The following is a brief outline of some stochastic actuarial and financial models that can be used for a quantitative analysis of the economic rationale for reinsurance. Such a careful quantification of the microeconomics of reinsurance and the tangible value created for the client lies at the very heart of Swiss Re’s value proposition for reinsurance. This technical note is written for practitioners with a mathematical background. We have tried to keep the mathematics as simple and straightforward as possible. Furthermore, for the reader’s convenience, we have listed a number of reference texts for further study at the end of this note. Also, all the models are available in software form (using Microsoft EXCEL as a user interface).

We start our quantitative analysis of the economic rationale for reinsurance by considering one line of business. Here, line of business means some homogeneous block of insurance or reinsurance business, for example UK Life, US Medex, P&C Asia Pacific, etc., depending on the level of detail at which the economic rationale and the value proposition for reinsurance are to be quantified. In a first step, we develop a dynamic stochastic model that describes the evolution of claims, premiums, reserves and expenses for this line of business over time. We then focus our attention on the estimation of claims severity, the estimation of claims frequency, the variation of risk propensity due to trends (e.g., improvements in mortality rates in life insurance, worsening claims experience in some areas of health insurance, etc.) and cycles (e.g., flu epidemics, etc.), and finally, on the effect of the three major reinsurance arrangements: quota share reinsurance, surplus reinsurance and stop loss reinsurance.

On the company level, we first develop a dynamic stochastic model for the solvency margin or risk reserve which allows us to determine the capital at risk as a function of the reinsurance arrangements in force and the client’s strategic allocation of assets. Note that we do not treat assets and liabilities separately here; for both the same simple and consistent methodological framework is used. This is very much in line with today's state-of-the-art asset / liability management strategies. Furthermore, the model can be applied to quantify the effects of reinsurance (and, more generally, of asset / liability management strategies) on a cedent, an insurer and a reinsurer simultaneously. Having determined capital at risk, we show how to implement a consistent system of performance measurement (“efficient frontier” approach) for insurance and reinsurance companies and that reinsurance creates tangible value by moving the insurer’s efficient frontier in the direction of higher returns and lower risk. As a final point in this section of the technical note, we show how the same methodology can be used to (a) develop a consistent system of performance measurement

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for internal operational units of the insurer and the reinsurer and (b) develop a consistent, best practice pricing approach for insurance and reinsurance contracts.

In a final section of this document, we look at how the mathematical models used to quantify the economic rationale for reinsurance can also be used to implement new value proposition (VP) based client solutions and can therefore provide the reinsurer with new market and profit growth opportunities.

A Single Line of Business

In a first approximation, any line of business can be characterized by the following market data (we consider three specific lines of business called A, B and C for illustration purposes throughout this section):

<table>
<thead>
<tr>
<th>Lines of Business</th>
<th>Line 1</th>
<th>Line 2</th>
<th>Line 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Description</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>Earned Premium (B)</td>
<td>150.00</td>
<td>110.00</td>
<td>180.00</td>
</tr>
<tr>
<td>Sum Assured</td>
<td>1500.00</td>
<td>770.00</td>
<td>2000.00</td>
</tr>
<tr>
<td>Number of Policies</td>
<td>1000.00</td>
<td>300.00</td>
<td>1500.00</td>
</tr>
<tr>
<td>Average Sum Assured</td>
<td>1.50</td>
<td>2.57</td>
<td>1.33</td>
</tr>
<tr>
<td>Distribution of Sum Assured</td>
<td>6.91</td>
<td>8.00</td>
<td>6.00</td>
</tr>
<tr>
<td>Average Reserve (R)</td>
<td>1200.00</td>
<td>1000.00</td>
<td>1500.00</td>
</tr>
<tr>
<td>Pricing Rate ((\lambda))</td>
<td>1.00%</td>
<td>1.00%</td>
<td>1.00%</td>
</tr>
<tr>
<td>Profit Loading ((\Lambda))</td>
<td>25.00%</td>
<td>20.00%</td>
<td>25.00%</td>
</tr>
<tr>
<td>Consistency Test</td>
<td>Technical Premium (P)</td>
<td>120.00</td>
<td>91.67</td>
</tr>
<tr>
<td></td>
<td>Loss Portfolio Mean ((E[X]))</td>
<td>132.00</td>
<td>101.67</td>
</tr>
<tr>
<td></td>
<td>Claims Frequency (n)</td>
<td>88.00</td>
<td>39.61</td>
</tr>
<tr>
<td></td>
<td>Expected Claims Rate</td>
<td>8.80%</td>
<td>13.20%</td>
</tr>
</tbody>
</table>

Fig. 1: Characterization of a Line of Business (Figures in MCHF)

By just noting the basic relationships

(a) \[ \text{Technical Premium} = \frac{\text{Earned Premium}}{1 + \text{Profit Loading}} \]

(b) \[ \text{Loss Portfolio Mean} = \text{Technical Premium} + \text{Pricing Rate} \times \text{Average Reserve} \]

(c) \[ \text{Claims Frequency} = \frac{\text{Loss Portfolio Mean}}{\text{Average Sum Assured}} \]

we can then derive a simple consistency test for any stochastic model of such a line of business. This consistency test states conditions for the expected claims frequency \(E[k] = n\), the expected claims severity \(E[Z] = m\), the loss portfolio mean \(E[X] = nm\) and the corresponding technical or risk premium \(P\). The distribution of the sum assured states a further condition to be satisfied by some percentile of the claims severity distribution \(S_z(z)\).
(these conditions can indeed be used to fit a claims severity distribution, for example of the Pareto type, see further below).

Moreover, any stochastic model of a line of business probably has the following standard structure:

\[ X(t) = \sum_{i=1}^{k(t)} Z_i(t), Z_i(t) \text{ iid. } Z(t) \]

denote the compound (mixed) Poisson aggregate claim amount in time period \([t-1,t]\) of a line of business (and, more generally, any risk portfolio). Furthermore, let \( S_t(z) = S^{(t)}_Z(z) \) and \( F_t(x) = F^{(t)}_X(x) \) be the corresponding distribution functions and \( s_t(z) = \frac{dS_t(z)}{dz} \) and \( f_t(x) = \frac{dF_t(x)}{dx} \) the associated densities. We interpret the random variable \( X(t) \) to denote claims incurred in time period \([t-1,t]\).

\[(1)\] In a first approximation, we assume a Pareto severity distribution (PD)

\[ S_t(z) = S^{(t)}_Z(z) = 1 - \left( \frac{D_t + \beta_t}{z + \beta_t} \right)^{\alpha_t}, \quad z \geq D_t. \]

The shape parameter \( \alpha_t \) controls how heavy the tail of the distribution is

**Fig. 2a:** The Pareto Shape Parameter \( \alpha_t \) (\( D_t = 0, \beta_t = 1 \))

while the parameter \( \beta_t \) influences the left-hand range of the distribution and does not change its tail for \( z \gg \beta_t \).
Pareto Distribution

The Pareto Parameter \( \beta_i \) (\( D_i = 0, \alpha_i = 1 \))

and the parameter \( D_i \) defines the threshold of the distribution, i.e., the Pareto distribution is only considered for \( z \geq D_i \).

Fig. 2b: The Pareto Parameter \( \beta_i \) (\( D_i = 0, \alpha_i = 1 \))

The moments of the Pareto distribution are

(a) \( m_i = m_Z^{(i)} = E[Z(t)] = \frac{\alpha_i D_i + \beta_i}{\alpha_i - 1}, \quad \alpha_i > 1 \) (mean)

(b) \( \sigma_i = \sigma_Z^{(i)} = \sqrt{\frac{\alpha_i (D_i + \beta_i)^2}{(\alpha_i - 1)^2 (\alpha_i - 2)}}, \quad \alpha_i > 2 \) (standard deviation)
(c) \( \gamma_z = \gamma_z^{(t)} = 2 \frac{\alpha_t + 1}{\alpha_t - 3} \sqrt[\alpha_t - 2]{\alpha_t}, \alpha_t > 3 \) (skewness)

while the limited expected value (LEV) function is

\[
L_t(M) = L_t^{(t)}(M) = E\left[ \min\{M, Z(t)\} \right] = \int_0^M zdS_t(z) + M\left[1 - S_t(M)\right]
\]

(d) \[
\frac{\alpha_t D_t + \beta_t - (M + \beta_t) \left( D_t + \beta_t \right)^{\alpha_t}}{\alpha_t - 1}, \quad M \geq D_t.
\]

---

**Fig. 3a:** The Pareto Mean \( m_t^{(t)}(D_t = 0) \)

**Fig. 3b:** The Pareto Standard Deviation \( \sigma_t^{(t)}(D_t = 0) \)
While the Pareto distribution is a simple and convenient model for claims severity, note that at the very heart of any capital at risk (or risk based capital, or risk adjusted capital) calculation (which is our main task in the quantification of the economic rationale for reinsurance and one of the main topics of this document) is the notion of an extremal event.

According to overwhelming theoretical and empirical evidence, extreme value theory (EVT) is a much more suitable, however a little bit more complex, actuarial framework for modelling extremal events (i.e., the tails of loss severity distributions). The implications on both the actual results (risk capital estimates, quantile estimates, etc.) and their stability (and reliability and trustworthiness) are profound. Therefore, we have also implemented our stochastic models in an EVT framework, see further below. The above chosen simple Pareto distribution is with the “right” choice of parameters usually a reasonably good approximation to the EVT distribution of the generalized Pareto (GPD) type.
If \( B(t) \) denotes **premium earned** in time period \([t-1,t]\), \( B'(t) \) **premium written** and \( X'(t) \) **claims paid**, respectively, and furthermore \( V(t) \) **unearned premium reserves** at time \( t \) and \( C(t) \) **outstanding claims reserves** at time \( t \), then the relationships

\[
V(t) = V(t-1) + B'(t) - B(t) \quad \text{and} \quad C(t) = C(t-1) + X(t) - X'(t)
\]

hold and

\[
T(t) = V(t) + C(t)
\]

are the **technical reserves** at time \( t \).

Following standard actuarial tradition, we also assume

\[
B(t) = [1 + \lambda(t)]P(t) + E(t) \quad \text{with} \quad P(t) = \mathbb{E}[X(t)] - i(t)R(t)
\]

the **technical or risk premium** in time period \([t-1,t]\), \( \lambda(t) \) the **profit loading coefficient**, \( E(t) \) the expense amount (**e.g., operating expenses, taxes, dividends, etc.**), \( i(t) \) the **pricing rate** and \( R(t) > T(t) \) the **total reserves**, respectively.

<table>
<thead>
<tr>
<th>Total Reserves ([R(t)])</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Risk Reserves</strong></td>
</tr>
<tr>
<td><strong>Minimum Required Capital</strong></td>
</tr>
<tr>
<td><strong>Technical Reserves ([T(t)])</strong></td>
</tr>
<tr>
<td>- Unearned Premium Reserves ([V(t)])</td>
</tr>
<tr>
<td>- Outstanding Claims Reserves ([C(t)])</td>
</tr>
</tbody>
</table>

Fig. 4: Total Reserves

Standard actuarial methods and **statutory minimum solvency margins** determine technical reserves and minimum required capital for a line of business. The determination of the corresponding **risk reserves (on top of the minimum required capital)** is an important topic addressed in this note however, see further below.

Note that when we consider the stochastic evolution of the above quantities over just one time period, we usually drop the time index \( t \) in order to make our notation simpler.

1. **Estimation of Claims Severity: The Classical Approach**

There are **two classical actuarial techniques** of estimating claims severity:

(1) **Direct numerical estimation of the Pareto parameters** \( D, \alpha, \beta \) **from the above market data** characterizing a line of business by using EXCEL's solver routine and the simple consistency test derived from the market information provided. For our three example lines of insurance or reinsurance business A, B and C, the procedure works as follows (**the systemic risk part** has to do with **parameter uncertainty inherent in our frequency estimates** derived from the provided market data and is explained further below):
Lines of Business

<table>
<thead>
<tr>
<th>Description</th>
<th>Line 1</th>
<th>Line 2</th>
<th>Line 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earned Premium (Br)</td>
<td>150.00</td>
<td>110.00</td>
<td>180.00</td>
</tr>
<tr>
<td>Sum Assured</td>
<td>1500.00</td>
<td>770.00</td>
<td>2000.00</td>
</tr>
<tr>
<td>Number of Policies</td>
<td>1000.00</td>
<td>300.00</td>
<td>1500.00</td>
</tr>
<tr>
<td>Average Sum Assured</td>
<td>1.50</td>
<td>2.57</td>
<td>1.33</td>
</tr>
<tr>
<td>Distribution of Sum Assured</td>
<td>6.91</td>
<td>8.00</td>
<td>6.00</td>
</tr>
<tr>
<td>Average Reserve (Ri)</td>
<td>1200.00</td>
<td>1000.00</td>
<td>1500.00</td>
</tr>
<tr>
<td>Pricing Rate (i)</td>
<td>1.00%</td>
<td>1.00%</td>
<td>1.00%</td>
</tr>
<tr>
<td>Profit Loading (Lambda_i)</td>
<td>25.00%</td>
<td>20.00%</td>
<td>25.00%</td>
</tr>
<tr>
<td>Consistency Test</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Technical Premium (Pt)</td>
<td>120.00</td>
<td>91.67</td>
<td>144.00</td>
</tr>
<tr>
<td>Loss Portfolio Mean (E(Xi))</td>
<td>132.00</td>
<td>101.67</td>
<td>159.00</td>
</tr>
<tr>
<td>Claims Frequency (ni)</td>
<td>88.00</td>
<td>39.61</td>
<td>119.25</td>
</tr>
<tr>
<td>Expected Claims Rate</td>
<td>8.80%</td>
<td>13.20%</td>
<td>7.95%</td>
</tr>
<tr>
<td>Loss Portfolio (Xi)</td>
<td>Mean</td>
<td>132.04</td>
<td>99.02</td>
</tr>
<tr>
<td></td>
<td>Std</td>
<td>1162.15</td>
<td>392.66</td>
</tr>
<tr>
<td>Claims Frequency</td>
<td>Mean</td>
<td>88.00</td>
<td>39.61</td>
</tr>
<tr>
<td></td>
<td>Std</td>
<td>9.38</td>
<td>6.29</td>
</tr>
<tr>
<td>Claims Severity</td>
<td>Mean</td>
<td>1.50</td>
<td>2.50</td>
</tr>
<tr>
<td></td>
<td>Std</td>
<td>1.50</td>
<td>1.49</td>
</tr>
<tr>
<td></td>
<td>Skew</td>
<td>2.00</td>
<td>46.65</td>
</tr>
<tr>
<td>Systemic Risk (qi)</td>
<td>Mean</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>Std</td>
<td>8.80</td>
<td>3.96</td>
</tr>
<tr>
<td>Severity Distribution (Pareto)</td>
<td>Alpha</td>
<td>12886486.86</td>
<td>3.10</td>
</tr>
<tr>
<td></td>
<td>Beta</td>
<td>19035910.67</td>
<td>0.26</td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>0.00</td>
<td>1.61</td>
</tr>
<tr>
<td></td>
<td>M</td>
<td>11.00</td>
<td>12.00</td>
</tr>
<tr>
<td></td>
<td>L(M)</td>
<td>1.50</td>
<td>2.48</td>
</tr>
<tr>
<td></td>
<td>z</td>
<td>1.50</td>
<td>2.50</td>
</tr>
<tr>
<td></td>
<td>S(z)</td>
<td>0.63</td>
<td>0.70</td>
</tr>
<tr>
<td>Percentile</td>
<td>99.00%</td>
<td>6.91</td>
<td>8.00</td>
</tr>
</tbody>
</table>

Fig. 5a: Direct Numerical Estimation of Pareto Parameters

Note that in this example the distribution of the sum assured equals the 99th percentile of the Pareto claims severity distribution and $D \geq 0$, $\alpha > 3$, $\beta \geq 0$ are additional conditions.
Once the claims severity distributions of our example lines of business are determined, a provision for the parameter uncertainty inherent in the corresponding claims frequency estimates has been made (for more details, see further below) and correlations.

