Valuation of Credit Default Swap and Parameter Estimation for Vasicek-type Hazard Rate Model

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ABSTRACT

It is a significant and difficult problem how default should be modeled for estimating the value of credit derivatives.

This article is made up of two parts --- theoretical one and empirical one. We first concentrate upon a kind of credit derivatives called credit default swap and discuss its valuation. The other objective is to present the method of estimating parameters on Vasicek type model such as stochastic differential equations models of term structure and illustrate some simulation examples.

(This article based on our contributing paper.)

1 Introduction

The starting point of our research is to consider how credit default swap, a kind of credit derivatives, is modeled for estimating the value. Roughly speaking, credit default swap is, like a sort of insurance, defined as the contract made with the holder of a defaultable bond, which obliges the counter party of the swap to compensate the loss that the holder suffers at the default in return for a regular premium income. Since the demand for such contract, for the purpose of hedging credit risk, is more and more increasing and appear some new versions, for example, one for more than two defaultable bonds and one with counter party risk,

There are several problems for modeling the credit derivatives, however. It is in general the most significant the way the event of default should be characterized. Duffie and Singleton(1994) propose the model in which the default is specified by a hazard rate process deeply connected with the distribution of default time. In their model it is possible to price the defaultable claims almost the same way as non-default claims. Davis and Mavroidis(1997) utilize the above framework to study the valuation of credit default swap by supposing that the hazard rate is a Gaussian model with time-dependent deterministic drift. In this article we also consider the default model which takes the hazard rate as principal factor following these previous model.

This paper consists of two parts --- theoretical one and empirical one. The first half is the theoretical one which explains the scheme of credit default swap that we take up (Section 2) and gives the general valuation of default swap which contains the case of so-called basket type and counter party risk type (Section 3). The evaluation formula obtained there seems to be too complicated for numerical computation, but under the hypothesis that the hazard rate is modeled in the form of affine type or quadratic Gaussian type term structure model such as Vasicek model, CIR model and so on, the expectation can be numerically calculated by solving some Riccaci type ordinary differential equations. It is often effective in thinking the case where there is a correlation among the issuers of defaultable bonds. (See [3],[4] and [9].) Anyway it is concluded that the choice of hazard rate model is important in order to evaluate the value of credit default swap. Mathematical discussion is omitted to the utmost extent.

In the latter half, we devote ourselves to estimate parameters of hazard rate model. It is suitable for the setting that the hazard rate is specified in terms of a stochastic differential equation such as term structure models, say, Vasicek model, CIR model and so on. The first is the question if such SDE models are really suitable to hazard rate process. We believe that this approach is rather efficient since it is possible to see the hazard rate indirectly through the spread between riskless interest rate and default-adjusted interest rate. Besides the hazard rate is expected to have such properties as risk-free interest rate, for instance, mean-reverting and positive. Vasicek model is often used as risk-free interest rate model since it is mean-reverting and Gaussian, although it has a serious defect of taking negative value with a positive probability. It is also noted that the mean-reverting property lessens the possibility of negative hazard rate in comparison with the non mean-reverting Gaussian case. That is why Vasicek type is allowed to use as hazard rate model in simulation and is our main object. On the other hand, hazard rate is guaranteed to be positive, for example, when choosing CIR model or Quadratic-Gaussian model. CIR model is also widely used because it is mean-reverting and affine as well as positive. Quadratic-Gaussian model is tractable when considering the correlation among riskless interest rate and each issuer's hazard rate. The way of parameter estimation for these models is analogue to that for Vasicek type.

The second problem is whether the parameter estimation on such models is sufficiently practicable. We often have to estimate parameters from scanty sample data since exists the case that available data on the underlying market are limited. For the purpose, it is necessary to establish the method of estimation using only a few market data. It seems an interesting problem from a statistical viewpoint. In fact, the investigation of better method still remains as our task. In the present paper we principally apply the moment method to parameter estimation of Vasicek type models. Some simulation results using past Japanese market data are given as well.

The remains of the present paper are organized as follows. In Section 4, discussed in the case of Vasicek type hazard rate is the relation between hazard rate and credit spread that is required for converting the credit spread data gotten directly from the market into the hazard rate data. Section 5 presents the methods of parameter estimation on Vasicek and the modified version and illustrates some simulation results based on Japanese market data. Conclusion and some comments are seen in Section 6. Appendix gives discussion about the way of setting a standard hazard rate for each grade.

