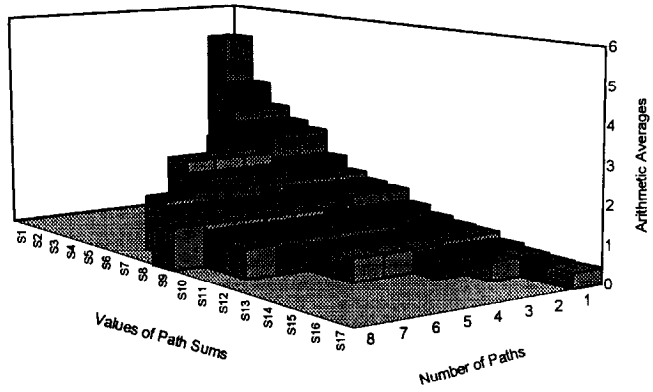Figure 1: A Bundle and its Sub-bundles for B(8, 4); $\sigma = 0.80$.
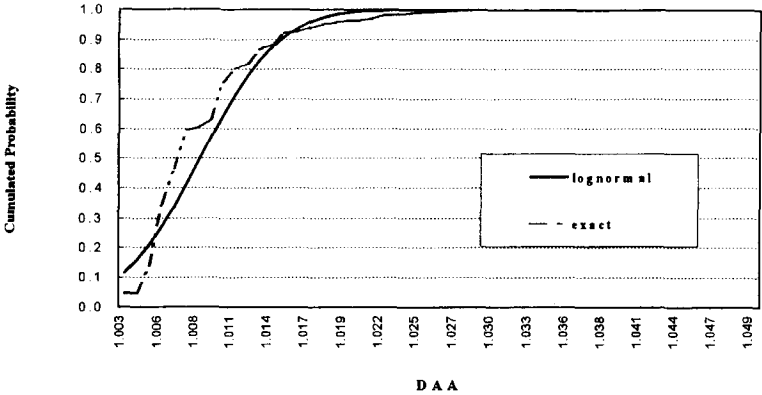
Figure 2 The Cumulated Probabilities of The Exact Distinct Arithmetic Averages Distribution in B(20,10,105) and The Lognormal Distribution
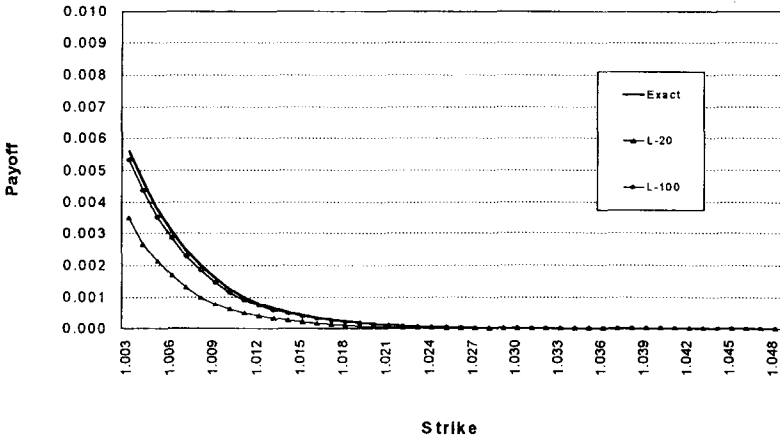

Notes: The cumulated probabilities of the exact didtinct arithmetic average(DAA) is calculated by using our forward induction procedure; the lognormal distribution is one with the same mean and variance as those of the exact distribution.
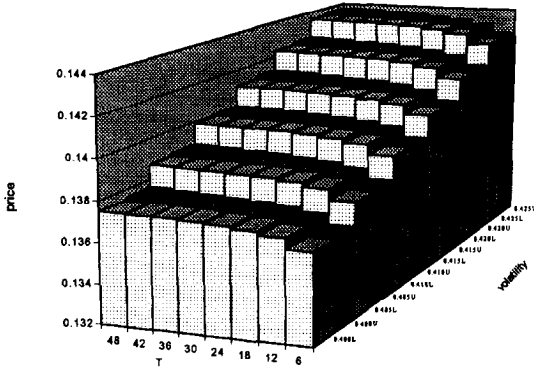
Figure 3 Upper Bounds and Lower Bounds in B(20,10,105)
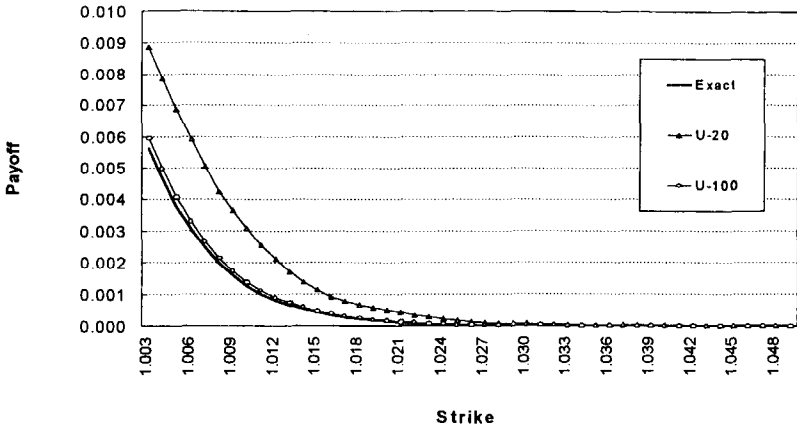
(a) upper bounds

(b) lower bounds

Notes: "Exact" is the subbundle expected payoffs as a function of the strike price "U-L" , "L-L" are the upper bound and lower bound calculated by our tight bound approach with the number of the intervals to be L, L=20, 100.

Figure 4 Asian Option Prices via Volatilities and Time-steps

Notes: The Asian option is an European call with 1.5 years to maturity. The initial stock price is $1, the strike price is $1, the interest rate is 10% per year, the volatilities from 40% to 42.5% per year are used with an increment of 0.5%. T is the number of time steps in the binomial tree, which is increased up to 48.