**Correlation Structure**

<table>
<thead>
<tr>
<th></th>
<th>Line 1</th>
<th>Line 2</th>
<th>Line 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line 1</td>
<td>1.00</td>
<td>0.20</td>
<td>0.10</td>
</tr>
<tr>
<td>Line 2</td>
<td>0.20</td>
<td>1.00</td>
<td>0.30</td>
</tr>
<tr>
<td>Line 3</td>
<td>0.10</td>
<td>0.30</td>
<td>1.00</td>
</tr>
</tbody>
</table>

**Variance/Covariance Matrix**

<table>
<thead>
<tr>
<th></th>
<th>Line 1</th>
<th>Line 2</th>
<th>Line 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line 1</td>
<td>1,350,587.62</td>
<td>912,662.27</td>
<td>220,364.90</td>
</tr>
<tr>
<td>Line 2</td>
<td>912,662.27</td>
<td>1,541,834.49</td>
<td>2,233,681.14</td>
</tr>
<tr>
<td>Line 3</td>
<td>220,364.90</td>
<td>2,233,681.14</td>
<td>3,595,522.90</td>
</tr>
</tbody>
</table>

**Fig. 5b:** Pareto Distribution and LEV Function for the Example Lines of Business

**Fig. 6a:** Correlation Structure for the Example Lines of Business
between the lines of business specified, the total book of business is characterized by:

<table>
<thead>
<tr>
<th>Total Book of Business</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earned Premium (B)</td>
</tr>
<tr>
<td>Technical Premium (P)</td>
</tr>
<tr>
<td>Profit Loading (Lambda)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Loss Portfolio (X)</th>
<th>Mean</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>392.67</td>
<td>2484.01</td>
</tr>
</tbody>
</table>

Fig. 6b: Total Book of Business (Example Lines of Business A, B and C)

(2) If enough individual claims data per line of business is available, then maximum likelihood estimation (MLE) and the corresponding Kolmogorov-Smirnov (KS) goodness-of-fit test can be applied to get the associated optimal Pareto parameters D, α, β. This is the standard technique implemented in well-known statistical software packages like BestFit. To illustrate the procedure, we look at a US accident & health book of business (in 1996 USD):

<table>
<thead>
<tr>
<th>Accident Year</th>
<th>Claims Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>83</td>
<td>1</td>
</tr>
<tr>
<td>84</td>
<td>4</td>
</tr>
<tr>
<td>85</td>
<td>12</td>
</tr>
<tr>
<td>86</td>
<td>14</td>
</tr>
<tr>
<td>87</td>
<td>15</td>
</tr>
<tr>
<td>88</td>
<td>20</td>
</tr>
<tr>
<td>89</td>
<td>20</td>
</tr>
<tr>
<td>90</td>
<td>21</td>
</tr>
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<td>91</td>
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</tr>
<tr>
<td>92</td>
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<tr>
<td>93</td>
<td>7</td>
</tr>
<tr>
<td>94</td>
<td>2</td>
</tr>
</tbody>
</table>

Fig. 7a: Claims Frequency of a US Accident & Health Book of Business

Comparison of Input Distribution and Pareto(0.45,3.22e+4)

Fig. 7b: Claims Severity of a US Accident & Health Book of Business (BestFit)
Fig. 7c: Statistics of the MLE Fit (BestFit)
Note that we set $\beta = 0$ as we are only interested in the tail of the severity distribution. BestFit then determines $D = 32200$, $\alpha = 0.45$. The resulting Q-Q plot tells us that the fitted Pareto distribution has a heavier tail than the input distribution (claims data) while the difference (in excess of USD 1M) and the P-P plot seem to be all right. This leads us to the idea of repeating the above procedure, but this time only considering accident & health claims in excess of USD 1M (i.e., only 12 claims instead of 153):
Q-Q Comparison Between Input Distribution and Pareto(2.04,1.1e+6)

Note the big difference in the Pareto shape and threshold parameters, now D = 1110000, α = 2.04, and the corresponding change in the limited expected value (LEV) function in excess of USD 1.11M:

Fig. 7d: Claims Severity in Excess of USD 1M (BestFit)

The Q-Q plot seems to have improved while the other plots are still all right. We could now try to further improve the fit by using a more sophisticated KS optimization routine or a parameter estimation technique based on the limited expected value (LEV) comparison test (for details, see further below). Instead, we are going to ask ourselves the following more fundamental questions:

(a) What kind of distribution(s) do claims in excess of some threshold have? How arbitrary is our above choice of the Pareto distribution? Is there a unique class of severity distributions for such claims?
Is there a way of estimating the parameters of such a claims severity distribution in a "stable" way given the fact that our information about claims is usually incomplete (i.e., claims data might be missing) and, what is even worse, to a certain extent incorrect (i.e., claims data might have been changed during the reporting process, for instance because of typing errors, etc.)? What about parameter uncertainty in both claims severity and claims frequency? What about cycles and trends (i.e., variations in risk propensity) in claims experience?

How can such a sophisticated claims modelling methodology be implemented so that it is useful in practical applications (e.g., in the development and successful marketing of new value proposition based reinsurance solutions)?

2. Estimation of Claims Severity: Extreme Value Techniques

The above fundamental questions of insurance and reinsurance claims modelling can be discussed and answered very precisely in the general framework of extreme value theory (EVT):

(1) As noted above, claims histories might be incomplete, e.g., in the case where only losses in excess of a so-called displacement $\delta > 0$ are reported. Let therefore $(Y_i)$ be an i.i.d. sequence of ground-up losses, $(Z_i)$ be the associated loss amounts in the target layer $M \leq z \leq Q$ and $X = \sum_{i=1}^{k} Z_i$ the corresponding aggregate loss. Similarly, let $(\bar{Y}_i)$, $\bar{Y}_i = Y_i 1_{Y_i > \delta}$, be the losses greater than the displacement $\delta$, $(\bar{Z}_i)$ be the associated loss amounts in the target layer $M \leq z \leq Q$ and $\bar{R} = \sum_{i=1}^{g} \bar{Z}_i$, $\bar{R} = \sum_{i=1}^{k} Y_i 1_{Y_i > \delta}$, the corresponding aggregate loss amount.

Some quite elementary mathematical considerations then show that $F_X \equiv F_{\bar{R}}$ holds for the aggregate loss distributions, provided that $\delta < M$.

(2) The peaks-over-thresholds model (Pickands-Balkema-de Haan theorem) on the other hand says that the exceedances of a high threshold $t < M$ are approximately $G_{\xi,t,\sigma}(z)$ distributed, where $G_{\xi,t,\sigma}(z)$ is the generalized Pareto distribution with shape $\xi$, location $t = \mu$, and scale $\sigma > 0$.

(3) The threshold $t < M$ is chosen in such a way that in a neighbourhood of $t$ the MLE-estimate of $\xi$ (and therefore the limited expected value function and the associated insurance or reinsurance treaty premium) remains reasonably stable.

(4) The generalized Pareto distribution (GPD) is defined by

$$G_{\xi,t,\sigma}(z) = \begin{cases} \frac{1}{1 + \xi} \left(1 + \frac{z - \mu}{\sigma} \right)^{-\frac{1}{\xi}} & \xi \neq 0 \\ 1 - e^{-\frac{z - \mu}{\sigma}} & \xi = 0 \end{cases}$$
where $z \geq \mu$ for $\xi \geq 0$ and $\mu \leq z \leq \mu - \frac{\sigma}{\xi}$ for $\xi < 0$. Compare this with the ordinary Pareto distribution (PD) defined above:

$$S(z) = 1 - \left( \frac{D + \beta}{z + \beta} \right)^{\alpha}, \quad z \geq D.$$  

(5) Reinsurance policies are often on a **multiple year basis**, some of them with a **built-in minimum guaranteed performance** (e.g., with an embedded option on some financial markets variable). Assuming relative constancy of the underlying risk distribution and exposure base for a particular line of insurance or reinsurance business is consequently too simplistic an approach. **In order to capture the risk dynamics of a policy or line / book of business, a sequence of standardized and adjusted loss scenarios should be developed for the corresponding policy term or the duration** (for details, see further below) **of the line / book of business under consideration.** Five kinds of **general threat scenarios** following such a schedule should be developed in principle:

(a) **adjustment scenarios** showing the effects of an increase in claims adjustment factors or an adverse development in the financial markets variables to which a policy or line / book of business is linked;

(b) **frequency scenarios** showing the effects of a higher claims frequency;

(c) **severity scenarios** showing the effects of a higher claims severity;

(d) **batch scenarios** showing the effects of claims series;

(e) **MPL scenarios** showing the effects of an extremely adverse maximum potential loss (MPL) estimate.

**Bootstrapping** is the applied statistical / actuarial methodology. However, according to the authors' experience in the application of extreme value techniques, under normal circumstances only an adjustment scenario has to be explicitly considered. The other scenarios just introduce additional parameter uncertainty into the original historical loss information and can therefore be replaced by a simulation approach (for details, see further below) to calculating aggregate loss distributions that allows for (e.g., normally distributed) parameter uncertainty. For example, consider the shape parameter $\xi$ of the generalized Pareto (GPD) distribution and assume that the extreme value techniques estimate of $\xi$ is $\xi_0$. **We assume then that $\xi \equiv \xi(\omega)$ is a normally distributed random variable with mean $\xi_0$ such that**

$$\text{Prob}\{0.75\xi_0 \leq \xi \leq 1.25\xi_0\} \geq 0.95$$

and say that the shape parameter uncertainty is 25% at the 95th percentile. The same approach is taken to model parameter uncertainty in the (Poisson) frequency parameter $n$ and the other (GPD) severity parameters $\mu$ and $\sigma$.

(6) We have implemented a number of tools that make the above outlined extreme value theory approach to claims modelling directly applicable in practice. These tools are available in the form of a corresponding **extreme value techniques toolbox (EVT) for insurance and reinsurance applications** that runs under Windows 3.1, 95, NT 3.51 and NT 4.0:
Extreme value techniques have within Swiss Re so far been applied very successfully in the development of Swiss Re's recently launched "Beta" program for high-excess property and casualty layers and for the development of advanced value proposition based reinsurance solutions to asset/liability management problems of pension funds (see further below for an example).

Using this toolbox for an analysis of our above US accident & health book of business yields the following interesting results which conclude this paragraph on extreme value techniques (see also the references section of this document):

(a) The Q-Q plot of the claims data indicates a Pareto distribution (concave shape). Visual inspection of the shape by threshold plot indicates a choice of threshold $\mu_0 = 0.22M$ with a corresponding interval $[\mu_0 - 0.10M, \mu_0 + 0.10M]$ of stability. Note that $\mu_0$ could have been chosen anywhere within the above interval of stability and that we assume a parameter uncertainty of 25% at the 95th percentile in the sequel.

(b) With this choice of threshold, the corresponding optimal generalized Pareto distribution is characterized by a shape $\xi_0 = 0.65$ and a scale $\sigma_0 = 1.55M$ (again, parameter uncertainty of 25% at the 95th percentile is assumed). Note the close match of the approximating Pareto distribution with parameters $D = 0.22M$, $\alpha = 1.49$, $\beta = 0$. The extreme value techniques toolbox also contains a sophisticated KS optimizer that could at this stage be used to further improve the estimates for both the GPD and PD shape parameters. This is however only necessary in a case where really sophisticated coverage structures are considered or very tight pricing is required.

(c) The most important goodness-of-fit test (that we always use) is the limited expected value (LEV) comparison test which is also supported (for both the generalized Pareto distribution and the ordinary Pareto distribution) by the extreme value techniques toolbox. It works as outlined below.
Consider the following quantities and basic relationships:

(i) **Empirical Limited Expected Value Function** for Sample $z_1, \ldots, z_n$

$$E_n(d) = \frac{1}{n} \sum_{i=1}^{n} \min(z_i, d)$$

(ii) **Limited Expected Value Function**

$$E[Z; d] = \int_{0}^{d} zS_z(z)dz + d[1 - S_z(d)]$$

(iii) **Empirical Mean Residual Life (= Sample Mean Excess) Function** for Sample $z_1, \ldots, z_n$

$$e_n(d) = \frac{\sum_{i=1}^{n} \max(z_i - d, 0)}{\sum_{i=1}^{n} 1\{z_i > d\}}$$

(iv) **Mean Residual Life Function**

$$e(d) = E[Z - d|Z \geq d] = \int_{d}^{\infty} (z - d) \frac{S_z(z)}{\text{Prob}\{Z \geq d\}} dz$$

$$= \frac{\int [1 - S_z(z)] dz}{1 - S_z(d)}, \text{ provided that } \lim_{z \to \infty} (d - z)[1 - S_z(z)] = 0$$

(v) **Loss Elimination Ratio**

$$L(d) = \frac{E[Z; d]}{E[Z]}$$

(vi) **Excess Ratio for Retention**

$$R(d) = \frac{e(d)}{E[Z]}$$

(vii) **Basic Relationships**

$$E[Z] = E[Z; d] + e(d)[1 - S_z(d)]$$

$$L(d) + R(d)[1 - S_z(d)] = 1$$

Then:

(viii) **Limited Expected Value Comparison Test** for Sample $z_1, \ldots, z_n$

$$\frac{E[Z; z_i] - E_n(z_i)}{E[Z; z_i]}$$

<table>
<thead>
<tr>
<th>Size of Loss</th>
<th>GPD LEV Comparison Test</th>
<th>PD LEV Comparison Test</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>YUSD</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5000</td>
<td>0.2056</td>
<td>0.2002</td>
</tr>
<tr>
<td>1.0000</td>
<td>0.2324</td>
<td>0.2245</td>
</tr>
<tr>
<td>1.5000</td>
<td>0.2286</td>
<td>0.2206</td>
</tr>
<tr>
<td>2.0000</td>
<td>0.2227</td>
<td>0.2147</td>
</tr>
<tr>
<td>2.5000</td>
<td>0.2234</td>
<td>0.2157</td>
</tr>
<tr>
<td>3.0000</td>
<td>0.2267</td>
<td>0.2256</td>
</tr>
<tr>
<td>3.5000</td>
<td>0.2366</td>
<td>0.2317</td>
</tr>
</tbody>
</table>
Fig. 9a: Q-Q Plot and GPD Shape by Threshold Plot

Data: GPD Fit to 98 exceedances

Fig. 9b: Optimal GPD (Solid Line) and PD (Dotted Line) Fit
Having determined the frequency and severity parameters of the US accident & health book of business under consideration together with the associated parameter uncertainty, we can now use the extreme value techniques toolbox for a detailed analysis of the underwriting risks involved on a one year or a multiple year basis (in our example below, we consider the one year aggregate loss distribution for 1996 together with the associated risk landscapes below and above a sequence of attachment points ranging from USD 1M to 100M). Note that the results of such an analysis do not any longer depend on the chosen severity distribution (there is only one: GPD) nor do they change very much as a function of the chosen parameters as we have incorporated parameter uncertainty (of 25%) into the calculations.
3. Estimation of Claims Frequency, Variations of Risk Propensity

A more traditional actuarial methodology used for modelling frequency parameter uncertainty as well as variations of risk propensity due to cycles and trends in a line / book of business is the mixing variable technique. We assume that the frequency parameter $k$ is conditionally Poisson distributed, i.e.,

$$p_i(q) = \text{Prob}(k = i|q) = e^{-\mu q} \frac{(\mu q)^i}{i!},$$

Fig. 10b: Associated Risk Landscapes (GPD)
where the mixing variable \( q \) is \( H(q) \) distributed\(^3\) with \( E[q] = 1 \), i.e.,

\[
p_i = \text{Prob}\{k = i\} = E[p_i(q)] = \int_0^\infty \frac{(nq)^i}{i!} e^{-nq} dH(q).
\]

\( H(q) \) is called a structure function and has various other applications in collective risk theory, for instance in credibility and rate-making.

Let now:

(a) \( n = E[k] \) and \( m = E[Z] \)

(b) \( a_2 = E[Z^2] = \sigma^2 + E[Z]^2 \) and \( r_2 = \frac{a_2}{m^2} \)

Then:

(c) \( E[X] = nm \) (aggregate mean)

(d) \( \sigma_x = \sqrt{na_2 + E[X]^2} \) \( \sigma^2 + E[Z]^2 = \sqrt{na_2 + (nm\sigma_q)^2} \) (standard deviation)

(e) \( \frac{\sigma_x}{E[X]} = \sqrt{\frac{na_2}{E[X]^2} + \sigma^2} = \sqrt{\frac{r_2}{n} + \sigma^2_q} \) (coefficient of variation)

(f) \( P = E[X] - iR = nm - iR \) (technical premium)

(recall that \( i \) is the pricing rate and \( R \) the total reserves for line / book of business \( X \)). The first term inside the square root in (e) above (coefficient of variation) arises from the compound Poisson fluctuation, whereas the second term introduces the additional effect of the mixing variable \( q \) (parameter uncertainty, variations of risk propensity, etc.). The former decreases when the volume parameter \( n \) increases but the latter is independent of \( n \). This implies that in small risk collectives the pure Poisson random variation, together with the random variation of the individual claim sizes, has a more significant effect on the fluctuation of the aggregate claims amount, whereas in large risk collectives the effect of the mixing variable \( q \) predominates.