2 Formal Definition of Credit Default Swap

In the first place we give a formal definition of credit default swap. We begin with confirmation of the rule of default swap. It sounds so complicated, but it is indeed not so complicated. It is defined as a contract made between two parties --- one is a firm (called `A') which holds $q$ references (e.g. defaultable bonds issued by some firms) and the other is a bank (called `B'), for example. Each k-th reference (assumed to be a bond with coupon) that `A' holds pays a fixed coupon $b_j^{(k)}(k=1,\cdots,q, j=1,2,\cdots)$ at each fixed time $u_j^{(k)}$ $(k=1,\cdots,q\ , j=1,2,\cdots,\ 0\le u_1^{(k)}<u_2^{(k)}<\cdots)$ unless the default happens. We will call `C^k' the issuer of the k-th reference $(k=1,\cdots,q)$ held by `A'. In particular we often say ``basket type'' when $q\ge 2$. We will call ``single type'' the case of $q=1$.

The outline is as follows:

①T is the contract termination date. If the first default among the references --- `C^k''s occurs before T, the contract is then closed.

②`A' has to pay for `B' a fixed premium $c_i\ (i=1,\cdots,n)$ at each fixed time t_i $(i=1,\cdots n,\ 0\le t_1<\cdots<t_n\le T)$. (This cashflow is called the fixed side.)

③`B' pays an amount of the difference between the notional (we set 1) and the recovery if the default of some reference, say `C^k', occurs before T and it is the first default among the references `A' holds. The liquidation for fixing the insurance will be done just at the default time, but the actual payment date shall be $u_j^{(k)}$ if the first default happens between $u_{j-1}^{(k)}$ and $u_j^{(k)}$. For a mathematical technical reason, the following regulation is added. The insurance that is really paid from `B' to `A' should have the amount equal to the interest which could be received if the sum of the amount settled at the default time was deposited between the default time and payment date. This means that the loss due to a time lag between the clearance and payment will be completely avoided. Here we define the recovered amount from `C^k' by L^k times the immediate pre-default value of the bond, where L^k is a constant with $0 < L^k < 1$. (This stream of money is called the recovery side.)

We remark that the ways of defining the recovered amount from the bond issuer are variously considered. Here we make use of the recovery rate, which is defined as the ratio of the recovered amount to the pre-default reference value, though it is indeed so hard to obtain sufficient data on the recovery rate directly from the market. For all practical purposes, not the recovery rate but only the recovered amount from the reference issuer is reported.

Besides, the actual contract, unlike the above, often provides that the payment of the corresponding amount for the loss is within ε dates after the default, but it doesn't matter as we may replace the payment date with the first default time plus ε without difficulty.

In addition to the above basic scheme, we take account of counter party risk, that is, the possibility that `A' or `B' falls into default before clearing up the financial obligation. It is natural to close the contract if either is led into default. However, we remark the point that even if the default of `A' should happen between the first reference-default occurs and the payment date of the compensation, `B' has to pay the corresponding amount for `A' as long as `B' survives. We will call this ``general type''.

According to these formal rules, we will discuss the value of fixed side and recovery side respectively in Section 3.

3 Valuation of Credit Default Swap

The final goal of valuation of credit default swap is to get equilibrium premium c_i's paid regularly paid by the reference-holder, which is followed from the equality of the value between fixed side and recovery side.

We argue on the general type of credit default swap, that is, the basket type containing counter party risk.

Instantaneous riskless interest rate is given by the process r. We hereafter suppose that the riskless interest rate is independent of any factor related to the credit risk, such as hazard rates and default time. This assumption is slightly strong, but it enables us to represent the value of default swap in terms of the prices of risk-free zero coupon bonds as seen below.

For each $k=1,\cdots,q$, let $\tau^k (k=1,\cdots,q)$ be the default time of the k-th firm for $k=1,\cdots,q$, τ^A (resp. τ^B) the default time of `A'(resp. `B'). Furthermore denote by h^k the hazard rate process for τ^k. Intuitively speaking, for a small time interval of length Δ, $h^k(t)\Delta$ is viewed as the conditional probability at time t that the default of k-th firm occurs between t and $t+\Delta$, given the condition that the k-th firm survive at t. (both h^A and h^B are the same as the above.)

Besides, define by $\tilde{\tau}$ the first default time of the reference basket, that is, $\tilde{\tau} = \min_{k=1,\cdots,q} \tau^k$. We remark that the hazard rate process of $\tilde{\tau}$ is given by $h(t)=\sum_{k=1}^{q} h^k(t)$.

Assume that all the randomness is measured under a risk-neutral probability measure. (denote by P)

One can refer to [5] and [8] for standard argument on defaultable securities.