Note that we also use this simple technique to model parameter uncertainty, variations of risk propensity, etc. across the asset / liability boundary (for details, see further below). In fact, it is this methodology that provides a simple but consistent framework for a state-of-the-art asset / liability management (ALM) concept.

4. The Effect of Reinsurance Arrangements: Quota Share Reinsurance

Having adequately covered the important subject of modelling claims portfolios, we now turn our attention to the most important reinsurance arrangements.

\(^3\) Note that \( F_X(x) = \text{Prob}\{X \leq x\} \rightarrow H(x) \) as \( n = E[k] \rightarrow \infty \).
Quota share reinsurance is the simplest of all reinsurance arrangements in which the reinsurer simply pays a fixed share of all claims occurring under a given reinsurance treaty or in a given line/book of business.

Let:

(a) \( X \) be a given claims portfolio
(b) \( r \) be the cedent's quota share

Then (with the same notation as above):

(c) \( E_X^{\text{ced}}(r) = nm \) (aggregate mean)
(d) \( \sigma_X^{\text{ced}}(r) = \sqrt{na^2 + E_X^{\text{ced}}(r)^2 \sigma_q^2} \) (standard deviation)

characterize the cedent's risk portfolio and

(e) \( E_X^{\text{re}}(r) = nm(1 - r) \) (aggregate mean)
(f) \( \sigma_X^{\text{re}}(r) = \sqrt{na^2(1 - r)^2 + E_X^{\text{re}}(r)^2 \sigma_q^2} \) (standard deviation)

characterize the reinsurer's risk portfolio.

5. The Effect of Reinsurance Arrangements: Surplus Reinsurance

Surplus reinsurance works in a way similar to quota share reinsurance with the difference however that the cedent retains all losses below a specified retention. Above this retention, the cedent's quota share is

\[
\text{Retention} = \frac{\text{Upper Claims Limit}}{Q}.
\]

Let therefore:

(a) \( Q \) be the upper claims limit (e.g., maximum potential loss, MPL)
(b) \( M \) be the cedent's retention, \( M \leq Q \)

Then (similar to above):

(c) \( r = r(M, Q) = \frac{M}{Q} + \frac{L(M)}{L(Q)} \left( 1 - \frac{M}{Q} \right) \) [\( L(z) \): Limited Expected Value Function (see also below)]

is the cedent's quota share and

(d) \( E_X^{\text{ced}}(r) = nm \) (aggregate mean)
(e) \( \sigma_X^{\text{ced}}(r) = \sqrt{na^2 + E_X^{\text{ced}}(r)^2 \sigma_q^2} \) (standard deviation)

characterize the cedent's risk portfolio whereas
6. The Effect of Reinsurance Arrangements: Stop Loss Reinsurance

In stop loss reinsurance, the reinsurer pays all claims in a specified layer \( M \leq z \leq Q \). Sometimes, aggregate limits are also defined. Crucial in our considerations concerning stop loss reinsurance is the limited expected value function

\[
L(M) = L_z(M) = E\left[\min(M, Z)\right] = \int_M^z dS_z(z) + M\left[1 - S_z(M)\right]
\]

where for simplicity of presentation, we assume an ordinary Pareto distribution (PD), i.e.,

(a) \( S(z) = 1 - \left(\frac{D + \beta}{z + \beta}\right)^\alpha, \ z \geq D \)

(b) \( m = \frac{\alpha D + \beta}{\alpha - 1}, \ \alpha > 1 \)

(c) \( \sigma^2_z = \frac{\alpha(D + \beta)^2}{\alpha - 1}(\alpha - 2), \ \alpha > 2 \)

(d) \( L(M) = \frac{\alpha D + \beta - (M + \beta)[1 - S(M)]}{\alpha - 1}, \ M \geq D. \)

Let furthermore:

(e) \( Q \) be the upper claims limit

(f) \( M \) be the retention (i.e., we consider the layer \( Q - M \times M \))

Then:

(g) \( m_{\text{within}}(M) = E\left[\min(M, Z)\right] = L(M) \)

\[
a^2_{\text{within}}(M) = E\left[\min(M, Z)^2\right] = \int_M^z z^2 dS(z) + M^2\left[1 - S(M)\right]
\]

(h) 
\[
= -2\beta \frac{\alpha(D + \beta) - (M + \beta)[1 - S(M)]}{\alpha - 1} + \frac{\alpha(D + \beta)^2 - 2(M + \beta)[1 - S(M)]}{\alpha - 2}
\]

(i) \( E_{\text{within}}(M) = nE\left[\min(M, Z)\right] = nm_{\text{within}}(M) = nL(M) \) (aggregate mean)

(j) \( \sigma^2_{\text{within}}(M) = \sqrt{n a^2_{\text{within}}(M) + E_{\text{within}}(M)^2} \sigma^2_q \) (standard deviation)
characterize the claims portfolio $z \leq M$, whereas

(k) $n_{\text{excess}}(M) = n \Pr \{Z > M\} = n[1 - S(M)]$

(l) $m_{\text{excess}}(M) = \mathbb{E}[Z - M | Z > M] = \frac{m - L(M)}{1 - S(M)}$

(m) $a_2^{\text{excess}}(M) = \mathbb{E}[(Z - M)^2 | Z > M] = \frac{a_2^{\text{within}}(M) - 2M[L(M) - L(M)]}{1 - S(M)}$

(n) $E_x^{\text{excess}}(M) = n_{\text{excess}}(M)m_{\text{excess}}(M) = n[m - L(M)]$ (aggregate mean)

(o) $\sigma_x^{\text{excess}}(M) = \sqrt{n_{\text{excess}}(M)a_2^{\text{excess}}(M) + E_x^{\text{excess}}(M)^2 \sigma_q^2}$ (standard deviation)

characterize the claims portfolio $z > M$ and

(p) $a_2(M, Q) = \frac{a_2^{\text{within}}(Q) - a_2^{\text{within}}(M) - 2M[L(Q) - L(M)]}{1 - S(M)}$

$E_x^e(M, Q) = n \mathbb{E}[\min(Q - M, \max(Z - M, 0))]$ (aggregate mean)

(q) $\sigma_x^e(M, Q) = \sqrt{n_{\text{excess}}(M)a_2(M, Q) + E_x^e(M, Q)^2 \sigma_q^2}$ (standard deviation)

characterize the reinsurer's risk portfolio $M \leq z \leq Q$.

We have implemented all the above mentioned forms of reinsurance as part of the extreme value techniques toolbox:

<table>
<thead>
<tr>
<th>Reinsurance</th>
<th>Template</th>
<th>Template</th>
<th>Template</th>
</tr>
</thead>
<tbody>
<tr>
<td>Description</td>
<td>Quota Share</td>
<td>Surplus</td>
<td>Stop Loss</td>
</tr>
<tr>
<td>Upper Limit</td>
<td>$Q$</td>
<td>11.00</td>
<td>11.00</td>
</tr>
<tr>
<td>Retention</td>
<td>$M$</td>
<td>3.00</td>
<td>1.00</td>
</tr>
<tr>
<td>Cedent's Share</td>
<td>$(M, Q)$</td>
<td>0.30</td>
<td>0.27</td>
</tr>
<tr>
<td>Cedent's Technical Premium</td>
<td>$P_{\text{tech}}(M, Q, c)$</td>
<td>36.00</td>
<td>32.73</td>
</tr>
<tr>
<td>Reinsurer's Portfolio</td>
<td>$\text{Mean}(M, Q, c)$</td>
<td>39.01</td>
<td>36.01</td>
</tr>
<tr>
<td>Std$(M, Q, c)$</td>
<td>348.54</td>
<td>316.95</td>
<td>659.11</td>
</tr>
<tr>
<td>Reinsurer's Portfolio</td>
<td>$\text{Mean}(M, Q, c)$</td>
<td>92.43</td>
<td>96.03</td>
</tr>
<tr>
<td>Std$(M, Q, c)$</td>
<td>813.50</td>
<td>845.20</td>
<td>506.69</td>
</tr>
</tbody>
</table>

Fig. 11: Reinsurance Templates Implemented (Underlying: Line of Business A)

The toolbox also handles far more complicated reinsurance arrangements and allows its user to optimize coverage structures according to the individual needs of a specific client. Indeed, according to Swiss Re's main value proposition argument, optimal layers of reinsurance coverage are characterized by efficiency and cost transparency, a high degree of structural flexibility to optimally fit clients' asset / liability management (ALM) needs, significant capacity, long-term stability (AAA capacity) and high financial security (AAA capital base). Furthermore, Swiss Re's value proposition based risk transfer products may also include sophisticated financial markets components (balance sheet protection; for an example, see further below).
An Insurance / Reinsurance Company

On the company level, we start our analysis by considering the solvency margin or risk reserve at a given time $t$,

$$U(t) = A(t) - L(t),$$

where $A(t)$ is the value of assets at time $t$ and

$$L(t) = T(t) + L_0(t) = V(t) + C(t) + L_0(t)$$

are total liabilities (excluding liabilities to shareholders) at time $t$ with $L_0(t)$ liabilities other than technical reserves (e.g., sundry creditors, amounts due to reinsurers, etc.).

Whereas liabilities have been considered in some detail in the previous section, the transition equation for the assets is

$$A(t) = A(t-1) + U_{\text{new}}(t) + W_{\text{new}}(t) + B'(t) + J(t) - X'(t) - E(t)$$

with $U_{\text{new}}(t)$ the new equity capital issued in time period $[t-1, t]$, $W_{\text{new}}(t)$ the new debt capital issued and any other new borrowings and $J(t)$ the total investment income (e.g., capital gains / losses, interest, dividends, rental income, etc.), respectively.

A basic convention used in this technical note is that assets and liabilities of an insurance or reinsurance company are not treated separately, i.e., for both the same consistent methodological framework is used.

In such a framework, insurers and reinsurers hold reserves to meet expected future claims and as a buffer against unexpected poor experience.

![Total Reserves Diagram](image)

**Fig. 12:** Total Reserves

The assets backing these reserves yield investment returns, plus an extra profit from insurance or reinsurance operations.

### 1. Capital at Risk

As noted in the previous section, standard actuarial methods and statutory minimum solvency margins determine technical reserves and minimum required capital for an insurance or reinsurance company. The determination of the corresponding risk reserves (on top of the minimum required capital) is an important topic addressed in this paragraph.
These additional risk reserves are also called capital at risk, risk adjusted capital or risk based capital.

Let now

\[ U = U_0 + (1 + \lambda)P + J - X, \quad P = E[X] - \sum_{i=1}^{N} R_i \]

be the solvency margin of an insurance or reinsurance company with \( N \) lines of business after one (arbitrary) time period. Here, we have dropped the time index for simplicity of notation, furthermore \( R_i \) is the reserve of line of business \( i \) and \( p_i \) the corresponding pricing rate (see the previous section). Depending on whether we consider an insurer or a reinsurer, the risk portfolio \( X \) is characterized by

\[ E[X] = E_X^\pi(..., M_i, Q_i, r_i, ...) \]
\[ \sigma[X] = \sigma_X^\pi(..., M_i, Q_i, r_i, ...) \]

or by

\[ E[X] = E_X^\sigma(..., M_i, Q_i, r_i, ...) \]
\[ \sigma[X] = \sigma_X^\sigma(..., M_i, Q_i, r_i, ...) \]

with \(..., M_i, Q_i, r_i, ...\) determined by the reinsurance arrangements in force (see the previous section for details; for illustration purposes, we consider again the example lines of business A, B and C introduced there).

<table>
<thead>
<tr>
<th>Reinsurance Measures</th>
<th>Line 1</th>
<th>Line 2</th>
<th>Line 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Description</td>
<td>A</td>
<td>R</td>
<td>C</td>
</tr>
<tr>
<td>Upper Limit</td>
<td>( Q_i )</td>
<td>11.00</td>
<td>12.00</td>
</tr>
<tr>
<td>Retention</td>
<td>( M_i )</td>
<td>3.00</td>
<td>3.00</td>
</tr>
<tr>
<td>Cedant's Share</td>
<td>( r(M_i, Q_i) )</td>
<td>0.27</td>
<td>0.25</td>
</tr>
<tr>
<td>Cedant's Technical Premium</td>
<td>( P_{\text{T}}(M_i, Q_i) )</td>
<td>32.73</td>
<td>22.80</td>
</tr>
<tr>
<td>Cedant's Portfolio</td>
<td>Mean ( E_X^\pi(M_i, Q_i) )</td>
<td>96.01</td>
<td>24.70</td>
</tr>
<tr>
<td></td>
<td>Std ( \sigma_X^\pi(M_i, Q_i) )</td>
<td>316.95</td>
<td>98.17</td>
</tr>
<tr>
<td>Reinsurer's Portfolio</td>
<td>Mean ( E_X^\sigma(M_i, Q_i) )</td>
<td>96.09</td>
<td>74.27</td>
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<tr>
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<td>Std ( \sigma_X^\sigma(M_i, Q_i) )</td>
<td>845.20</td>
<td>294.50</td>
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<tr>
<td>Loss Portfolio after Reinsurance (X)</td>
<td>Mean</td>
<td>108.47</td>
<td></td>
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<tr>
<td></td>
<td>Std</td>
<td>718.12</td>
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<tr>
<td>Correlation Structure after Reinsurance</td>
<td>Line 1</td>
<td>Line 2</td>
<td>Line 3</td>
</tr>
<tr>
<td>Line 1</td>
<td>316.95</td>
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<tr>
<td>Line 2</td>
<td>0.20</td>
<td>98.17</td>
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<td>Line 3</td>
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<td>Variance/Covariance Matrix after Reinsurance</td>
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<td>Line 3</td>
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<td>10029.86</td>
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<tr>
<td>Line 2</td>
<td>6222.70</td>
<td>9636.47</td>
<td>16752.61</td>
</tr>
<tr>
<td>Line 3</td>
<td>18029.86</td>
<td>16752.61</td>
<td>325597.07</td>
</tr>
</tbody>
</table>

Fig. 13: The Risk Portfolio of an Insurer (Surplus Reinsurance for LoB A, B and C)

Given a ruin probability \( \varepsilon > 0 \), we then define the corresponding capital at risk \( U_0 \) such that

\[ \text{Prob}\{U - U_0 > -U_\varepsilon\} = 1 - \varepsilon. \]

\( U_0 \) is the minimum required capital for the time period under consideration (e.g., in the case of Swiss Re it would be the capital required for a BBB rating by S&P). If we now consider the variable

\[ Y = X - J, \]
then the **extreme value techniques toolbox** determines the value $Y_\varepsilon$ such that

$$F_Y(Y_\varepsilon) = \operatorname{Prob}\{Y \leq Y_\varepsilon\} = 1 - \varepsilon$$

(see the previous section for details) and we have

$$U_\varepsilon = Y_\varepsilon - (1 + \lambda)P.$$ 

Very often however, it is useful to have an **analytic expression for the capital at risk as a function of the reinsurance arrangements in force and the asset / liability management strategy followed** by an insurance or reinsurance company. In order to be able to derive such analytics, some approximation of the distribution function $F_Y(Y)$ is usually considered. Here is how a corresponding **NP-approximation**

$$F_Y(Y) \approx N(y) \frac{Y - E[Y]}{\sigma_Y} = y + \frac{\gamma_Y}{6} (y^2 - 1), \; Y > E[Y]$$

(*that accounts for skewness*) of the above general capital at risk formula works (alternatives would be the **Haldane approach** or the **WH-formula** which are slightly more complicated, however).

Let:

(a) $y_\varepsilon$ be such that with the standard normal distribution function $N(y_\varepsilon) = 1 - \varepsilon$

(b) $n = E[k]$

(c) $m = \frac{E[Y]}{n} = \frac{E[X] - E[J]}{n}$

(d) $r_2 = \frac{n}{k} \left( \frac{\sigma_Y}{E[Y]} \right)^2 - \sigma_q^2$, $\sigma_q \leq \frac{\sigma_Y}{E[Y]}$ (**coefficient of variation**)

(e) $r_3 = \left( \frac{Q}{m} \right)^2$ (**upper bound**, $Q = \max\{\ldots, Q_i, \ldots\}$)

Then:

$$\gamma_Y = \frac{r_1}{n^2} + 3 \cdot \frac{r_2}{n} + \frac{r_q}{n} \sigma_q^2 + \gamma_Y \sigma_q^2$$

(**skewness**)

(g) $U_\varepsilon = E[Y] + \sigma_Y y_\varepsilon - (1 + \lambda)P + R_\gamma$ (**capital at risk**)

(h) $R_\gamma = \frac{\sigma_Y \gamma_Y}{6} (y^2_\varepsilon - 1)$ (**correction term for skewness**)

Note that we have assumed $Y$ to be of the compound mixed Poisson type with the associated mean frequency determined by $X$ except for parameter uncertainty. The mixing variable $q$ then models parameter uncertainty, variations of risk propensity, etc. across the asset / liability boundary at the overall risk portfolio level. Capital at risk is always determined at the overall risk portfolio (i.e., total book of business) level, in our example:
<table>
<thead>
<tr>
<th>Result after Reinsurance</th>
<th>Mean</th>
<th>54.63</th>
</tr>
</thead>
<tbody>
<tr>
<td>Std</td>
<td>720.85</td>
<td></td>
</tr>
<tr>
<td>Premium</td>
<td>Earned Premium before Reinsurance</td>
<td>440.00</td>
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<tr>
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<td>Earned Premium (Market)</td>
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<tr>
<td></td>
<td>Earned Premium after Reinsurance (B)</td>
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<td></td>
<td>Target Earned Premium after Reinsurance</td>
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<tr>
<td>Expenses</td>
<td>50.00</td>
<td></td>
</tr>
<tr>
<td>Capital at Risk</td>
<td>Capital at Risk before Reinsurance</td>
<td>2500.00</td>
</tr>
<tr>
<td></td>
<td>Capital at Risk after Reinsurance (U)</td>
<td>1987.98</td>
</tr>
<tr>
<td>Risk Tolerance (Epsilon)</td>
<td>1.00%</td>
<td></td>
</tr>
<tr>
<td>Profit Loading (Lambdas)</td>
<td>27.71%</td>
<td></td>
</tr>
<tr>
<td>Reinsurance</td>
<td>Line 1</td>
<td>Line 2</td>
</tr>
<tr>
<td>Description</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper Limit</td>
<td>O</td>
<td>11.00</td>
</tr>
<tr>
<td>Retention</td>
<td>M</td>
<td>3.00</td>
</tr>
<tr>
<td>Cedent's Share</td>
<td>R(M,O)</td>
<td>0.27</td>
</tr>
<tr>
<td>Cedent's Technical Premium</td>
<td>P_{w}(M,O,u)</td>
<td>32.73</td>
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<tr>
<td>Cedent's Portfolio</td>
<td>Mean_{w}(M,O,u)</td>
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<tr>
<td></td>
<td>Std_{w}(M,O,u)</td>
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<td>Correlation Structure after Reinsurance</td>
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<tr>
<td>Line 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Line 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Line 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance/Covariance Matrix after Reinsurance</td>
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</tr>
<tr>
<td>Line 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Line 2</td>
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<td></td>
</tr>
<tr>
<td>Line 3</td>
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</tr>
<tr>
<td>Investment Portfolio (J)</td>
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<td></td>
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<tr>
<td></td>
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<td>Investment Opportunities</td>
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<td>Proportion</td>
<td>Mean</td>
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<tr>
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<tr>
<td>Stocks</td>
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<td>370.00</td>
</tr>
<tr>
<td>Other</td>
<td>10.00%</td>
<td>19.00</td>
</tr>
<tr>
<td>Correlation Structure</td>
<td>X</td>
<td>J</td>
</tr>
<tr>
<td></td>
<td>718.12</td>
<td>-0.20</td>
</tr>
<tr>
<td></td>
<td>-0.20</td>
<td>13.54</td>
</tr>
<tr>
<td>Variance / Covariance Matrix</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>X</td>
<td>J</td>
</tr>
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<td></td>
<td>-1944.10</td>
<td>183.22</td>
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<td></td>
<td>Skew</td>
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</tr>
<tr>
<td>Capital at Risk after Reinsurance (U)</td>
<td>Mean</td>
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<tr>
<td></td>
<td>n, m</td>
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</tr>
<tr>
<td></td>
<td>r2</td>
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</tr>
<tr>
<td></td>
<td>r3</td>
<td>375758.71</td>
</tr>
<tr>
<td></td>
<td>Skew</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>R_{max}</td>
<td>296.25</td>
</tr>
</tbody>
</table>

Fig. 14a: The Capital at Risk of an Insurer (Surplus Reinsurance for LoB A, B and C)
The important question of how to allocate the determined overall capital at risk to the individual lines of business (e.g., for performance measurement purposes) or even to individual insurance or reinsurance policies (e.g., for pricing purposes) is discussed in some detail further below.

As can be seen from our simple example above, insurers and reinsurers typically hold high levels of total reserves. Investment income is consequently a very important element of insurance and reinsurance profits and even a small change in the return made on the assets backing a (re)insurer's total reserves has a substantial impact on the (re)insurer's profitability. Return on capital employed $K(t)$, where

$$K(t) = R(t) - T(t),$$

i.e., capital employed equals total reserves minus technical reserves, falls as the level of capital increases. However, an insurer's or reinsurer's security increases with its capitalization. Efficient capital (i.e., asset / liability) management is therefore critical to balance the various pressures on capital from shareholders and regulators as well as financial and insurance and reinsurance markets.

2. Asset / Liability Management

As stated above, insurance and reinsurance companies invest large funds (i.e., their total reserves) in order to satisfy future liabilities resulting from the various contractual obligations entered into with their clients.

*Bonds* have cashflow characteristics that make them very attractive investments for these purposes (in our example, 80% of funds are invested in bonds). By monitoring credit risk and call risk and adequately diversifying a bond portfolio by type of issuer, a (re)insurer can expect its promised cashflows with a high degree of certainty.

The sources of return from investing in a bond are its coupon payments, the interest on these payments and potential capital gains over the investment horizon. *Holding aside credit risk and embedded options, there are therefore three components to evaluating the attractiveness of a bond: yield, duration and convexity.* If the bond's price is $P$ and its cashflows are $c_1, \ldots, c_T$ (where $T$ is the maturity period), then its *yield to maturity (YTM)* $y$ is defined by the equation

$$P = \sum_{t=1}^{T} \frac{c_t}{(1+y)^t}.$$
Given an investment horizon $H$, a realized end price $P_H$ of the bond and a set of reinvestment rates $r_1, ..., r_H$, the bond's realized compound yield (RCY) $Y$ is defined by the equation

$$P(1+Y)^H = P_H + c_H + \sum_{i=1}^{H-1} c_i (1+r_{i+1}) \cdots (1+r_H).$$

Bond portfolio managers typically use these simple yield measures as a basis for undertaking bond swaps in order to enhance the performance of their bond portfolio over some investment period. There are five basic types of bond swaps: pure yield pickup swaps, substitution swaps, interest rate anticipation swaps, intermarket spread swaps and tax swaps.

A rate anticipation swap involves the bond portfolio manager's expectations about future interest rate movements and the idea is to position the bond portfolio on the basis of its interest rate sensitivity (duration) to take advantage of anticipated shifts in market interest rates: if rates are expected to fall, the portfolio's duration is increased; if rates are expected to rise, high duration bonds in the portfolio are swapped for lower duration bonds in the market. The Macaulay duration $D$ of a bond is

$$D = \frac{1}{P} \sum_{i=1}^{T} \frac{tc_i}{(1+y)^i}$$

and the relationship

$$\frac{dP}{P} = -D \frac{dy}{1+y}$$

shows that Macaulay duration is indeed a measure of its first order interest rate exposure. This equation also explains the above mentioned simple bond portfolio optimization strategy.

Convexity, the bond's second order interest rate sensitivity, is

$$C = \frac{1}{2P} \frac{d^2P}{dy^2} = \frac{1}{2P(1+y)^2} \sum_{i=1}^{T} \frac{(t+1)tc_i}{(1+y)^i}$$

and the relationship

$$\frac{AP}{P} = -D \frac{\Delta y}{1+y} + C \Delta y^2$$

shows that a high convexity bond outperforms a bond with the same yield and duration characteristics but lower convexity in all conceivable interest rate scenarios. Generally, therefore, high convexity bonds offer lower yields, that is, the market prices convexity. This yield discount can be substantial in times of high anticipated interest rate volatility. The modified duration

$$\bar{D} = D/(1+y)$$

and the convexity $C$ of a bond portfolio are value-weighted averages of the respective component quantities, i.e.,

$$\frac{1}{x_1P_1 + x_2P_2} \frac{d(x_1P_1 + x_2P_2)}{dy} = \frac{x_1P_1(\frac{1}{P_1} \frac{dP_1}{dy}) + x_2P_2(\frac{1}{P_2} \frac{dP_2}{dy})}{x_1P_1 + x_2P_2}.$$

This fact is the basis for another class of simple bond portfolio optimization strategies the objective of which is to improve portfolio performance (RCY) while keeping interest rate
exposure (duration and convexity) at the same level. These so called **duration-equivalent portfolio swaps** typically replace a bond currently held (bullet) with a synthetic security (barbell) consisting of two bonds in amounts \( x_1, x_2 \) chosen according to the equation system

\[
\frac{x_1 p_{1} + x_2 p_{2} \tilde{D}_1}{x_1 p_{1} + x_2 p_{2}} = \tilde{D} \\
\frac{x_1 p_{1} C_{1} + x_2 p_{2} C_{2}}{x_1 p_{1} + x_2 p_{2}} = C
\]

and work well under parallel shifts of the current term structure of interest rates.

**Duration adjustments of bond portfolios are usually carried out by using bond futures contracts instead of the bonds themselves.** The futures price \( F \) [where \( S < T \) is the contract maturity period] is related to the bond's **spot price** \( P \) via the equation

\[
\frac{F}{(1+y)^S} = P - \sum_{i=1}^{S} \frac{c_i}{(1+y)^i}
\]

from which the contract's **modified duration and convexity** can immediately be calculated, i.e.,

\[
\tilde{D}_F = \frac{1}{P} \frac{dF}{dy} = \frac{\sum_{i=1}^{S} \frac{t c_i}{(1+y)^i}}{\sum_{i=1}^{S} \frac{c_i}{(1+y)^i} - \frac{S}{1+y}} \\
C_F = \frac{1}{2P} \frac{d^2F}{dy^2} = \frac{\sum_{i=1}^{S} \frac{(t+1) t c_i}{(1+y)^i}}{\sum_{i=1}^{S} \frac{c_i}{(1+y)^i} - \frac{S}{1+y}} + \frac{(S-1) S}{2(1+y)^2}
\]

The futures position \( x \) is then chosen according to the equation system

\[
\frac{x F \tilde{D}_F + V_B \tilde{D}_B}{x F + V_B} = \tilde{D} \\
\frac{x F C_F + V_B C_B}{x F + V_B} = C
\]

where \( V_B, \tilde{D}_B \) and \( C_B \) are the bond portfolio value, modified duration and convexity.

The same technique can also be used by managers of **mixed asset portfolios** (bonds and stocks; in our example above, 10% of funds are held in stocks) to change their optimal asset allocation on a duration-equivalent basis. The bond futures position \( x \) is in this case chosen according to the equation system

\[
\frac{x F \tilde{D}_F + V_1 \tilde{D}_1}{V_T} = \tilde{D}_T = \tilde{D}_1 \\
\frac{x F C_F + V_1 C_1}{V_T} = C_T
\]

where \( V_1, V_T, \tilde{D}_1, \tilde{D}_T, C_1 \) and \( C_T \) are the initial and target bond portfolio values, modified durations and convexities.

**Efficient duration adjustment techniques are very important for insurers and reinsurers because of regulatory requirements to match the durations of assets and liabilities to a certain extent.** The above outlined simple methods are readily applicable for this purpose if we write a insurer's or reinsurer's total liabilities as a sum of discounted future cashflows,
\[ L(t) = \sum_{i=1}^{M} e^{-\delta(t+\Delta t_i) \lambda_i} \{ c(t + \Delta t_i) \}, \]

with \( M, \Delta t_i, c(t + \Delta t_i) \) and \( \delta(t + \Delta t_i) \) random variables. Using this approach, we can then define the (stochastic) duration

\[ D(t) = D_L(t) = \frac{\sum_{i=1}^{M} (t + \Delta t_i) e^{-\delta(t+\Delta t_i) \lambda_i} c(t + \Delta t_i)}{L(t)} \]

of a (re)insurer’s risk portfolio. Note that the asset yield \( y \) is of course a component of the stochastic liability discount rate \( \delta(t + \Delta t_i) \). Using the extreme value techniques toolbox, loss event scenario contingent values of \( L(t) \) and \( D(t) \) can readily be determined (see the previous section for details) and a corresponding match

\[ A(t) = L(t) \]

\[ D_A(t) = D_L(t) \]

easily and efficiently effected with bond futures. Note in this context also that the (stochastic) convexity of a (re)insurer’s risk portfolio is

\[ C(t) = C_L(t) = \frac{1}{2L(t)} \frac{\partial^2 L(t)}{\partial y^2} \]

(with the partial derivative taken pathwise).

Unlike the simple swap strategies described so far, structured bond portfolio management strategies do not rely on expectations of interest rate movements or changes in yield spread relationships. Instead, the objective is to design a portfolio that will achieve the performance of some predetermined benchmark (indexing) or finance a single future liability (immunization) or an entire future liability stream (cashflow matching). If \( P_1^a, \ldots, P_t^a, \ldots, P_N^a \) and \( P_1^b, \ldots, P_t^b, \ldots, P_N^b \) are the asked and the bid prices, respectively, of the bonds currently available in a specific bond market and if \( c_1, \ldots, c_i, \ldots, c_H \) [where \( H \geq T_j \) is the relevant investment horizon] are the (adjusted) cashflows of bond \( j \), \( 1 \leq j \leq N \), during the investment period under consideration, then the general structured bond portfolio management problem can be stated in the form

\[ \max_{x_i^a, x_i^b} \sum_{i=1}^{N} x_i^a P_i^a - x_i^b P_i^a \]

\[ u_1 \leq \sum_{i=1}^{N} [x_i^a - x_i^b] c_i^a \leq v_1 \]

\[ u_2 \leq \sum_{i=1}^{N} [x_i^a - x_i^b] [(1 + r_2) c_i^a + c_i^b] \leq v_2 \]

\[ u_3 \leq \sum_{i=1}^{N} [x_i^a - x_i^b] [(1 + r_3) [(1 + r_2) c_i^a + c_i^a] + c_i^b] \leq v_3 \]

\[ \vdots \]

\[ u_H \leq \sum_{i=1}^{N} [x_i^a - x_i^b] [(1 + r_H) [(1 + r_{H-1}) (...) + c_i^{a_{H-1}}] + c_i^a] \leq v_H \]

where \( r_2, \ldots, r_i, \ldots, r_H \) are the consecutive implied one period forward rates in the market and the respective long and short bond portfolio positions satisfy \( 0 \leq x_i^a \leq a_i \) and \( 0 \leq x_i^b \leq b_i \),
$1 \leq j \leq N$. If $R_1, \ldots, R_i, \ldots, R_N$ is the term structure of simple interest rates per trading period, then the above forward rates satisfy

$$1 + r_2 = \frac{(1 + R_2)^2}{1 + R_1}, \ldots, 1 + r_N = \frac{(1 + R_N)^N}{(1 + R_{N-1})^{N-1}}.$$

This linear program has two interpretations: (a) from a bond arbitrage point of view the objective is to maximize the current market value of the portfolio (by exploiting relative mispricing - e.g., as a result of different tax brackets - of bonds in the market) while at the same time constraining the risk exposure of the arbitrage transactions (in terms of their implications on the future portfolio cashflows) to values within a specified tolerance band $(u_i, v_i)$; (b) from a term structure estimation point of view by solving the associated dual problem

$$\min_{d, \leq 0, j \in (0, 2)} J \sum_{j=1}^{2N} y_j$$

$$P^b_j \leq \sum_{i=1}^{H} d_i c_i^j + y_j, \quad 1 \leq j \leq N$$

$$\sum_{i=1}^{H} d_i c_i^j - y_{N+j} \leq P^s_j, \quad 1 \leq j \leq N$$

$$d_0 = 1, \quad 0 \leq d_i - (1 + \rho)d_{i+1}, \quad 0 \leq t < H$$

a corresponding (tax-specific) term structure $d_0, d_1, \ldots, d_t, \ldots, d_N$ of discount factors and associated simple interest rates per trading period

$$R_1 = \frac{1}{d_1} - 1, \quad R_2 = \frac{1}{d_2} - 1, \ldots, \quad R_N = \frac{1}{d_N} - 1$$

that is consistent with a given exogenous (minimal) one period reinvestment rate $\rho$ and prices all bonds in the market within their respective bid/offer spreads $(P^b_j, P^s_j)$ can be obtained. Loss event scenario contingent values of the upper and lower bounds $u_i, v_i$ for the future cashflows of the bond portfolio can again be determined with the extreme value techniques toolbox.