3.1 Fixed side

Since all the premiums are measured at the initial level and no premium is paid after any default happens, the current value P_F of the fixed side is naturally defined by

$$P_F = E\left[\sum_{i=1}^{n} c_i \cdot \exp\left(-\int_0^{t_i} r_u du\right) 1_{\{t_i < \tilde{\tau}\wedge\tau^A\wedge\tau^B\}}\right],$$

where $E[\cdot]$ is the expectation under P and $a\wedge b\cdots$ stands for the minimum among $a,b,\cdots$.

Then it follows from standard argument about default securities that

$$P_F = \sum_{i=1}^{n} c_i \cdot Z(0,t_i)\cdot E\left[\exp\left(-\int_0^{t_i}\left(h(u)+h^A(u)+h^B(u)\right)du\right)\right], \tag{1}$$

where $z(s,t)$ stands for the price at time $s(<t)$ of a t-maturity riskless zero-coupon bond.

3.2 Recovery side

We define the cum-coupon value of the underlying k-th reference by

$$B^{(k)}(t) = E_t\left[\sum_{u_i^{(k)}\geq t} b_i^{(k)} \cdot \exp\left(\int_t^{u_i^{(k)}} \left\{r_s + (1-L^k)h^k(s)\right\} ds\right)\right],$$

where $E_t[\cdot]$ is the conditional expectation given by the information up to time t.

It is presumed that the counter party of the reference-holder can recover $L^k Y^k(\tau^k)$ when the issuer of k-th reference happens to go into default and it is the first default among q issuers.

Hence we define the value P_R of the recovery side by

$$P_R = E\left[\sum_{k=1}^{q} \exp\left(-\int_0^{\tau^k} r_s ds \right) \left(1 - L^k Y^k(\tau^k)\right) \cdot 1_{\{\tilde{\tau} \le T, \tau^A > \tilde{\tau}, \tau^B > u_k(\tilde{\tau})\}} 1_{\{\tilde{\tau} = \tau^k\}} \right],$$

where $u^k(t) = \min\left\{ u_j^{(k)} \middle| u_j^{(k)} > t \right\}$.

It follows from standard argument and the given assumptions that

$$P_R = \sum_{k=1}^{q} \int_0^T Z(0,t) \cdot E\left[\xi^k(t) \cdot h^k(t)\right] dt \tag{2}$$

$$- \sum_{k=1}^{q} L^k \sum_{i=1} b_i^{(k)} Z(0, u_i^{(k)}) \int_0^{T \wedge u_i^{(k)}} E\left[E_t\left[\exp\left(-\int_t^{u_i^{(k)}} (1 - L^k) \cdot h^k(s) ds \right) \right] \xi^k(t) h^k(t) \right] dt.$$

where

$$\xi^k(t) = \exp\left(-\int_0^t \left(h(s) + h^A(s)\right) ds - \int_0^{u^k(t)} h^B(s) ds \right).$$

4 Estimation Models with hazard rate process

The previous section gives the general evaluation of credit default swap. Now we consider how to estimate the value from the market data under some specific hazard rate models. We suppose hereafter that the riskless interest rate r is independent of all the hazard rates, that is, the happening of default is not correlated with riskless bond prices. The hypothesis implies that it is not the level of riskless interest rate but some factors of the issuer's own that should cause the default.

For simplicity, we also assume that the reference is single and it is a defaultable zero-coupon bond with maturity T and notional 1. This means that the bond price at initial time $P(0,T)$ is given by

$$P(0,T) = Z(0,T) \cdot E\left[\exp\left(-\int_0^T (1 - L) h(s) ds \right) \right],$$

where $h(t)$ is the hazard rate with respect to the single reference and L is the recovery rate. In other words, the credit spread at time t, the difference between the default-adjusted interest rate and the risk-free one, is given by $(1 - L)h(t)$.

It is an advantage for estimation of parameters that the relation between the credit spread and the factor on default is observable. However it is impossible to estimate L and h separately from the credit spread, so given L (estimated by another means), we estimate the parameters of the hazard rate h.

Fix $T > 0$. The process $\{y(t,T)\}_{t\in[0,T]}$ is called the credit spread at time t for the bond with maturity T if $\{y(t,T)\}_{t\in[0,T]}$ satisfies the following relation

$$\exp(-y(t,T)(T-t)) = \frac{P(t,T)}{Z(t,T)}$$
$$= E_t\left[\exp\left(-\int_0^T (1-L)\cdot h(s)ds\right)\right] \quad \text{for } t\in[0,T].$$

Note that if the hazard rate is constant, $h(t)\equiv h_0 > 0$, then $y(t,T) = (1-L)\cdot h_0$ since $\exp(-y(t,T)(T-t)) = \exp(-(1-L)h_0(T-t))$.

We use this as fundamental data to calculate the estimators.

In the remainder of this section We illustrate the valuation and estimation in the case that Vasicek model is assumed as the hazard rate. It has a mean-reverting property and is tractable due to its Gaussian property. It goes without saying that we must take notice of the fact that the hazard rate may come into negative, that is, the survival probability at some time may exceed one! However the procedure of valuation and estimation is not so different from models and Vasicek model is often used as risk-free short rate model, so we manage them under this hypothesis. (That is, we accept the existence of negative hazard rate to some extent.)

Define the hazard rate process h by

$$dh(t) = c(m-h(t))\,dt + \sigma\, dW_t \ , \ h(0) = h_0 > 0,$$

where c, m and σ are positive constants and W is a one dimensional standard Brownian motion.

As is well known, c, m and σ are regarded as the parameters that stand for the reverting velocity to the long-term average, the long-term average, the volatility respectively.

Let τ be the default time of the reference issuer. Then

$$P(\tau > t) = E\left[\exp\left(-\int_0^t h(s)ds\right)\right]$$
$$= \exp\left[\frac{1}{c}(e^{-ct}-1)h_0 - \frac{1}{c}(e^{-ct}-1)\left(m + \frac{\sigma^2}{4c^2}(e^{-ct}-3)\right) - t\left(m - \frac{\sigma^2}{2c^2}\right)\right]. \quad (3)$$

The first equality is based on the fundamental property between default time and hazard rate. The second equality follows from the usual Feynman-Kac solution to the corresponding PDE. (See Nakagawa(1998) for an affine class.)

Now we present the relation between the credit spread and the parameters of hazard rate. Given the credit spread $y(t,T)$, by the Feynman-Kac approach, we have

$$\exp(-y(t,T)(T-t))$$

$$= \exp\left(\frac{1}{c}(e^{-c(T-t)} - 1)(1-L)\left(\left(h(t)-m\right) - \left(e^{-c(T-t)} - 1\right)\Phi(t)\right) \right), \tag{4}$$

equivalently,

$$h(t) = \frac{c}{1-L}\left(\frac{-y(t,T)(T-t)}{e^{-c(T-t)} - 1} - \Phi(t) \right) + m, \tag{5}$$

where

$$\Phi(t) = \frac{(T-t)(1-L)\left\{ m - \frac{\sigma^2}{2c^2}(1-L) \right\}}{e^{-c(T-t)} - 1} + \frac{\sigma^2}{4c^3}(1-L)^2(e^{-ct} - 3).$$

The value of credit default swap, both fixed side and recovery side, can be calculated from (1) and (2) if the law of hazard rate, that is, each parameter of the model is specified. Thus the evaluation of the swap is reduced to the problem how the parameters are computed.

This scheme remains unchanged when other hazard rate models are used.

5 Method of Parameter Estimation with Vasicek Model and Some Results

As seen in the last section, the parameters of hazard rate process can be estimated from the credit spread calculated by the market data. In the case such as Vasicek model, besides the recovery rate L, we have to obtain the estimator for c, m and σ. That means that we need at least four different spread data. However it is not simple since only a few kinds of the bonds are issued by each firm and the number traded in the market is not large enough to estimate all the parameters. Therefore we cannot help give some parameters exogenously and compute the relatively important ones impliedly.

We consider the estimation procedure when the bond which the firm issues is only one or the number of the reference bonds that is satisfyingly traded in the market is only one. As the hazard rate model, no mean-reverting Gaussian type and Vasicek type are supposed below.

5.1 Characteristics of parameters

Before discussing the method, we note again that the distribution of the default time (3) may be not monotonously decreasing, that is, the probability of default may be over one if using the model like Vasicek type.

Figure 1,2 and 3 show the probability of the firm's survival in twenty years from now, where the volatility is given by 5%, 1% and 0.5% respectively and the initial annual hazard rate is 1% in common. Three curves in each figure respectively illustrate Gaussian type with zero drift, Vasicek type with mean-reverting velocity $c=0.2$ and with $c=0.5$.

The curves of zero-drift Gaussian imply that the survival function is increasing on the way and such a trend is conspicuously observed when the volatility is high. It is unfavorable that the probability of default increases in time, so this model is difficult to apply to the case of long period and high volatility.

On the other hand, from the result of Vasicek model, it follows that the misgivings that the probability proceeds one is getting less than zero-drift Gaussian and that the larger the mean-reverting speed c is, the less the possibility is. It is still necessary to set a desirable value of the speed carefully since a higher mean-reverting speed removes rid of the effect of volatility, but it is concluded that Vasicek type is improved in comparison with zero-drift Gaussian type.