For more sophisticated asset / liability management approaches like structured bond portfolio management (indexing, immunization, cashflow matching) discussed above, we have implemented a financial / (re)insurance techniques toolbox (FRT).

The extreme value techniques toolbox (EVT) handles the liability side while on the asset side multivariate stochastic models of the (jump) diffusion type are used for the evolution of the main financial markets variables like interest rates, stocks, stock indices and foreign currencies. A (re)insurer's assets and liabilities are then claims contingent on these stochastic dynamic variables describing the financial and insurance and reinsurance markets. Note that modern value proposition (VP) based reinsurance solutions very often require sophisticated financial engineering, too (for more details, see further below). FRT also provides the necessary quantitative support for the design, implementation and successful marketing of such new Swiss Re risk transfer products.

This toolbox again runs under Windows 3.1, 95, NT 3.51 and NT 4.0:
3. Implementing Stochastic Dynamic Models for Assets and Liabilities (FRT)

Lattices (see Fig. 16 below) and matrices are the main information processing structures used in value proposition based corporate and investment banking and (re)insurance applications. These structures tend to be quite large and have to be accessed and updated many times to obtain the results needed in quantitative financial (re)insurance decision making. The PC is widely used as a convenient low-cost financial services platform in modern investment banking and (re)insurance. Its main limitations are the small 64 KB data segment size, the typically insufficient RAM size and the usually rather slow and limited hardisk. One of the main objectives in the design of FRT was consequently to overcome these architectural limitations and to allow networked PCs to process very large financial (re)insurance information structures as efficiently as possible. A direct node access capability and a fast direct data access capability are the two key features which were built into the lattice manager (Lattices) and the virtual memory manager (VML) to achieve this goal. Given the time / state coordinates \((i, j)\) of a lattice node, its address in virtual memory (VML) is looked up in a lattice access structure (LAS) with a binary search algorithm and the node is then directly accessed with one physical memory (RAM, disk or network) operation. Dynamic programming procedures that operate on the lattice are considerably speeded up with the help of a bounds access structure (BAS) which stores the consecutive upper and lower lattice bounds over time. These acceleration structures themselves run on corresponding VML kernels (VMLAS and VMBAS). Given the address in virtual memory (VML) of a data element (lattice node), its address in physical memory (RAM, disk or network drive) is looked up in an area access structure (IASL) and an address access structure (KASL) which
both again use the services of a corresponding VML kernel (VMIASL and VMKASL). The data element is then, as mentioned above, directly accessed with one physical memory (context or cache) operation. A similar concept was used to implement large matrices (Matrices, VMM, etc.). With the above two design ideas, the processing of large financial information structures on a PC network is always almost at the speed of RAM although the data may actually be stored on disk or even on a (remote) network drive.

Fig. 16: FRT Lattice

The financial / (re)insurance techniques toolbox (FRT) determines the current price and the current sensitivities (derivatives risk parameters) of a contingent claim (a (re)insurer's assets and liabilities are interest rate contingent claims) as well as their future evolution over the claim's entire lifetime by using a dynamic programming procedure that operates on the underlying (binomial or trinomial) lattice. Each node in this lattice represents a potential financial and insurance and reinsurance market state at a given future time and the root describes the current market conditions that are relevant in a asset / liability management or value proposition based (re)insurance product design / marketing context.

As stated above, because of the high levels of total reserves held, investment income is a very important element of insurance and reinsurance profits and even a small change in the return made on the assets backing a (re)insurer's total reserves has a substantial impact on the (re)insurer's profitability. FRT supports a corresponding sensitivity analysis of a (re)insurer's profitability in several ways. Here are the two examples that are most important in practice:

(I) Usually, a (re)insurer's (substantial) bond portfolio can be tracked by a benchmark bond. Any changes in this bond's value (as determined by FRT with the dynamic programming techniques described above) then directly translate into corresponding changes in the (re)insurer's investment income and profitability. For example, in the case
of the Swedish pension fund discussed in the next section, the corresponding SEK benchmark bond was:

- **Notional Principal:** SEK 100.00
- **Coupon:** 5.5% p.a., paid annually
- **Coupon Date:** 12 April
- **Maturity Date:** 12 April 2002

On 1 October 1997, the market ("dirty") price of the bond was SEK 102.53 and a SEK 1.00 change in the bond's value or, equivalently, a 25 BP (where 1 BP = 0.01%) change in SEK interest rates translated into a SEK 60.00 million change in the value of the pension fund's bond portfolio which had a total market value of SEK 6.15 billion. An **FRT bond portfolio analysis** involving this benchmark bond (BND) and a December 1998 European put option (EPB) on this bond (providing a hedge at SEK 102.00) is then based on the following lattice structure (see Fig. 17 below): DF0 - DF4 are the discount functions prevailing in the SEK interest rate scenarios under study, P0 - P4 are the prices of the securities under the above mentioned interest rate conditions, alpha and beta are model sensitivities with respect to potential estimation errors in the relevant model parameters (we use the **extended Ho & Lee interest rate model** (Ho), see below) and D0 and D1 (delta), gamma and theta are the contingent claim sensitivities (defined as rates of change of the contingent claim value with respect to instantaneous changes in the underlying initial term structure of SEK interest rates and conditionally expected rates of change of the contingent claim value with respect to changes in time) in the given interest rate scenarios. With this information about the future dynamics of the underlying securities market variables, corresponding (consistent within an arbitrage pricing theory framework) contingent claim **price and sensitivity (derivatives risk exposure) forecasts**, i.e., expected values

\[ E[x_i] = \sum_{j=0}^{l} \pi_{ij} x_{ij} \]

and standard deviations

\[ D[x_i] = \sqrt{V[x_i]} = \sqrt{E[(x_i - E[x_i])^2]} = \sqrt{E[x_i^2] - E[x_i]^2} \]

\[ = \sqrt{\sum_{j=0}^{l} \pi_{ij} x_{ij}^2} \left( \left( \sum_{j=0}^{l} \pi_{ij} x_{ij} \right)^2 \right) \]

[\pi_{ij} is the time / state probability^4 associated with node \((i, j)\) and \(x_{i0}, x_{i1}, \ldots, x_{iu}\) are the time realizations of the stochastic process \((x_i)\) denoting the discrete price or derivatives risk parameter dynamics over time], are possible (see Fig. 18 below).

---

^4 In a asset / liability management context, the financial time / state probabilities (risk-neutral) are usually modified to take loss scenario probabilities (risk-averse) on the liability side into account. Lattices are very convenient for such applications, as they can store the necessary information on the associated Girsanov transformation of probability measure in each node (see the literature mentioned in the references section of this document for more details).
### BND / EPB Lattice Nodes (April / May 1997)

![Diagram of BND / EPB Lattice Nodes]

<table>
<thead>
<tr>
<th>Node</th>
<th>i-Coordinate</th>
<th>j-Coordinate</th>
<th>PARENTS</th>
<th>CHILDREN</th>
<th>LABELS</th>
<th>Term Structures</th>
<th>DF0</th>
<th>DF1</th>
<th>DF2</th>
<th>DF3</th>
<th>DF4</th>
<th>DF5</th>
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</thead>
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<td>0.99628541</td>
<td>0.99626222</td>
<td>0.99245991</td>
<td>0.98833385</td>
<td>0.98840042</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td>0.98833385</td>
<td>0.98840042</td>
<td>0.98840042</td>
<td>0.98840042</td>
<td>0.98840042</td>
<td>0.98840042</td>
</tr>
</tbody>
</table>

### Underlying Variables

**BND**

- Price: 98.46056949
- P1: 98.46056949
- P2: 98.46056949
- P3: 98.46056949
- P4: 98.46056949
- Alfa: 0.00000000
- Beta: 0.00000000
- DeltaD: 99.99999999
- DeltaD: 99.99999999
- DeltaD: 99.99999999
- Gamma: 99.99999999

**EPB**

- Price: 98.46056949
- P1: 98.46056949
- P2: 98.46056949
- P3: 98.46056949
- P4: 98.46056949
- Alfa: 0.00000000
- Beta: 0.00000000
- DeltaD: 99.99999999
- DeltaD: 99.99999999
- DeltaD: 99.99999999
- Gamma: 99.99999999

Fig. 17: BND / EPB Lattice Nodes (April / May 1997)

- 37 -
Fig. 18a: Contingent Claim Prices (April 1997 - April 2002 / December 1998)
Fig. 18b: Instantaneous Investment Risk (April 1997 - April 2002 / December 1998)
Fig. 18c: Future Risk Dynamics (April 1997 - April 2002 / December 1998)
Fig. 18d: Value Appreciation Dynamics (April 1997 - April 2002 / December 1998)
These forecasts (which have an **adaptive update property**) can then be used as an effective quantitative guideline in (conventional) every-day hedging decisions as well as in the design and implementation of longer-term asset / liability management strategies. Furthermore, this data can be stored in a relational database system and on demand be consolidated into appropriate risk management reports for a book of business, a department and the entire company. FRT’s simple **portfolio management component (Basket)** supports these tasks on the user interface level. FRT also contains all the necessary **functionality for derivatives risk management and hedging strategy evaluation (LRA, QP, LP, Simplex)** and uses **EXCEL as a user interface (EXCEL, Access)**.

(2) The simple FRT bond portfolio analysis discussed above operates within the extended **Ho & Lee interest rate model**

\[ dr = \varphi(t)dt + \sigma dz \]

\[ \varphi(t) = \frac{\partial f}{\partial t}(0, t) + \sigma^2 t \]

(see the references section of this document for details) [where \( f(t,T) \) is the instantaneous forward rate at time \( t \) for an investment at time \( T \)]. In discrete-time we have then: (1) Security markets clear at time points \( 0,1,2,...,i,...,H \) (where \( H \) is the given investment horizon) which are separated into regular intervals (model time periods). For each of these time points \( i \) the initial discount factor \( P(i) \) (relative to the time origin \( 0 \)) is known. Furthermore, at each time point \( i = 1,2,...,FH \) [where \( FH < H \) is the relevant forward horizon] there are \( i + 1 \) possible future discount functions \( P_{ij}(k), j = 0,1,...,i \) and \( k = 0,1,...,MH \) [where \( MH = H - FH \) is the associated maturity horizon]. (2) The evolution of the term structure of interest rates over the investment period \([0,H]\) is modelled by a recombining binomial lattice with root

\[ P_{\infty}(k) = P(k) \]

and branching process

\[ P_{i+1j}(k) = h_u(k) \frac{P_{ij}(k)}{P_{ij}(1)} \]

\[ P_{i+1j}(k) = h_d(k) \frac{P_{ij}(k)}{P_{ij}(1)} \]

where

\[ h_u(k) = \frac{1}{p + (1-p)d^k} \] and \[ h_d(k) = \frac{d^k}{p + (1-p)d^k} \]

are the corresponding upward and downward perturbation functions, the model probability is \( p \) and the model delta is \( d \), \( 0 \leq p,d \leq 1 \). With the length \( \Delta t \) of the model time periods we have then

\[ r = -\frac{1}{\Delta t} \log(P(1)) \] and \[ \sigma = \sqrt{\frac{p(1-p)\log(d)^2}{r^2\Delta t^3}} \]
for the current short-term interest rate and the term structure volatility. (3) The *risk-neutral pricing formula* is in this context

$$v_{ij} = \max \left[ L(i, j), \min \left[ P_i(j), \left[ p \left( v_{i+1,j+1} + X(i+1, j+1) \right) \right] + (1-p) \left( v_{i+1,j} + X(i+1, j) \right) \right], U(i, j) \right] \quad 0 \leq i \leq T-1$$

$$v_{ij} = F(j) \quad 0 \leq j \leq T$$

where $T \leq H$ is the contingent claim maturity. (4) With the forward rates

$$r_{ij}(k) = -\log(P_i(k))$$

we define the *derivatives risk parameters* (contingent claim sensitivities) as follows

$$\delta_{ij} = p \frac{v_{i+1,j+1} - v_{ij}}{r_{i+1,j+1}(1) - r_{ij}(1)} + (1-p) \frac{v_{i+1,j} - v_{ij}}{r_{i+1,j}(1) - r_{ij}(1)}$$

$$\gamma_{ij} = p \frac{\delta_{i+1,j+1} - \delta_{ij}}{r_{i+1,j+1}(1) - r_{ij}(1)} + (1-p) \frac{\delta_{i+1,j} - \delta_{ij}}{r_{i+1,j}(1) - r_{ij}(1)}$$

$$\theta_{ij} = p \frac{v_{i+1,j+1} - v_{ij}}{\Delta t} + (1-p) \frac{v_{i+1,j} - v_{ij}}{\Delta t}$$

(conditionally expected rates of change of the option value with respect to the underlying short-term interest rate and time). (5) The time / state probabilities associated with FRT's (extended) Ho & Lee lattice are

$$\pi_{ij} = \frac{(i!)p^i(1-p)^{i-j}}{\sum_{j=0}^{i} \pi_j v_{ij}} \quad \sigma_i^\gamma = \sqrt{\sum_{j=0}^{i} \pi_j v_{ij}^2 - \mu_i^\gamma^2} \quad \text{(price)}$$

$$\mu_i^\delta = \sum_{j=0}^{i} \pi_j \delta_{ij} \quad \sigma_i^\delta = \sqrt{\sum_{j=0}^{i} \pi_j \delta_{ij}^2 - \mu_i^\delta^2} \quad \text{(delta)}$$

$$\mu_i^\gamma = \sum_{j=0}^{i} \pi_j \gamma_{ij} \quad \sigma_i^\gamma = \sqrt{\sum_{j=0}^{i} \pi_j \gamma_{ij}^2 - \mu_i^\gamma^2} \quad \text{(gamma)}$$

$$\mu_i^\theta = \sum_{j=0}^{i} \pi_j \theta_{ij} \quad \sigma_i^\theta = \sqrt{\sum_{j=0}^{i} \pi_j \theta_{ij}^2 - \mu_i^\theta^2} \quad \text{(theta)}$$

(and similar for higher order moments of the corresponding distributions). (7) The parameters of the above outlined Ho & Lee interest rate model are $p$ (model probability), $d$ (model delta) and $R(1), R(2), \ldots, R(i), \ldots, R(H)$ (initial term structure of simple, annualized interest rates). We have then

$$r(i) = \log(1+R(i)) \quad \text{and} \quad P(i) = e^{-r(i)\Delta t} = \frac{1}{(1+R(i)\Delta t)^i}$$

for the corresponding continuously compounded interest rates and discount factors. In addition to these parameters we now also consider the quantities

$\Delta p$ (probability increment), $\Delta d$ (delta increment)

$\Delta R(1), \Delta R(2), \ldots, \Delta R(i), \ldots, \Delta R(H)$ (interest rate increments)

and construct a recombining binomial lattice for the term structures

$^5$ $X$ (intertemporal cashflows) and $F$ (terminal condition) characterize the contingent claim. $L \leq v \leq U$ are boundary conditions for its price process (see the references section of this document for details).
For an interest rate contingent claim we then calculate the corresponding scenario dependent prices
\( P_{ij}^{(p,d,R)}(k), P_{ij}^{(p+d,p,d,R)}(k), P_{ij}^{(p,d,R+\Delta R)}(k), P_{ij}^{(p,d,R+2\Delta R)}(k) \) and
\( P_{ij}^{(p,d,R+2\Delta R)}(k) \).

For an interest rate contingent claim we then calculate the corresponding scenario dependent prices
\( v_{ij}^{(p,d,R)}(k), v_{ij}^{(p+d,p,d,R)}(k), v_{ij}^{(p,d,R+\Delta R)}(k) \) and
\( v_{ij}^{(p,d,R+2\Delta R)}(k) \)
and sensitivities
\[
\begin{align*}
\sigma_{ij}^{(p,d,R)} &= \frac{v_{ij}^{(p+d,p,d,R)} - v_{ij}^{(p,d,R)}}{\Delta p} \\
\beta_{ij}^{(p,d,R)} &= \frac{v_{ij}^{(p,d,R+\Delta R)} - v_{ij}^{(p,d,R)}}{\Delta d} \\
\delta_{ij}^{(p,d,R+\Delta R)} &= \frac{v_{ij}^{(p,d,R+2\Delta R)} - v_{ij}^{(p,d,R+\Delta R)}}{\Delta R} \\
\delta_{ij}^{(p,d,R+2\Delta R)} &= \frac{v_{ij}^{(p,d,R+2\Delta R)} - v_{ij}^{(p,d,R+\Delta R)}}{\Delta R}
\end{align*}
\]

Note that the probability and delta exposure
\[
\alpha = \frac{\partial v}{\partial p} \quad \text{and} \quad \beta = \frac{\partial v}{\partial d}
\]
of a contingent claim in the Ho & Lee interest rate model can be written in the form
\[
\alpha = v \frac{\partial \sigma}{\partial p} + ... \quad \beta = v \frac{\partial \sigma}{\partial d} + ...
\]
where
\[
\frac{\partial \sigma}{\partial p} = \frac{(1-2p)\log(d)^2}{2\sigma^2\Delta t^3} \quad \frac{\partial \sigma}{\partial d} = \frac{p(1-p)\log(d)}{\sigma^2\Delta t^3}
\]
holds. This extension of the original Ho & Lee interest rate model can still be easily calculated (simultaneous scenario analyses) and is very well suited for asset / liability management and product development applications under varying securities market scenarios.