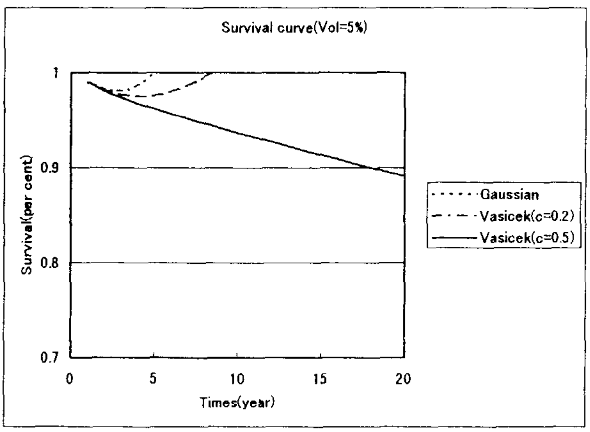

Figure 1: Survival Probabilities with $\sigma=5\%$

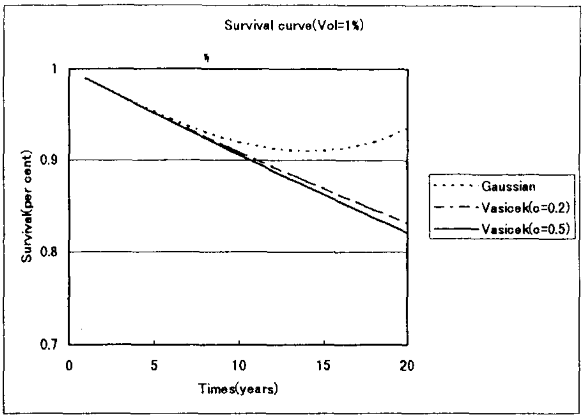

Figure 2: Survival Probabilities with $\sigma = 1\%$

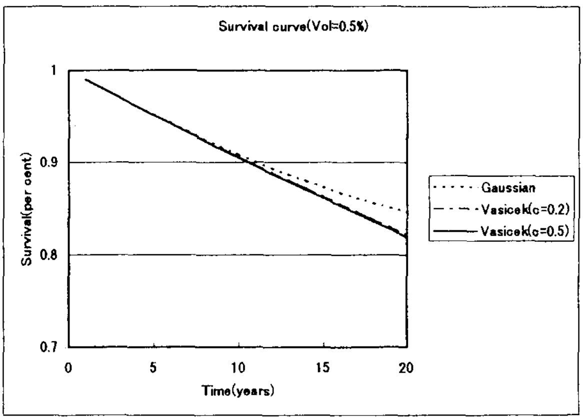

Figure 3: Survival Probabilities with $\sigma = 0.5\%$

5.2 Estimating parameters

The way of implicitly estimating the parameters, such as the implied volatility often used in option valuation, is strongly wanted, while they are estimated from the historical data. Here we take Vasicel model as the hazard rate model, but the same procedure is applicable to other models.

①Settlement of recovery rate L.

Moody's database contains the bond price after default, to be precise, the market price of the bond at one month after the issuer fell into a default, which is regarded as the recoverable amount from the default bond. Naturally the surplus of debt differs from firm to firm, but we assume that the recovery rates of the same type of industry and the same financial rating are common and view as the referred firm's recovery rate L, the mean value calculated from Moody's data of recovery rate every type of industry and rating.

②Estimation of long-term mean hazard rate m.

Under the hypothesis that the long-term mean hazard rate coincides for the same type of industry and the same rating, we utilize as the estimator of m the mean value of probability of default on each type of industry and rating reported by Moody's or S & P.

③Estimation of men-reverting speed c and volatility σ.

It is difficult to obtain the implied estimators of c and σ separately from data of the spread upon only one reference bond, so we pay attention to the limit distribution of Vasicek hazard rate and try to apply the moment method.

That is, as Vasicek hazard rate $h(t)$, satisfies as $t \to \infty$,

$$h(t) \sim N\left(m, \frac{\sigma^2}{2c}\right),$$

we presume that this homogenization holds universally in t. In order to practice the moment method, we use the values at n points $t_1, \cdots, t_n$ calculated from (5) as data. Then we have

$$\frac{1}{n}\sum_{i=1}^{n} h(t_i) = m, \tag{6}$$

$$\frac{1}{n}\sum_{i=1}^{n} h(t_i)^2 = m^2 + \frac{\sigma^2}{2c}. \tag{7}$$

Hence we have only to obtain c and σ which satisfy (6) and (7) simultaneously, for example, by solving the following non linear programming problem. Let

$$z_1 = \frac{1}{n}\sum_{i=1}^{n} h(t_i) - m,$$

$$z_2 = \frac{1}{n}\sum_{i=1}^{n} h(t_i)^2 - m^2 - \frac{\sigma^2}{2c}.$$

The problem is to minimize z_1^2 and z_2^2 subject to $c > 0$, $\sigma \geq 0$. In short, we seek c and σ that minimize both object functions z_1^2 and z_2^2 at the same time.

5.3 Validity of moment method

As is well realized, while the moment method is rather easy to use, the estimator may not be unique and tend to contain larger error than other estimation methods. We analyze how much error may happen to our case by using Monte Carlo simulation. It is not rare that only twelve historical data (one data per month) are gotten for pricing the credit derivatives. Therefore we examine validity of applying the moment method as the way of estimation when twelve historical data are given. The procedure is as follows. And, SAS (Release 6.12 for Windows) helps our analysis and NLP Procedure in SAS ([10]) also does solution of non-linear programming problem.

1. Begin with formation of hazard rate sequences. The sequences of hazard rate is recursively generated by the following discretized model:
   $$h(t) = h(t-1) + c\big(m - h(t-1)\big)\,\Delta t + \sigma\sqrt{\Delta t}\,\varepsilon,$$
   where ε is a standard normal random variable and $\Delta t = \dfrac{1}{12}$ due to monthly data.
   Nine patterns are given by combination of parameters c, m and σ in the following way and 10,000 sequences are created as discrete 30-year(360-month) hazard rates for each combination of given parameters.
   $$m = 0.1\,,\ \begin{cases} c = 0.05 \\ c = 0.20 \\ c = 0.50 \end{cases},\ \begin{cases} \sigma = 0.5\% \\ \sigma = 1.0\% \\ \sigma = 5.0\% \end{cases}$$
   The initial hazard rate h_0 is given by $m = 0.1$. Moreover, let the bond maturity $T = 30$ (year) and the recovery rate $L = 0.3$.
2. On the other hand, pick out twelve different-time market data for each combination of parameters above. Estimate m and σ by using the moment method for twelve hazard rate data calculated from the relation (5).
3. Compare the estimates with the parameters given ahead.

Table 1 shows the result on estimation for c and Figure 5 displays standard deviations of the estimated mean-reverting speed $\hat{c}$. It is guessed that the standard deviation of $\hat{c}$ is getting larger as the given parameter c becomes larger and the given volatility σ does smaller.

Table 2 shows the result on estimation for σ and Figure 4 displays standard deviations of the estimated volatility $\hat{\sigma}$. We may consider that the standard deviation of $\hat{\sigma}$ is getting larger as the given parameter c becomes larger, but hardly influenced by the size of the given volatility σ.

Table 1: Estimation of mean-reverting speed

Mean-reverting Speed		Volatility		
		0.5%	1.0%	5.0%
0.05	Mean	0.0500	0.0500	0.0500
	Sta. Dev	0.0003	0.0001	0.0001
0.20	Mean	0.2000	0.2000	0.2000
	Sta. Dev	0.0007	0.0003	0.0001
0.50	Mean	0.4999	0.5000	0.5000
	Sta. Dev	0.0049	0.0025	0.0005

Table 2: Estimation of volatility

Mean-reverting Speed		Volatility		
		0.5%	1.0%	5.0%
0.05	Mean	0.5000	1.0001	5.0001
	Sta. Dev	0.0008	0.0008	0.0008
0.20	Mean	0.5000	1.0000	5.0000
	Sta. Dev	0.0012	0.0012	0.0012
0.50	Mean	0.4999	1.0000	0.5000
	Sta. Dev	0.0045	0.0045	0.0045

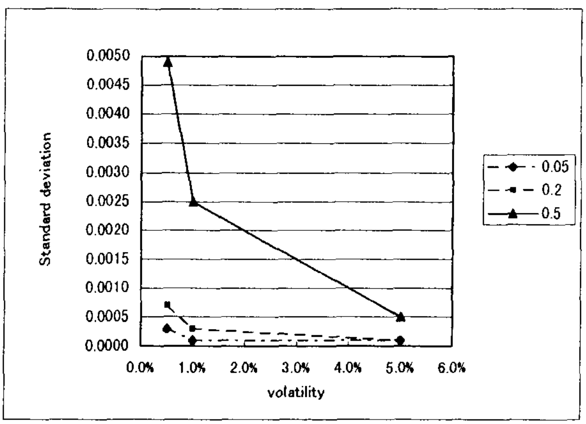

Figure 4: The standard deviation of volatility

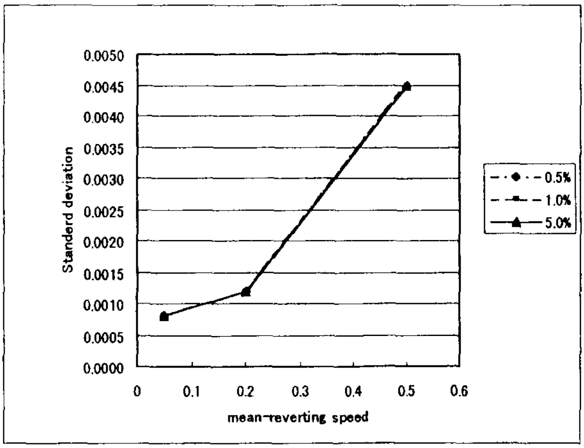

Figure 5: The standard deviation of mean-reverting speed

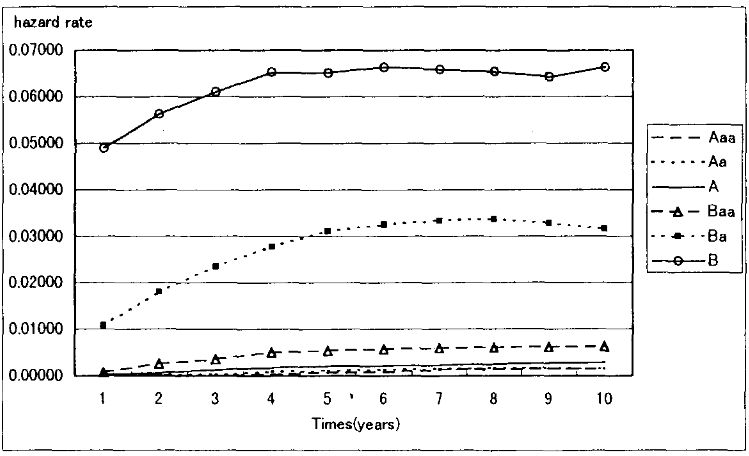

Figure 6: The estimated hazard rate for rach rating
(Average of ten years : 1979-1988(reference years))

5.4 Estimation with modified Vasicek model

The last section illustrates the estimation results for parameters of Vasicek model, using the Moody's data from January 1970 to January 1998. However it is observed that there are some unusual aspects in the data for past about one year. The first is that the ratings of many companies change (almost drops) for the last one year. This fact shakes the premise that the hazard rate has the tendency to revert a constant mean since small rating change for some company generally causes the sharp fluctuation of its hazard rate. The other point is that it is necessary to take notice of Japan premium or the original premium for each firm as well as the rating when analyzing actual Japanese market. In a word, the default probability may be miscalculated if one presumes that the hazard rate is dependent only upon the firm's rating.

Considering these points, we give Vasicek hazard model a modification of making the mean rate time-dependent. (we will call this modified Vasicek.) That is,

$$dh(t) = c\big(m(t) - h(t)\big)\,dt + \sigma\,dW_t \;,\; h(0) = h_0 \tag{8}$$

Suppose that the mean hazard rate consists of the following two factors $m(t) = h^{(k)}(t) + \tilde{h}$, where $h^{(k)}(t)$ is the hazard rate at time t for firms with the rating k and $\tilde{h}$ is the additional premium such as Japan premium or the original one of the company. The way of setting the hazard rate for each rating is described in Appendix.

We demonstrate the parameter estimation on modified Vasicek hazard model by using moment method and solving non-linear programming problem numerically. Here we utilize the Moody's data for 605 kinds of debentures issued by Japanese companies between March 1997 to April 1998. The recovery rate L is given by 0.312, which is the average of recovery rate of defaulted bonds between 1970 and 1998, reported by Moody's. Thorough the relation (5)(replacing m with $m(t)$), we get the hazard rate data from credit spread $y(t,T)$ in the same way as the last section.

Denote by $t_{-n}, t_{-(n-1)}, \cdots, t_{-1}, t_0 = 0$ times at which available data were achieved. (The current time is set 0.) The additional premium is supposed to be calculated from the value of hazard rate at time t_{-n}.

The moment method now leads to

$$\frac{1}{n}\sum_{j=-n}^{0} h(t_j) = \frac{1}{n}\sum_{j=-n}^{0} m(t_j), \tag{9}$$

$$\frac{1}{n}\sum_{j=-n}^{0} h(t_j)^2 = \left(\frac{1}{n}\sum_{j=-n}^{0} m(t_j)\right)^2 + \frac{\sigma^2}{2c}. \tag{10}$$

Next the non-linear programming problem to solve is given below. Let

$$z_1 = \frac{1}{n}\sum_{j=-n}^{0} h(t_j) - \frac{1}{n}\sum_{j=-n}^{0} m(t_j),$$

$$z_2 = \frac{1}{n}\sum_{j=-n}^{0} h(t_j)^2 - \left(\frac{1}{n}\sum_{j=-n}^{0} m(t_j)\right)^2 - \frac{\sigma^2}{2c},$$

$$z_3 = -\frac{1}{c}(e^{ct_{-n}} - 1)(1-L)\big(h(t_{-n}) - m(t_{-n})\big).$$

The aim is to minimize z_1^2, z_2^2 and z_3^2 simultaneously subject to the constrains $0 < c < 1,\; 0 \le \sigma < 50,\; \tilde{h} < 50$.

Before giving the simulation results, we remark that for fifteen bonds among 605, this method tried but failed to be used for convergence calculation.

6 Conclusion

The valuation formula for each side of credit default swap is given in Section 3 in a quite general form, that is, it contains basket type or the type with counter party risk. It implies that if the laws of all the hazard rates upon the underlying firms are known, we can achieve the explicit value.

We particularly utilized Vasicek type term structure model and the modified version for modeling hazard rate since it has some desirable properties --- mean-reverting and Gaussian ---. In the second half of this article, we proposed one estimation procedure when the available data are not so rich and gave some simulation results. The relation among survival probability and parameters such as mean-reverting speed and volatility became apparent to some extent.

On the other hand, many problems remains unsolved for modeling the credit risk and so on. One of them is under which measure the calculation is executed. We assume that the original measure is risk-neutral and all the computation should be done under the measure, but the validity of the assumption is another problem. Besides, as a practical matter, the modeled market is often incomplete, that is, the risk-neutral probability is not unique. As another subject, it is not well-known how the chain default should be considered. These are our present and future tasks.

A Setting Hazard Rate for Each Rating

In this section, we discuss how to analyze the hazard rate for each rating made by Moody's that is used for parameter estimation on modified Vasicek hazard rate in Subsection 5.4. For the purpose, we refer to *Moody's Investors Service, Global Credit Research, "Historical Default Rates of Corporate Bond Issuers, 1920-1997"*. This report divides companies into 6 grades and presents the transitions of cumulative default rate for each rating group ranked at a reference year (see Table 3 for the data of 1988). We have to note that in fact each firm's rating may change as time passes. For example, some firms of the Aaa class at a reference point may fall into the lower grade class, so the average rating of this class seems to get decreasing as time goes by. On the contrary, the average rating of the low grade class at first may be increasing as time goes by since the class is divided into the group which falls into default early and one that gets a upper grade.

Table 3: Transition of cumulative default rate for each group divided by the rating given in 1988

Rating/year	89	90	91	92	93	94	95	96	97	98
Aaa	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Aa	0.00	0.33	0.67	0.67	0.67	0.67	0.67	1.14	1.14	1.14
A	0.00	0.38	0.98	1.40	1.40	1.40	1.40	1.40	1.40	1.40
Baa	0.00	0.33	1.03	2.49	3.65	4.50	4.50	4.50	4.50	5.05
Ba	1.44	7.14	12.97	20.99	24.01	27.18	28.08	29.07	30.16	31.37
B	6.31	13.34	25.95	37.19	41.36	46.46	47.32	50.36	50.36	51.62

Picking up ten data on cumulative default rates whose reference years correspond to each year from 1979 to 1988, we apply the exponential hazard rate process to them. Denote by $F(n)$ the cumulative default rate after n years.($n = 1, 2, \cdots, 10$) Then the hazard rate for the rating k up to that time, $h_0^{(k)}$, is calculated separately by

$$h_0^{(k)} = -\log \frac{1 - F(n)}{n}.$$

The result implies that the hazard rates obtained above are quite different from year to year. In order to remove the peculiar effect dependent upon each year data, we take their simple mean (see Figure 6) and choose as our basic hazard rate for each rating the maximum of ten average hazard rates corresponding to the number of years after the reference point. The result is shown in the below table.

Table 4: Transition of cumulative default rate for each group divided by the rating given in 1988}

Rating	Aaa	Aa	A	Baa	Ba	B
Hazard rate	0.00146	0.00160	0.00282	0.00626	0.003363	0.06628

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