(3) For a more general sensitivity analysis of investment income and profitability with FRT, we assume again extended Ho & Lee interest rate dynamics
\( dy = \phi(t)dt + \sigma dz \)
(as above) but consider now a (re)insurer’s overall asset / liability portfolio \( P = P(t, y) \) with (stochastic) duration \( D(t) \) and (stochastic) convexity \( C(t) \). Then an application of Ito’s formula\(^6\) yields the risk portfolio dynamics
\[
dP = \frac{\partial P}{\partial t} + \phi(t) \frac{\partial P}{\partial y} + \sigma^2 \frac{\partial^2 P}{\partial y^2} dt + \sigma \frac{\partial P}{\partial y} dz
\]
with time first and second order characteristics
\[
E_t[dP] = \left[ \frac{\partial P}{\partial t} - \frac{\sigma^2}{2} \right] + \sigma \frac{\partial P}{\partial y} dz
\]
and
\[
\sigma_t[dP] = \frac{\sigma D(t)}{1 + y} \sqrt{dt}.
\]
More sophisticated yield dynamics would be
\[
dy = \left[ \theta(t) - \phi(t)y \right] dt + \sigma y^\beta dz
\]

\(^6\) Let \( y = f(t, x) \), \( dx(t) = a(t, x(t))dt + b(t, x(t))dz(t) \), \( a(t, x) \in \mathbb{R}^m \), \( b(t, x) \in \mathbb{R}^{mx} \). Then (Ito’s formula): \( dy = \left[ f_t + a^T \nabla_x f + 0.5tr(bb^T \nabla_x^2 f) \right] dt + (\nabla_x f)^T bdz \).
(Hull & White interest rate model, see the references section of this document for details). In this case, the asset/liability portfolio dynamics are

\[ dP = \left[ \frac{\partial P}{\partial t} + [\theta(t) - \phi(t)y] \frac{\partial P}{\partial y} + \frac{(\sigma y^\theta)^2}{2} \frac{\partial^2 P}{\partial y^2} \right] dt + \sigma y^\theta \frac{\partial P}{\partial y} dz \]

and the corresponding time \( t \) first and second order characteristics

\[ E_t[dP] = \left[ \frac{\partial P}{\partial t} - \lambda(1+y) \left( D(t) \frac{\theta(t) - \phi(t)y}{1+y} + C(t)(\sigma y^\theta)^2 \right) \right] dt \quad \text{and} \quad \sigma_t[dP] = \frac{D(t)\sigma y^\theta}{1+y} \sqrt{dt}. \]

Having outlined a set of (hopefully) fairly simple financial and actuarial models that allow the determination of a (re)insurer's risk reserves (or more specifically, capital at risk or risk adjusted capital or risk based capital) and an in-depth analysis of their sensitivity with respect to changes in the financial and insurance and reinsurance markets, we now turn our attention to the questions of risk capital allocation and risk adjusted performance measurement.

4. Performance Measurement: The Efficient Frontier

Modern portfolio theory (MPT, see Fig. 19 below) is a commonly accepted framework for measuring the performance of assets in the capital markets. As stated at the outset of this section, a basic convention used in this technical note is that assets and liabilities of an insurance or reinsurance company are not treated separately, i.e., for both the same consistent methodological framework is used. After a brief outline of classical MPT as it was introduced by Harry M. Markowitz, we therefore show in this paragraph how MPT can be extended to provide a suitable performance measurement framework for insurance and reinsurance companies and, in particular, that reinsurance creates tangible value in that it enables insurers to perform more efficiently than by using any other available form of risk transfer.

**Efficient Frontier:**

![Efficient Frontier Diagram](image-url)

If only risky assets are being considered, then the efficient frontier, i.e., the set of all efficient investment opportunities available in the capital markets, will be a curved line (defined by corner portfolios) that is upward sloping and concave.

With the addition of risk-free investment opportunities, the efficient frontier becomes a straight line. All the portfolios on this straight line involve combining one portfolio consisting of just risky assets (tangency portfolio) with either risk-free borrowing or lending.

**Fig. 19a:** Modern Portfolio Theory (MPT): The Efficient Frontier
Markowitz portfolio selection is the standard approach to structured (one period) asset management under uncertainty (diversification). Given an initial wealth \( v > 0 \), an investor in common stocks \( S_1, \ldots, S_i, \ldots, S_N \) with portfolio management objectives \( U(V) \) [where
\[
A(V) = -\frac{U''(V)}{U'(V)} \quad \text{and} \quad R(V) = -V \frac{U''(V)}{U'(V)}
\]
are the associated coefficients of absolute and relative risk aversion], that is, a strictly increasing (non-satiation), strictly concave (risk aversion) and continuously differentiable utility function for final wealth \( V \), determines (with FRT) the optimal portfolio positions.
\[ x = \left[ x_1 \quad K \quad x_j \quad \Lambda \quad x_N \right]^T \text{ where } x_j \geq 0, \ 1 \leq j \leq N, \text{ and } \sum_{j=1}^{N} x_j = 1 \]

(proportions of initial wealth invested in the available stocks) to be held over the relevant investment period \([0, H]\) such that expected utility of final wealth

\[
E[U(V)] = U(\mu_V) + \frac{\sigma^2_V}{2}U''(\mu_V) + \sum_{k=3}^{\infty} \frac{E[(V - \mu_V)^k]}{k!}U^{(k)}(\mu_V)
\]

\[
= U(\mu(1 + \mu_R)) + \frac{\sigma^2}{2}U''(\mu(1 + \mu_R))
\]

\[
= f(\sigma_R, \mu_R), \quad R = \frac{V - \mu}{\sigma}
\]

is maximized. Note that above \(E[U(V)] = f(\sigma_R, \mu_R)\) holds for quadratic utility functions \(U(V)\) or normally distributed portfolio returns \(R\) and that in such a case the investor's indifference curves

\[
E[U(V)] = f(\sigma_R, \mu_R) = c
\]

in \((\sigma_R, \mu_R)\) space are convex (risk aversion). The reason for expressing the portfolio positions as proportions of initial wealth \([i.e., as x_j = \frac{n_j S^j}{\n} \text{ and } V = \sum_{j=1}^{N} n_j S^j]\), and not simply as the numbers \(n_j\) of chosen stocks \(S^j\) lies in the fact that the return of a portfolio of stocks is a value-weighted average of the individual component returns, i.e.,

\[
\frac{(n_1 S^1_H + n_2 S^2_H) - (n_1 S^1_H + n_2 S^2_H)}{(n_1 S^1_H + n_2 S^2_H)} = \frac{n_1 S^1_H - S^1_H}{n_1 S^1_H + n_2 S^2_H} + \frac{n_2 S^2_H - S^2_H}{n_1 S^1_H + n_2 S^2_H}
\]

\[
= x_1 \frac{S^1_H - S^1_H}{S^1_H} + x_2 \frac{S^2_H - S^2_H}{S^2_H}.
\]

With the first two moments

\[
M = E[r] \quad \Sigma = E[(r - M)(r - M)^T]
\]

of the component return vector

\[
r = \left[ r_1 \quad K \quad r_j \quad \Lambda \quad r_N \right]^T \text{ where } r_j = \frac{S^j_H - S^j_H}{S^j_H}, \ 1 \leq j \leq N,
\]

we can write the mean and variance of the portfolio return \(R\) in the form

\[
\mu_R = \mu_x = x^T M \quad \sigma^2_R = \sigma^2_x = x^T \Sigma x
\]

and therefore the Markowitz portfolio selection problem is

\[
\max_{x \geq 0} U(\nu(1 + x^T M)) + \frac{\nu^2 x^T \Sigma x}{2} U''(\nu(1 + x^T M))
\]

\[
x^T 1_N = 1
\]

in general and

\[
\max_{x \geq 0} (\alpha - 2 \beta \nu)x^T M - \beta vx^T (MM^T + \Sigma)x
\]

\[
x^T 1_N = 1
\]

for quadratic utility \(U(V) = \alpha V - \beta V^2\). A feasible portfolio \(x \in X\) [where \(X \subseteq \mathbb{R}^N\) is the investor's opportunity set that is usually characterized by general linear constraints of the form \(Ax + b \leq 0\) and \(Cx + d = 0\)] is called efficient if
\[ \mu_\chi = \max \{ \mu_y : y \in X, \sigma_y^2 = \sigma_\chi^2 \} \quad \text{and} \quad \sigma_\chi^2 = \min \{ \sigma_y^2 : y \in X, \mu_y = \mu_\chi \} \]

holds and the set \( X^* \subseteq X \) of all efficient portfolios [in \((\sigma_y, \mu_y)\) space, i.e., under the assumption of normally distributed portfolio returns \( R \)] is called the *efficient frontier*. The efficient frontier, on which the optimal portfolio \( \chi \) is necessarily (risk aversion) situated, is concave and generated by a finite sequence \( x_0^*, x_1^*, \ldots, x_M^* \) of corner portfolios where \( x_0^* \) is the maximum expected return and \( x_M^* \), the minimum variance portfolio [that are the solutions of the optimization programs]

\[
\begin{align*}
\max & \quad x^T M \\
\text{s.t.} & \quad Ax + b \leq 0 \\
\end{align*}
\]

respectively. Two special cases of the general (quadratic utility) Markowitz model of optimal (one period) investment in shares of common stock (with normally distributed returns) are

\[
\begin{align*}
\min & \quad x^T \Sigma x \\
\text{s.t.} & \quad \mu_b \leq x^T M \leq \mu_a \\
& \quad \sigma_b^2 \leq x^T \Sigma x \leq \sigma_a^2 \\
& \quad Ax + b \leq 0 \\
& \quad Cx + d = 0
\end{align*}
\]

where the former emphasizes the risk minimization aspect in the investment decision and the latter takes a limited risk arbitrage point of view. Markowitz portfolio analysis can be substantially simplified by further assuming a *linear factor structure* of common stock returns. i.e.,

\[
C\{\varepsilon_j, F_k\} = \mu E\{\varepsilon_j\} = 0 \\
\text{and} \\
C\{\varepsilon_j, \varepsilon_j\} = 0, \quad j \neq j_2 \\
C\{F_k, F_k\} = 0, \quad k_1 \neq k_2
\]

[where

\[
C[x, y] = E[(x - E[x])(y - E[y])] \quad \text{and} \quad R[x, y] = \frac{C[x, y]}{D[x]D[y]}
\]

are the covariance and correlation coefficient of the random variables \( x \) and \( y \). Note first that these assumptions imply

\[
\alpha_j = E[r_j] - \sum_{k=1}^K \beta_j \mu_k \\
\beta_j = \frac{C[r_j, F_k]}{V[F_k]}
\]

for the relevant *factor sensitivities* and *intercept terms*. Furthermore, if now \( x \in X \) is any feasible portfolio, then we have

\[ \mu_x = x^T (\alpha + \beta M_p), \quad M_p = E[F] \]

\[ \sigma_x^2 = x^T (\beta \Sigma_p \beta^T + \Sigma_\varepsilon) x, \quad \Sigma_p = V[F], \quad \Sigma_\varepsilon = V[\varepsilon] \]

where

\[
\Sigma_p = \begin{bmatrix} \sigma_{F_1}^2 & 0 & \Lambda & 0 \\ 0 & 0 & 0 & M \\ \Lambda & M & O & 0 \\ 0 & 0 & \sigma_{F_k}^2 & 0 \end{bmatrix} \quad \text{and} \quad \Sigma_\varepsilon = \begin{bmatrix} \sigma_{\varepsilon_1}^2 & 0 & \Lambda & 0 \\ 0 & 0 & 0 & M \\ M & 0 & 0 & 0 \\ 0 & \Lambda & 0 & \sigma_{\varepsilon_w}^2 \end{bmatrix}
\]

and with the *portfolio beta*

\[ \beta_x = \beta^T x \]
therefore a decomposition
\[ \sigma_x^2 = \sigma_{px}^2 + \sigma_{ex}^2 \]
\[ \sigma_{px}^2 = \beta_x^T \Sigma \beta_x \quad \sigma_{ex}^2 = \alpha_x \Sigma_e \alpha_x \]
of portfolio risk (\( \sigma_x^2 \)) into a systematic component (\( \sigma_{px}^2 \)) and a diversifiable component (\( \sigma_{ex}^2 \)). Moreover, we can represent the above moments of the portfolio return \( R \) in the (diagonal variance/covariance matrix) form
\[ \mu_p = \mu_x = \begin{bmatrix} x^T \\ \beta_x \\ M_F \end{bmatrix} \begin{bmatrix} \alpha \\ \Sigma_e \end{bmatrix} \text{ and } \sigma_p^2 = \sigma_x^2 = \begin{bmatrix} x^T \\ \beta_x \\ 0 \end{bmatrix} \begin{bmatrix} \Sigma_e & 0 \\ 0 & \Sigma_F \end{bmatrix} \begin{bmatrix} x \\ \beta_x \end{bmatrix}. \]

Diversification:

<table>
<thead>
<tr>
<th>Security / Portfolio Total Risk</th>
<th>( \sigma_x = \sqrt{\beta^2 \sigma_{iw}^2 + \sigma_e^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio Market Risk</td>
<td>( \beta_x^2 \sigma_{iw}^2 - X_1 (\beta_1 \sigma_{iw}) + X_2 (\beta_2 \sigma_{iw}) + X_3 (\beta_3 \sigma_{iw}) + \ldots + X_N (\beta_N \sigma_{iw}) )</td>
</tr>
<tr>
<td>Portfolio Unique Risk</td>
<td>( \sigma_{ip} = \frac{1}{N} \left[ \sigma_{e1}^2 + \sigma_{e2}^2 + \sigma_{e3}^2 + \ldots + \sigma_{eN}^2 \right] )</td>
</tr>
</tbody>
</table>

Diversification leads to averaging of market risk and can substantially reduce unique risk.

Fig. 19d: Modern Portfolio Theory (MPT): Diversification

(2) The key paradigm in our extension of the above MPT framework for performance measurement in the capital markets to include the insurance and reinsurance markets is that we consider reinsurance as an additional investment opportunity for risk adjusted capital. We illustrate the corresponding general concepts and ideas by using our example insurer with lines of business A, B and C (see Fig. 5 and Fig. 14 above):

(a) Total reserves are CHF 3700.00 million, of which CHF 2500.00 million is the risk adjusted capital (or capital at risk or risk based capital).

(b) Capital markets investment opportunities for this risk capital fall into three classes: bonds, stocks and others (more categories can easily be added if required).

(c) Reinsurance markets investment opportunities are surplus treaties for the three lines of business of the example insurer considered (any combination of quota share, surplus and stop loss reinsurance can be used in general).

(d) The example insurer’s MPT choice of optimal portfolio then depends on the following parameters (see Fig. 14 above):
Reinsurance Premium (Market) Expenses Capital at Risk before Reinsurance
$x_1 = x_2 = x_3 =$

300.00 50.00 2500.00

Risk Tolerance (Epsilon) Systemic Risk (q) Utility Function
$x_4 = x_5 = x_6 = x_7 = x_8 =$

1.00% 5.00 2.00 50.00 0.01

Fig. 20a: MPT for (Re)insurance: Choice Parameters

Recall that the mixing variable $q$ models parameter uncertainty, variations of risk propensity, etc. across the asset/liability boundary at the overall risk portfolio level. Moreover, the example insurer's utility function (portfolio management objectives) has the following characteristics:

$$U(V) = \alpha V - \beta V^2 \quad U'(V) = \alpha - 2\beta V > 0 \quad U''(V) = -2\beta < 0$$

$$A(V) = \frac{2\beta}{\alpha - 2\beta V} \quad R(V) = \frac{2\beta V}{\alpha - 2\beta V}$$

Fig. 20b: MPT for (Re)insurance: Portfolio Management Objectives

(e) Starting with the current asset/liability management strategy, i.e., the surplus reinsurance arrangements and asset allocation in force
the example insurer can then by using the financial / (re)insurance techniques toolbox (FRT) optimize its overall asset / liability management strategy according to the above stated portfolio management objectives and the usual constraints

\[ 0 \leq M_i \leq Q_i, \ i = 1,2,3 \]

as well as the risk capital constraint

Capital at Risk after Reinsurance \leq\ Capital at Risk before Reinsurance:

\[
\begin{array}{c|c|c|c|c}
\text{Retention} & \text{Investment Opportunities} \\
M_1 & M_2 & M_3 & \text{Stocks} & \text{Other} \\
x_9 = x_{10} = x_{11} = x_{12} = x_{13} = & 3.00 & 3.00 & 3.00 & 10.00\% & 10.00\% \\
\end{array}
\]

Note the substantial effect this MPT optimization of the example insurer's asset / liability management strategy has on its result:

\[
\begin{array}{c|c|c|c|c|c|c}
\text{Expected Utility} & \text{Capital at Risk after Reinsurance (Ur)} & \text{Result after Reinsurance} \\
x_{14} = & x_{15} = & y = & z = & & & \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\text{Result after Reinsurance} & \text{Mean} & \text{Std} \\
1987.98 & 94.83 & 720.95 \\
\end{array}
\]

The tangible value created by reinsurance (optimal asset / liability management) then lies in the fact that the example insurer's mean result is now 140.97\% of the original mean result of 94.83 at only 50.43\% of the original result standard deviation of 720.95.
To summarize: In our extension of modern portfolio theory (MPT) to include the insurance and reinsurance markets, reinsurance represents an additional investment opportunity for asset/liability management decisions that is not or only weakly correlated with the capital markets investment opportunities and therefore moves a insurer's efficient frontier in the direction of higher expected returns at lower return standard deviations. Moreover, reinsurance improves the diversification of an insurer.

This is then the most important quantitative argument of the value proposition (VP) of reinsurance (and also of insurance). It is based on the notion of risk adjusted capital in a unified asset/liability management framework and holds over a wide range of insurers’ risk tolerance levels:

<table>
<thead>
<tr>
<th>Risk Tolerance (Epsilon)</th>
<th>M_1</th>
<th>M_2</th>
<th>M_3</th>
<th>Investment Opportunities</th>
</tr>
</thead>
<tbody>
<tr>
<td>x_4 = 10.00%</td>
<td>5.14</td>
<td>2.13</td>
<td>5.97</td>
<td>Stocks: 86.54% Other: 8.24%</td>
</tr>
<tr>
<td></td>
<td>9.00%</td>
<td>4.96</td>
<td>2.13</td>
<td>Stocks: 76.22% Other: 8.27%</td>
</tr>
<tr>
<td></td>
<td>8.00%</td>
<td>4.72</td>
<td>2.14</td>
<td>Stocks: 66.48% Other: 8.31%</td>
</tr>
<tr>
<td></td>
<td>7.00%</td>
<td>4.43</td>
<td>2.13</td>
<td>Stocks: 57.22% Other: 8.31%</td>
</tr>
<tr>
<td></td>
<td>6.00%</td>
<td>4.12</td>
<td>2.12</td>
<td>Stocks: 48.47% Other: 8.32%</td>
</tr>
<tr>
<td></td>
<td>5.00%</td>
<td>3.79</td>
<td>2.11</td>
<td>Stocks: 40.13% Other: 8.34%</td>
</tr>
<tr>
<td></td>
<td>4.00%</td>
<td>3.40</td>
<td>2.09</td>
<td>Stocks: 31.99% Other: 8.37%</td>
</tr>
<tr>
<td></td>
<td>3.00%</td>
<td>3.01</td>
<td>2.10</td>
<td>Stocks: 24.06% Other: 8.46%</td>
</tr>
<tr>
<td></td>
<td>2.00%</td>
<td>2.61</td>
<td>2.18</td>
<td>Stocks: 16.20% Other: 8.78%</td>
</tr>
<tr>
<td></td>
<td>1.00%</td>
<td>2.03</td>
<td>2.85</td>
<td>Stocks: 8.83% Other: 10.59%</td>
</tr>
</tbody>
</table>

Varying the Insurer's Risk Tolerance: Optimal MPT Portfolio Parameters
Fig. 21b: Varying the Insurer's Risk Tolerance: Optimal MPT Result Parameters

Fig. 21c: Varying the Insurer's Risk Tolerance: Result after Reinsurance

Fig. 21d: Varying the Insurer's Risk Tolerance: Efficient Frontier
5. Performance Measurement: The Return on Risk Adjusted Capital

In the last two paragraphs of this section, we are briefly considering the effect of the value proposition (VP) of reinsurance (i.e., the globally optimal MPT positioning of the insurer along the efficient frontier for both the capital markets and the insurance and reinsurance markets) on the internal performance measurement and the pricing of the insurer. We illustrate the corresponding concepts and ideas by again using our example insurance company with lines of business A, B and C (see above). Having optimally positioned this insurer at the 1.00% risk tolerance level along the globally efficient frontier (see Fig. 21 above), we use the covariance principle to allocate risk adjusted capital to individual operational subunits of this insurance company:

(1) Let \( R_1, \ldots, R_i, \ldots, R_M \) be the MPT optimal results of \( M \) operational subunits (e.g., lines of business, profit centers, departments, etc.) of a (re)insurance company and \( R = \sum_{i=1}^{M} R_i \) be its overall MPT optimal result. Furthermore, let \( RAC[R_i] \) denote the risk adjusted capital allocated to operational subunit \( i \) and \( RAC[R] \) denote the overall risk adjusted capital (or capital at risk or risk based capital) of the (re)insurer. Then the allocation rule is

\[
RAC[R_i] = \frac{C[R_i, R]}{V[R]} RAC[R],
\]

(2) Performance measurement on a risk adjusted capital basis then considers the return on risk adjusted capital (RORAC) for each operational subunit \( i \),

\[
RORAC[R_i] = \frac{R_i}{RAC[R_i]},
\]

and for the whole company,

\[
RORAC[R] = \frac{R}{RAC[R]}.
\]

(3) In the case of our example insurer, we look at the following operational subunits:

(a) **Investment:** Recall that \( RAC[Overall \ Result] = \text{CHF} \, 2500.00 \text{ million} \) and \( V[Overall \ Result] = 132183.14 \). The variance / covariance matrix relevant for the allocation of risk capital to investment and underwriting is:

\[
\begin{array}{ccc}
X & \text{X} & 130181.45 \\
J & -918.48 & -918.48 \\
\end{array}
\]

(b) **Underwriting:** Similarly, by using the above variance / covariance matrix
again, we obtain \( C[\text{Underwriting Result, Overall Result}] = 131099.90 \) and therefore
\[
RAC[\text{Underwriting}] = \text{CHF } \frac{131099.90}{132183.14} \times 2500.00\text{ million} = \text{CHF } 2479.56\text{ million.}
\]

(i) **Line of Business A**: The relevant variance / covariance matrix for the allocation of the total underwriting RAC of CHF 2479.56 million to the individual lines of risk business is:

<table>
<thead>
<tr>
<th></th>
<th>Line 1</th>
<th>Line 2</th>
<th>Line 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line 1</td>
<td>46100.68</td>
<td>4009.99</td>
<td>4609.66</td>
</tr>
<tr>
<td>Line 2</td>
<td>4009.99</td>
<td>8720.05</td>
<td>6014.45</td>
</tr>
<tr>
<td>Line 3</td>
<td>4609.66</td>
<td>6014.45</td>
<td>46092.53</td>
</tr>
</tbody>
</table>

**Fig. 22b**: Variance / Covariance Matrix for Lines of Business A, B and C

We have then \( V[\text{Total Book of Business}] = 130181.45 \) and \( C[\text{Line of Business A, Total Book of Business}] = 54720.33 \) and consequently
\[
RAC[\text{Line of Business A}] = \text{CHF } \frac{54720.33}{130181.45} \times 2479.56\text{ million} = \text{CHF } 1042.26\text{ million.}
\]

(ii) **Line of Business B**: Similarly, we have
\[
C[\text{Line of Business B, Total Book of Business}] = 18744.49
\]
and consequently
\[
RAC[\text{Line of Business B}] = \text{CHF } \frac{18744.49}{130181.45} \times 2479.56\text{ million} = \text{CHF } 357.03\text{ million.}
\]

(iii) **Line of Business C**: Finally, we have
\[
C[\text{Line of Business C, Total Book of Business}] = 56716.64
\]
and consequently
\[
RAC[\text{Line of Business C}] = \text{CHF } \frac{56716.64}{130181.45} \times 2479.56\text{ million} = \text{CHF } 1080.28\text{ million.}
\]

**Note**: The fact that the insurer under consideration is optimally positioned along the global efficient frontier of the capital markets and the insurance and reinsurance markets has important implications on the performance of its subunits. The same statement holds true for the prices of its value proposition (VP) based risk transfer solutions.

6. Pricing on a Risk Adjusted Capital Basis

Following standard actuarial tradition, we apply the **premium calculation principle**
\[ B(t) = [1 + \lambda(t)]P(t) + E(t) \]

with \( P(t) \) the technical or risk premium in time period \([t-1, t]\), \( \lambda(t) \) the profit loading coefficient and \( E(t) \) the expense amount (e.g., operating expenses, taxes, dividends, etc.), respectively (see also the previous section). Considering a fixed but otherwise arbitrary time period and dropping the time index again for simplicity of notation, it is specifically the profit loading
\[ \lambda P \]
for that time period which we are interested in here.

We assume that the (re)insurer under consideration has assumed its MPT optimal position along the globally efficient frontier for the capital markets and the insurance and reinsurance markets.

Modern value proposition (VP) based risk transfer solutions \( \mathcal{X} \) require the following lower bound for the associated profit loading coefficient \( \lambda \):

(1) Using the covariance principle again, the risk adjusted capital allocated to the VP based risk transfer solution \( \mathcal{X} \) is
\[
\text{RAC}[\mathcal{X}] = \frac{C[X,R]}{\text{V[R]}} \text{RAC}[R]
\]
where \( \text{RAC}[R] \) denotes the overall risk adjusted capital (or capital at risk or risk based capital) of the (re)insurer.

(2) If now \( \rho \) is the rate of return on risk adjusted capital required by management for the corresponding line of (re)insurance business, then
\[
\lambda P \geq \rho \text{RAC}[\mathcal{X}]
\]
\[
= \rho \frac{C[X,R]}{\text{V[R]}} \text{RAC}[R]
\]
or, equivalently, \( \lambda \geq \frac{C[X,R]}{\text{PV}[R]} \text{RAC}[R] \)

must hold.

(3) For VP based risk transfer solutions that contain a financial markets risk part, we shall be able to also give an upper bound for the associated profit loading coefficient (see further below).

Implementing New VP-based Client Solutions

In this final section of the technical note, we look at how the above mathematical models and the corresponding financial / (re)insurance techniques toolbox (FRT) used to quantify the economic rationale for reinsurance can also be used to implement new value proposition (VP) based client solutions and can therefore provide the (re)insurer with new market and profit growth opportunities. The example considered is derived from a real Swiss Re deal in which the models described here were successfully applied. The reinsurance coverage proposed by Swiss Re was of the dual trigger stop loss type:

(1) Under segment A of the reinsurance protection, a stop loss treaty protected the pension fund against excessive losses from its total book of risk business during 1998.
(2) Under segment B of the reinsurance protection, a stop loss treaty protected the pension fund against an excessive drop in the market value of its substantial bond portfolio during 1998.

(3) A loss event under the dual trigger stop loss reinsurance protection was defined by the simultaneous activation of both the segment A trigger and the segment B trigger.

1. Claims Data and Parameter Estimation for Segment A

Using the claims listing up to August 12, 1997, provided by the Swedish pension fund, we applied extreme value techniques to determine the 1998 aggregate loss distribution for the stop loss cover under segment A of the reinsurance protection. Specifically, we considered two scenarios (loss adjustment at 5% and 10%, respectively) with the same occurrence probability of 50%. Within each scenario, we estimated loss severity with the generalized Pareto distribution (GPD) and considered normally distributed parameter uncertainty of 25% at the 95th percentile for both frequency and severity. We believe that this is a prudent but not overly pessimistic approach under the given circumstances. The 1 Y (1998) aggregate loss distributions are then (aggregate loss in MSEK):

![Aggregate Loss Distributions](image)

Fig. 23: 1 Y Aggregate Loss Distributions (Loss Adjustment at 5% and 10%)

2. The Conditional Premium for Segment A

Let

\[ L \] be the 1998 aggregate loss under segment A

\[ P \] be the 1998 gross earned premium under segment A

and

\[ TA = \max(4P, 1000\text{MSEK}) \]
Here, $TA$ is the trigger threshold and $XA$ the corresponding cashflow under segment $A$ of the proposed dual trigger stop loss treaty. Furthermore, let us assume that $4P$ is normally distributed with mean 700MSEK (estimated 1997 gross premium income before reinsurance is 250MSEK) and standard deviation 100MSEK ($\approx 14\%$).  

Then, under segment $A$, the trigger is 

$$ L > TA, $$

the corresponding trigger probability 

$$ PA = \text{Prob}\{L > TA\} = \int_{0}^{1000\text{MSEK}} \left[1 - F_{L}(x)\right] f_{4p}(x) dx + \left[1 - F_{L}(1000\text{MSEK})\right] \int_{0}^{0} f_{4p}(x) dx $$

and the conditional net premium

$$E[XA|L > TA] = \int_{L>TA,P>0} XA(L,P)f_{L}(L)f_{4p}(P)dLdP $$

with standard deviation

$$\sigma[XA|L > TA] = \sqrt{\int_{L>TA,P>0} \left(XA(L,P) - E[XA|L > TA]\right)^{2} f_{L}(L)f_{4p}(P)dLdP}. $$

Here, 

$$f_{4p}(x) = \frac{1}{\sqrt{2\pi\sigma^{2}}} \exp\left(-\frac{(x - \mu)^{2}}{2\sigma^{2}}\right), \mu = 700\text{MSEK}, \sigma = 100\text{MSEK} $$

and (the density) $f_{L}(L)$ is a numerical approximation of the derivative of the ($1 \text{ Y}$ aggregate loss) distribution function $F_{L}(L)$.

$FRT$ then calculates the above quantities as follows:

$$\begin{align*}
PA &= 9.16\% \\
E[XA|L > TA] &= 12.76\text{MSEK} \\
\sigma[XA|L > TA] &= 43.30\text{MSEK}
\end{align*}$$

3. SEK Interest Rates and Interest Rate Volatilities

For our analysis of segment $B$ of the proposed dual trigger stop loss treaty, we note the following SEK interest rates (as of 2 October 1997)

---

7 These estimates were provided by the underwriters involved after discussion with both the broker and the Swedish pension fund.  
8 For such deals, Swiss Re sets the pricing rate to zero, $i = 0$, and the technical premium is taken to be equal to the expected loss under the reinsurance treaty.  
9 Financial markets information was provided by the investment bankers involved.
(here, the usual cubic spline-interpolation method is used; for a more sophisticated approach that also allows to incorporate the specifics of a potential client, e.g., tax brackets, corporate debt structures, etc., see the previous section) and SEK interest rate volatilities (as of 2 October 1997)
to which we (with FRT) then fit an extended Cox, Ingersoll and Ross (CIR) interest rate model\(^{10}\) (modelling 1M STIBOR; for details, see the literature mentioned in the references section of this document):

\(^{10}\) The Hull & White class of (one-factor) interest rate models has the generic representation

\[ dr(t) = \eta(\theta(t), \phi(t), r(t), t)dt + \sigma(t)dz(t) \]

and is rich enough to model a wide variety of different interest rate scenarios occurring in practical corporate and investment banking as well as financial (re)insurance applications. Here, we are especially interested in the model

\[ dr(t) = (\theta(t) - \phi(t)r(t))dt + \sigma(t)dz(t) \]

of the extended Cox, Ingersoll and Ross (CIR) type (belonging to the Hull & White class of interest rate models just as the above mentioned Ho & Lee model)

\[ dr = \phi(t)dt + \sigma(t)dz \]

\[ \phi(t) = \frac{d\theta}{dt}(0, t) + \sigma^2(t) \]

The generic class of Hull & White models has a very efficient (recombining) trinomial lattice implementation and is able to fit

(a) the current term structure of short-term interest rates [specification of the function \(\theta(t)\) which is deterministic and varies with time \(t\)];

(b) the current volatility structure of short-term interest rates [specification of the function \(\phi(t)\) which is also deterministic and varies with time \(t\)].

The risk-neutral pricing formula and the contingent claim sensitivities (derivatives risk parameters) are defined as in the Ho & Lee model above - with the obvious modifications to take the trinomial structure of the underlying lattice implementation into account. Note that the term structure estimation approach outlined in the previous section can be used to provide customized (i.e., taking client-specific tax-brackets, corporate debt structures, etc. into account) estimates of the initial term structure of short-term interest rates for both the Hull & White and the Ho & Lee model. Current volatilities are usually estimated from historical (discount) bond yield data.
The specification of the SEK benchmark bond is:

- **Maturity Date:** 12 April 2002
- **Coupon Date:** 12 April
- **Coupon:** 5.5% p.a., paid annually
- **Nominal Principal:** SEK 100.00

Note that we consider 60 monthly periods starting on 12 April 1997 and ending at the SEK benchmark bond's maturity on 12 April 2002.

**Risk Management Report (Standard Deviations)**

**Risk Management Report (Expectations)**
On 1 October 1997, the market ("dirty") price of the bond was SEK 102.53 and a SEK 1.00 change in the bond's value or, equivalently, a 25 BP (where 1 BP = 0.01%) change in SEK interest rates translated into a SEK 60.00 million change in the value of the pension fund's bond portfolio which had a total market value of SEK 6.15 billion.

B. SEK Short-term Rate (1M STIBOR) Distribution on 1 January 1998.

The corresponding mean is 5.1267% and the standard deviation 0.7932%.

C. SEK Short-term Rate (1M STIBOR) Distribution on 31 December 1998.

The mean is now 5.2445% and the standard deviation 1.1951%.
4. SEK/CHF Exchange Rate and Futures & Options Based Hedge Strategies

Apart from SEK interest rates, we also consider the SEK/CHF exchange rate very briefly here. On a monthly basis, the SEK/CHF exchange rate from 2 October 1990 to 2 October 1997 evolved as follows:

Exchange rate volatility during the above time period was 9.70%, the expected return 2.73%. In order to be able to develop SEK/CHF hedging strategies, we have to consider the differential between SEK and CHF interest rates during the 15 month period from October...
1997 to December 1998. Using the same spline-interpolation approach as above for SEK yields, we obtain the CHF term structure of interest rates (as of 2 October 1997)

**CHF Term Structure**

<table>
<thead>
<tr>
<th>Time Period (Years)</th>
<th>Interest Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0%</td>
</tr>
<tr>
<td>2.0</td>
<td>0.5%</td>
</tr>
<tr>
<td>3.0</td>
<td>1.0%</td>
</tr>
<tr>
<td>4.0</td>
<td>1.5%</td>
</tr>
<tr>
<td>5.0</td>
<td>2.0%</td>
</tr>
<tr>
<td>6.0</td>
<td>2.5%</td>
</tr>
<tr>
<td>7.0</td>
<td>3.0%</td>
</tr>
<tr>
<td>8.0</td>
<td>3.5%</td>
</tr>
</tbody>
</table>

**SEK Term Structure Volatility**

<table>
<thead>
<tr>
<th>Time Period (Years)</th>
<th>Bond Yield Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.2%</td>
</tr>
<tr>
<td>2.0</td>
<td>0.4%</td>
</tr>
<tr>
<td>3.0</td>
<td>0.6%</td>
</tr>
<tr>
<td>4.0</td>
<td>0.8%</td>
</tr>
<tr>
<td>5.0</td>
<td>1.0%</td>
</tr>
<tr>
<td>6.0</td>
<td>1.2%</td>
</tr>
<tr>
<td>7.0</td>
<td>1.4%</td>
</tr>
<tr>
<td>8.0</td>
<td>1.6%</td>
</tr>
</tbody>
</table>

**Fig. 26a:** CHF Bond Yields

and the corresponding term structure of CHF interest rate volatilities (as of 2 October 1997)

**Fig. 26b:** CHF Bond Yield Volatilities

With SEK interest rates at 5.0% and CHF interest rates at 1.5%, the interest rate differential is 3.5%. With these market parameters, the December 1998 SEK/CHF futures and corresponding European and American put option prices (in SEK) are:
Note finally the probability distribution of the December 1998 SEK/CHF exchange rate (with mean 4.6308 and standard deviation 0.5822):

<table>
<thead>
<tr>
<th>Futures</th>
<th>Futures/Option Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.6308</td>
<td>4.6308</td>
</tr>
<tr>
<td>5.0000</td>
<td>0.3927</td>
</tr>
<tr>
<td>5.1000</td>
<td>0.4628</td>
</tr>
<tr>
<td>5.2000</td>
<td>0.5330</td>
</tr>
<tr>
<td>5.3000</td>
<td>0.6096</td>
</tr>
<tr>
<td>5.4000</td>
<td>0.6904</td>
</tr>
<tr>
<td>5.5000</td>
<td>0.7715</td>
</tr>
<tr>
<td>5.6000</td>
<td>0.8560</td>
</tr>
<tr>
<td>5.7000</td>
<td>0.9434</td>
</tr>
<tr>
<td>5.8000</td>
<td>1.0308</td>
</tr>
<tr>
<td>5.9000</td>
<td>1.1192</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>European Put Option</th>
<th>Futures/Option Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0000</td>
<td>0.5752</td>
</tr>
<tr>
<td>5.1000</td>
<td>0.6752</td>
</tr>
<tr>
<td>5.2000</td>
<td>0.7752</td>
</tr>
<tr>
<td>5.3000</td>
<td>0.8752</td>
</tr>
<tr>
<td>5.4000</td>
<td>0.9752</td>
</tr>
<tr>
<td>5.5000</td>
<td>1.0752</td>
</tr>
<tr>
<td>5.6000</td>
<td>1.1752</td>
</tr>
<tr>
<td>5.7000</td>
<td>1.2752</td>
</tr>
<tr>
<td>5.8000</td>
<td>1.3752</td>
</tr>
<tr>
<td>5.9000</td>
<td>1.4752</td>
</tr>
</tbody>
</table>

5. The Benchmark Bond and Corresponding Futures & Options

The SEK benchmark bond mentioned in segment B of the proposed dual trigger stop loss treaty as a tracking variable for the substantial bond portfolio of the Swedish pension fund then has the following characteristics (calculated by FRT):
Note that we consider 60 monthly periods starting on 12 April 1997 and ending at the benchmark bond’s maturity on 12 April 2002. The expected price, SEK 100.97, predicted by the model for 1 October 1997 matches the corresponding trading price, SEK 102.53, very closely (the inaccuracy is only about 1%). The December 1998 benchmark bond futures and corresponding European and American put option prices (in SEK) are:

<table>
<thead>
<tr>
<th>December 98</th>
<th>Strike Price</th>
<th>Futures/Options Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Futures</td>
<td></td>
<td>101.9904</td>
</tr>
<tr>
<td>European Put Option</td>
<td>101.0000</td>
<td>0.7514</td>
</tr>
<tr>
<td></td>
<td>103.0000</td>
<td>1.6724</td>
</tr>
<tr>
<td></td>
<td>105.0000</td>
<td>3.0275</td>
</tr>
<tr>
<td>American Put Option</td>
<td>101.0000</td>
<td>2.7124</td>
</tr>
<tr>
<td></td>
<td>103.0000</td>
<td>4.3810</td>
</tr>
<tr>
<td></td>
<td>105.0000</td>
<td>6.2113</td>
</tr>
</tbody>
</table>

Alternatively, monthly resettled December 1998 caps and floors on the SEK short-term interest rate (1M STIBOR) would cost:
We mention these financial markets techniques here because they could be considered as alternative ways of insuring the pension fund’s bond portfolio against a rise in SEK interest rates (and as a benchmark for our approach). As outlined below, by comparing segment B of our dual trigger stop loss treaty with these financial markets portfolio insurance schemes, we will be able to derive a “market-implied” profit loading coefficient as an upper bound for the profit loading coefficient of our stop loss treaty. Recall from the previous section that the VP based profit loading coefficient is a corresponding lower bound.

B. Delta.
C. Gamma.

![Risk Management Report (Expectations) Graph]

D. Theta.

![Risk Management Report (Expectations) Graph]
E. Price Distribution on 1 January 1998 (mean = 102.2022, std = 2.5526).

Note that the above European and American put option strike prices of SEK 101.00, SEK 103.00 and SEK 105.00 correspond to the expected value, the 80th percentile and the 95th percentile of the January 1, 1998, benchmark bond value distribution.

6. The Segment B Trigger and Conditional Premium

The Swiss Re contract proposal for the dual trigger stop loss treaty under consideration here states the segment B trigger in the form

\[ i_1 > i_0 + 150 \text{BP}, \text{ where } 1 \text{BP} = 0.01\% . \]

\( i_0 \) is the benchmark bond yield on 1 January 1998, whereas \( i_1 \) is the benchmark bond yield on 31 December 1998. Let now

\[
\begin{align*}
B_0 & \text{ be the traded price of the benchmark bond on 1 January 1998} \\
B_1 & \text{ be the traded price of the benchmark bond on 31 December 1998}
\end{align*}
\]

and

\[
\begin{align*}
TB &= \max[(B_0 - B_1)60\text{MSEK} - 360\text{MSEK},0] \\
XB &= \min(TB,250\text{MSEK}).
\end{align*}
\]

Then the above stated trigger condition is equivalent to

\[ TB > 0 \text{ or } B_1 < B_0 - 6 \]

[with \( PB = \text{Prob}\{B_1 < B_0 - 6\} \)] and \( XB \) is the corresponding cashflow under segment B of the proposed dual trigger stop loss treaty. We price segment B of this treaty by using the loss event contingent valuation techniques described in the literature listed in the references section of this document. To this end, we consider the probability distribution of \( B_0 \) (see above) and derive the following valuation (loss event) scenarios
FRT then determines that

\[
\text{PR} = 11.31\% \\
\text{E}[XB|TB > 0] = 4.67\text{MSEK} \\
\sigma[XB|TB > 0] = 11.52\text{MSEK}
\]

(for more details, see the references section of this document). In addition, we also get the main characteristics of both the trigger TB (FPGTB) and the loss event contingent cashflow XB (FPGCB):

A. Trigger and Cashflow Value Process.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Number & Probability & Description \\
\hline
1 & 0.9900% & 93.6182 & 0.9900% \\
2 & 0.9500% & 94.1182 & 0.4400% \\
3 & 0.6100% & 96.1182 & 0.6500% \\
4 & 3.3000% & 96.6182 & 3.9500% \\
5 & 7.2000% & 97.1182 & 11.1500% \\
6 & 17.5000% & 98.1182 & 13.4500% \\
7 & 22.7000% & 98.6182 & 19.1700% \\
8 & 32.4000% & 100.1182 & 31.6000% \\
9 & 23.8000% & 101.1182 & 37.3400% \\
10 & 23.7800% & 103.1182 & 81.0200% \\
11 & 11.1000% & 104.1182 & 92.2100% \\
12 & 3.4600% & 105.1182 & 95.5900% \\
13 & 2.6600% & 105.6182 & 97.8500% \\
14 & 0.3800% & 106.6182 & 98.2300% \\
15 & 1.4100% & 108.1182 & 99.6400% \\
16 & 0.3600% & 110.6182 & 100.0000% \\
17 & & & \\
18 & & & \\
19 & & & \\
20 & & & \\
\hline
\end{tabular}
\end{table}

\footnote{Note that the protection of the pension fund's bond portfolio is activated with probability 11.31\%, i.e., \( \text{PB} = 11.31\% \). This could be changed by increasing/decreasing the trigger B attachment point above / below the level of 360MSEK.}
B. Trigger and Cashflow Time Value.
C. Trigger and Cashflow Delta.

D. Trigger and Cashflow Gamma.
E. Trigger and Cashflow Theta.

Risk Management Report (Expectations)

Risk Management Report (Standard Deviations)
7. The Total Premium of the Dual Trigger Stop Loss Treaty

With an actuarial loading factor \( k \) to be determined by the experienced underwriter (or by a VP based approach like the one described above), the total premium of the dual trigger stop loss treaty analyzed here is

\[
P = PB(E[X_A | L > TA] + k\sigma[X_A | L > TA]) + PA(E[X_B | TB > 0] + k\sigma[X_B | TB > 0]).
\]

Numerically (FRT):

<table>
<thead>
<tr>
<th>( k )</th>
<th>( P ) (MSEK)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>1.87</td>
</tr>
<tr>
<td>0.25</td>
<td>3.36</td>
</tr>
<tr>
<td>0.50</td>
<td>4.85</td>
</tr>
<tr>
<td>0.75</td>
<td>6.34</td>
</tr>
<tr>
<td>1.00</td>
<td>7.82</td>
</tr>
<tr>
<td>1.25</td>
<td>9.31</td>
</tr>
<tr>
<td>1.50</td>
<td>10.80</td>
</tr>
<tr>
<td>1.75</td>
<td>12.29</td>
</tr>
<tr>
<td>2.00</td>
<td>13.78</td>
</tr>
</tbody>
</table>

Of course, the standard deviation premium principle used here is equivalent to the traditional premium calculation principle introduced earlier in this document:

\[
\lambda E[X_A | L > TA] = k\sigma[X_A | L > TA] \quad \text{or} \quad \lambda = k \frac{\sigma[X_A | L > TA]}{E[X_A | L > TA]}
\]

Therefore, the VP based profit loading coefficient determined in the previous section provides a lower bound for the actuarial loading factor \( k \) considered here. We now show how a “market-implied” actuarial loading factor can be determined from financial market information. This profit loading coefficient can then be used as a corresponding upper bound (otherwise arbitrage between financial and financial reinsurance markets would be possible).

Basically, a financial markets solution (bond portfolio insurance scheme) for segment B of the dual trigger stop loss treaty would be based on December 1998 European put options or January 1998 to December 1998 SEK interest rate caps:

A. European Put Options.

Starting point for a put option hedge is the benchmark bond price distribution on 1 January 1998 (see above). Two reasonable strike prices for the bond portfolio insurance under segment B of the dual trigger stop loss treaty would then be SEK 101.00 and SEK 103.00
(note that the mean is SEK 102.20 and the standard deviation SEK 2.55). Per 100 notional, a corresponding December 1998 European put option would therefore cost

<table>
<thead>
<tr>
<th>Strike (SEK)</th>
<th>Option Price (SEK)</th>
</tr>
</thead>
<tbody>
<tr>
<td>101.00</td>
<td>0.7514</td>
</tr>
<tr>
<td>103.00</td>
<td>1.6724</td>
</tr>
</tbody>
</table>

which means that the entire bond portfolio insurance under segment B of the treaty would cost (multiply the above option prices with MSEK 60.00)

<table>
<thead>
<tr>
<th>Strike (SEK)</th>
<th>Bond Portfolio Insurance Cost (MSEK)</th>
</tr>
</thead>
<tbody>
<tr>
<td>101.00</td>
<td>45.08</td>
</tr>
<tr>
<td>103.00</td>
<td>100.43</td>
</tr>
</tbody>
</table>

and the corresponding market implied \( k \) s would be:

<table>
<thead>
<tr>
<th>Strike (SEK)</th>
<th>Market Implied ( k )</th>
<th>Stop Loss Treaty Price (MSEK)</th>
</tr>
</thead>
<tbody>
<tr>
<td>101.00</td>
<td>3.51</td>
<td>22.76</td>
</tr>
<tr>
<td>102.00</td>
<td>5.91</td>
<td>37.05</td>
</tr>
<tr>
<td>103.00</td>
<td>8.31</td>
<td>51.34</td>
</tr>
</tbody>
</table>

**B. SEK Interest Rate Caps.**

Starting point for an interest rate cap based bond portfolio insurance under segment B of the dual trigger stop loss treaty is the 1M STIBOR distribution on 1 January 1998 (see above). A reasonable cap rate for January 1998 to December 1998 would be 5.00% (note that the mean is 5.13% and the standard deviation 0.79%). Per 100 notional, a corresponding monthly resettled (i.e., 1M STIBOR) cap would then cost

<table>
<thead>
<tr>
<th>Strike Rate</th>
<th>Cap Price (SEK)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.00%</td>
<td>0.5798</td>
</tr>
</tbody>
</table>

which means that the entire bond portfolio insurance under segment B of the treaty would cost (multiply the above cap price with MSEK 60.00)

<table>
<thead>
<tr>
<th>Strike Rate</th>
<th>Bond Portfolio Insurance Cost (MSEK)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.00%</td>
<td>34.79</td>
</tr>
</tbody>
</table>

and the corresponding market implied \( k \) would be:

<table>
<thead>
<tr>
<th>Strike Rate</th>
<th>Market Implied ( k )</th>
<th>Stop Loss Treaty Price (MSEK)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.00%</td>
<td>2.61</td>
<td>17.41</td>
</tr>
</tbody>
</table>
References

References for the extreme value theory approach are:


Swiss Re applications (non-life, financial reinsurance, securitisation) are described in:


FRT is described in detail in:
