Stochastic Analysis of Life Insurance Surplus

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Abstract

The behaviour of insurance surplus over time for a portfolio of homogeneous life policies in an environment of stochastic mortality and rates of return is examined. We distinguish between stochastic and accounting surpluses and derive their first two moments. A recursive formula is proposed for calculating the distribution function of the accounting surplus. We then examine the probability that the accounting surplus becomes negative in a given insurance year. Numerical examples illustrate the results for portfolios of temporary and endowment life policies assuming a conditional AR(1) process for the rates of return.

Keywords: insurance surplus, stochastic rates of return, AR(1) process, stochastic mortality, distribution function.

1 Introduction

The surplus is an important indicator of an insurance company’s financial position. In this paper, we present a methodology for studying the insurance surplus for a homogeneous portfolio of life insurance policies in a stochastic mortality and rates of return environment.

The results are presented in a rather simplified framework with the main focus on the cash flows arising from just the benefit and premium payment streams driven by the mortality experience of the portfolio; i.e. expenses and other possible sources of decrements (e.g. lapses) are ignored. A similar approach for the investment component as in Parker (1997) is assumed. That is, a random global rate of return is used to determine values of the cash flows.

Although there are many limitations of our framework to more fully address complexity of the real world, the results and conclusions are believed to be useful in enhancing actuaries’ understanding of the stochastic behaviour underlying life insurance products.

The study of life contingencies in an environment of stochastic mortality and interest rates can be traced back to the 1970’s, and by now there exists a vast actuarial literature on the topic. In particular, some of the papers that consider portfolios of life insurance and life annuity contracts include Frees (1990), Parker (1994a, 1996, 1997) and Coppola et al. (2003). Marceau and Gaillardetz (1999) look at the reserve calculation for general portfolios of life insurance policies and examine the appropriateness of using the limiting portfolio approximation. For an extensive literature review on the subject, the reader is referred to Hoedemakers et al. (2005); the paper proposes an approximation for the distribution of the prospective loss for a homogeneous portfolio of life annuities based on the concept of comonotonicity.

In general, the above-mentioned papers deal with a stochastically discounted value of future contingent cash flows that are viewed and valued at the same point in time. This includes net single premium and reserve calculations.

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The problem we try to address is of a different nature. To illustrate, consider a closed block of life insurance business at its initiation, time 0. At each future valuation date, we are interested in the financial position of this block of business as measured by the amount of the surplus available at that time. Let us fix one of the future valuation dates, say corresponding to time $r$, and consider how the surplus can be described at this valuation date. Prior to time $r$, the insurer collects premiums and pays death benefits according to the terms of the contracts in the portfolio. The accumulated value to time $r$ of these cash flows represents the insurer’s retrospective gain or the assets accumulated from this block of business. After time $r$, the insurer continues to pay benefits as they come due and receive periodic premiums. This stream of payments, discounted to time $r$, constitutes the prospective loss, which represents net future obligations or liabilities of the insurer. This leads to one definition of the surplus as the difference between the retrospective gain and the prospective loss. We also consider an alternative definition of the surplus (referred to as the accounting surplus) with the liabilities given by the actuarial reserve as opposed to the prospective loss itself.

One can draw an analogy between our approach and the dynamic solvency testing (DST), which 'involves projecting a company’s solvency position into the future under varying assumptions' (from 1994 General Insurance Convention, by J. Charles). However, instead of considering several scenarios, we average over all possible scenarios by placing a distribution on them.

The first three sections set up our framework for studying homogeneous portfolios assuming random mortality and rates of return. In Sections 5, 6 and 7 we define the two types of surplus and derive their first two moments.

For insurance regulators it is important that insurance companies maintain an adequate surplus level. To represent actuarial liabilities, the insurers are required to report their actuarial reserves calculated in accordance with the regulations. So, when monitoring insurance companies, the regulators actually look at what we call the accounting surplus. In Section 8, we propose a formula for obtaining the distribution function of the accounting surplus at a given valuation date. One piece of information that is readily available from this distribution function is the probability that the surplus falls below zero. If this probability is too high, say above 5%, then perhaps the insurer should make some adjustments to the terms of the contract such as, for example, increasing the premium rate or raising additional initial surplus. In Section 8, a method for computing the distribution function of the accounting surplus is discussed. Finally, in Section 9, numerical examples for two portfolios of endowment and temporary life insurance policies are used to illustrate main results of this paper.

2 Homogeneous Portfolio

In this paper, we consider a portfolio of identical life policies issued to a group of $m$ policyholders all aged $x$ with the same risk characteristics. It is assumed that the future lifetimes of the policyholders in the portfolio are independent and identically distributed. Each policy pays a death benefit $b$ at the end of the year of death if death occurs within $n$ years since the policy issue date and a pure endowment benefit $c$ if the policyholder survives to the end of year $n$. The annual level premium $\pi$ is payable at the beginning of each year as long as the contract remains in force.
3 Cash Flows

To study the surplus for a portfolio, one approach would be to model the surplus for a single policy and then sum over the individual policies in the portfolio. However, this approach can be computationally time-consuming when dealing with large portfolios. Alternatively, one could model aggregate annual cash flows and sum over policy-years. This is the way we will proceed.

Consider a valuation date corresponding to time $r$. For the purpose of our further discussion, we distinguish between those cash flows that occur prior to time $r$ and those that occur after. In addition, we define the cash flows prior to time $r$ as the net inflows into the company and the cash flows after time $r$ as the net outflows.

More formally, let $RC^r_j$ denote the net cash inflow at time $j$ for $0 \leq j \leq r$ (i.e., it is the retrospective cash inflow for the valuation at time $r$). It is given by

$$RC^r_j = \sum_{i=1}^{m} \left[ \pi \cdot L_{i,j} \cdot 1_{\{j<r\}} - b \cdot D_{i,j} \cdot 1_{\{j>0\}} \right]$$

where

$$L_{i,j} = \begin{cases} 1 & \text{if policyholder } i \text{ is alive at time } j, \\ 0 & \text{otherwise,} \end{cases}$$

$$D_{i,j} = \begin{cases} 1 & \text{if policyholder } i \text{ dies in year } j, \\ 0 & \text{otherwise,} \end{cases}$$

and $1_{\{A\}}$ is an indicator function; it is equal to 1 if condition $A$ is true and 0 otherwise.

Notice that at $j = 0$, $RC^r_0$ is simply the sum of all the premiums collected at the issue date. Now consider what happens at the valuation date, $j = r$. Death benefits are paid at the end of year $r$ to everyone who dies during that year. So, this cash flow becomes a part of the retrospective cash inflow $RC^r_r$. However, premiums are collected at the beginning of the next year ($(r + 1)^{st}$ year) and, therefore, they contribute to the prospective cash outflow (defined below).

$$L_j = \sum_{i=1}^{m} L_{i,j} \text{ is the number of people from the initial group of } m \text{ policyholders who survive to time } j \text{ (i.e. it is the number of in-force policies at time } j) \text{ and } D_j = \sum_{i=1}^{m} D_{i,j} \text{ is the number of deaths in year } j.$$  

Let $PC^r_j$ denote the net cash outflow that occurs $j$ time units after time $r$ for $0 \leq j \leq n-r$ (i.e., it is the prospective cash outflow for the valuation at time $r$). Using notation introduced above, we have

$$PC^r_j = \sum_{i=1}^{m} \left[ b \cdot D_{i,(r+j)} \cdot 1_{\{j>0\}} + c \cdot L_{i,n} \cdot 1_{\{j=n-r\}} - \pi \cdot L_{i,(r+j)} \cdot 1_{\{j<n-r\}} \right]$$

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Calculation of the cash flow moments is straightforward. For example,

$$E[RC^r_j] = \pi \cdot E[L^r_j] \cdot 1_{\{j<r\}} - b \cdot E[D^r_j] \cdot 1_{\{j>0\}};$$
Var[RC\_j] = \pi^2 \cdot \text{Var}(\mathcal{L}_j) \cdot \mathbf{1}_{\{j<r\}} + b^2 \cdot \text{Var}(\mathcal{D}_j) \cdot \mathbf{1}_{\{j>0\}} - 2 \cdot \pi \cdot b \cdot \text{Cov}(\mathcal{L}_j, \mathcal{D}_j) \cdot \mathbf{1}_{\{0<j<r\}}
\text{and, for } i < j,
\begin{align*}
\text{Cov}[RC\_i, RC\_j] &= \pi^2 \cdot \text{Cov}[\mathcal{L}_i, \mathcal{L}_j] \cdot \mathbf{1}_{\{i<j\}} + b^2 \cdot \text{Cov}[\mathcal{D}_i, \mathcal{D}_j] \cdot \mathbf{1}_{\{i>0\}} \\
&\quad - \pi \cdot b \cdot \text{Cov}[\mathcal{L}_i, \mathcal{D}_j] \cdot \mathbf{1}_{\{0<i<j\}} - b \cdot \pi \cdot \text{Cov}[\mathcal{D}_i, \mathcal{L}_j] \cdot \mathbf{1}_{\{0<i<j\}} .
\end{align*}

Using the fact that, under our assumption of the homogeneous portfolio and independent future lifetimes\(^1\), \(\{\mathcal{D}_j\}_{j=1}^\infty \cup \{\mathcal{L}_i\}\) is multinomial \((m; 0, q_x, 1, q_x, \ldots, r-1, q_x, r_p_x)\), the moments of \(\mathcal{D}_j\) and \(\mathcal{L}_j\) can be calculated. For instance, refer to Equations (9) and (10) in Marceau and Gaillardetz (1999) for explicit expressions with \(r_i = 0\), since we study all cash flows as viewed from time 0.

4 Stochastic Rates of Return

For illustrative purposes, we choose to model the forces of interest by a conditional autoregressive process of order one, AR(1), given the current force of interest. This model was used by Bellhouse and Panjer (1981) and Marceau and Gaillardetz (1999) for instance.

Let \(\delta(k)\) be the force of interest in period \((k - 1, k]\), \(k = 1, 2, \ldots, n\), with a possible realization denoted by \(\delta_k\). Then, the forces of interest satisfy the following autoregressive model:
\begin{equation}
\delta(k) - \delta = \phi [\delta(k - 1) - \delta] + \sigma \varepsilon(k),
\end{equation}
where \(\{\varepsilon(k)\}\) is a sequence of independent and identically distributed standard normal random variables and \(\delta\) is the long-term mean of the process. We assume that \(|\phi| < 1\) to ensure covariance stationarity of the process.

For our further discussion, it is convenient to introduce a notation for the force of interest accumulation function, which is then used to study both discounting and accumulation processes.

Let \(I(s, r)\) denote the force of interest accumulation function between times \(s\) and \(r\), \((0 \leq s \leq r)\)\(^2\). It is given by
\begin{equation}
I(s, r) = \begin{cases} 
\sum_{j=s+1}^r \delta(j), & s < r, \\
0, & s = r.
\end{cases}
\end{equation}

The AR(1) model has already been extensively studied and many results about it are readily available in the literature. Following similar derivations as given, for example, in Cairns and Parker (1997), we get
\begin{equation}
\text{E}[I(s, r) | \delta(0) = \delta_0] = (r - s)\delta + \frac{\phi}{1 - \phi} (\phi^s - \phi^r)(\delta_0 - \delta)
\end{equation}
and
\begin{equation}
\text{Var}[I(s, r) | \delta(0)] = \\
= \frac{\sigma^2}{1 - \phi^2} \left[ r - s + \frac{2\phi}{1 - \phi} \left( r - s - 1 - \phi \right) \left( 1 - \phi^{r-s-1} \right) - \left( \frac{\phi}{1 - \phi} \right)^2 (\phi^s - \phi^r)^2 \right].
\end{equation}

\(^1\)If \(K_x\) denotes the cutate-future lifetime of a person aged \(x\), then in the standard actuarial notation 
\(P(K_x = k) = k_{x|q_x}\) and \(P(K_x > k) = k_{x-p_x}\).

\(^2\)This notation is an extension of the notation introduced in Marceau and Gaillardetz (1999).
When two force of interest accumulation functions are involved, it will be necessary to distinguish three cases for the times between which the accumulation occurs. Suppose we are interested in obtaining the value at time $r$ of two cash flows occurring at times $s$ and $t$.

1. If $s < t < r$ (i.e., both cash flows occur prior to time $r$), the values at time $r$ of these cash flows need to be accumulated using $I(s, r)$ and $I(t, r)$;

2. If $r < s < t$ (i.e., both cash flows occur after time $r$), the values at time $r$ of these cash flows need to be discounted using $I(r, s)$ and $I(r, t)$;

3. If $s < r < t$ (i.e., one cash flow occurs before time $r$ and the other one occurs after time $r$), the values at time $r$ of these cash flows need to be accumulated and discounted using $I(s, r)$ and $I(r, t)$ respectively.

Case 1 is considered in Cairns and Parker (1997), whereas Case 2 is a slight generalization of results presented in Marceau and Gaillardetz (1999). To summarize, the unconditional covariance terms, corresponding to the three cases for the force of interest accumulation functions mentioned above, are given by the following formulas:

**Case 1: $s < t < r$**

$$
\text{Cov}[I(s, r), I(t, r)] = \sum_{i=s+1}^{r} \sum_{j=t+1}^{r} \text{Cov}[\delta(i), \delta(j)] = \text{Var}[I(t, r)] + \frac{\sigma^2}{1 - \phi^2} \frac{\phi}{(1 - \phi)^2} (\phi^t - \phi^s)(\phi^{-t} - \phi^{-s});
$$

**Case 2: $r < s < t$**

$$
\text{Cov}[I(r, s), I(r, t)] = \text{Var}[I(r, s)] + \frac{\sigma^2}{1 - \phi^2} \frac{\phi}{(1 - \phi)^2} (\phi^s - \phi^t)(\phi^{-s} - \phi^{-r});
$$

**Case 3: $s < r < t$**

$$
\text{Cov}[I(s, r), I(r, t)] = \frac{\sigma^2}{1 - \phi^2} \frac{\phi}{(1 - \phi)^2} (\phi^r - \phi^t)(\phi^{-r} - \phi^{-s}).
$$

The corresponding conditional covariance terms can then be obtained using known results from the multivariate normal theory (e.g., see Johnson and Wichern (2002)). For instance, for $s < r < t$ (Case 3),

$$
\text{Cov}[I(s, r), I(r, t) | \delta(0)] = \text{Cov}[I(s, r), I(r, t)] - \frac{\text{Cov}[I(s, r), \delta(0)] \cdot \text{Cov}[I(r, t), \delta(0)]}{\text{Var}[\delta(0)]}, \quad (4.5)
$$

where

$$
\text{Cov}[I(s, r), \delta(0)] = \frac{\sigma^2}{1 - \phi^2} \frac{\phi}{1 - \phi} (\phi^s - \phi^r). \quad (4.6)
$$

Formulas for the other two cases are analogous.

We will also require the distribution of $I(r, t)$ for $r < t$ conditional on both $\delta(0)$ and $\delta(r)$. In this case, due to the Markovian nature of the rates of return process, we have

$$
E[I(r, t) | \delta(0) = \delta_0, \delta(r) = \delta_r] = E[I(0, t-r) | \delta(0) = \delta_r]
$$
and

\[ \text{Var}[I(r, t) | \delta(0), \delta(r)] = \text{Var}[I(0, t - r) | \delta(0)], \]

and Equations (4.3) and (4.4) can be applied.

To simplify notation, we will omit conditioning on \( \delta(0) \) when referring to moments of a function of \( I(s, r) \) (e.g., \( E[e^{I(s,r)} | \delta(0)] \) will be denoted by \( E[e^{I(s,r)}] \)).

In our notation, the accumulation function from time \( s \) to time \( r \) and the discount function from time \( t \) to time \( r \), for \( s < r < t \), are given by \( e^{I(s,r)} \) and \( e^{-I(r,t)} \), respectively. Each of them follows a lognormal distribution.

Using a well-known fact that if \( Y \sim N(\mu, \sigma^2) \), then the \( m^{th} \)-moment of \( e^Y \) is

\[ E[e^{mY}] = e^{E[Y] + \frac{m^2}{2} \text{Var}[Y]}, \]

we can find moments of \( e^{-I(r,t)} \) and \( e^{I(s,r)} \).

Next, we introduce two random variables, the retrospective gain and the prospective loss, which will be used to define and study the surplus.

5 Retrospective Gain

The retrospective gain at time \( r \) is the difference between the accumulated values to time \( r \) of past premiums collected and benefits paid. It can be expressed in terms of \( RC_j^r \) as follows:

\[ RG_r = \sum_{j=0}^{r} RC_j^r \cdot e^{I(j,r)}. \quad (5.1) \]

Then, assuming independence between future lifetimes and rates of return, we obtain

\[ E[RG_r] = \sum_{j=0}^{r} E[RC_j^r] \cdot E[e^{I(j,r)}] \quad (5.2) \]

and

\[ E[(RG_r)^2] = \sum_{i=0}^{r} \sum_{j=0}^{r} E[RC_i^r \cdot RC_j^r] \cdot E[e^{I(i,r)+I(j,r)}]. \quad (5.3) \]

6 Prospective Loss

The prospective loss at time \( r \) is the difference between the discounted values to time \( r \) as viewed from time \( 0 \) of future benefits to be paid and premiums to be collected. Using \( PC_j^r \), it is given by

\[ PL_r = \sum_{j=0}^{n-r} PC_j^r \cdot e^{-I(r,r+j)}. \quad (6.1) \]

The moments of \( PL_r \) can be calculated similarly to the moments of \( RG_r \).
When calculating actuarial reserves, it will be useful to know the moments of the prospective loss conditional on the number of in-force policies and the rate of return prevailing at the valuation date. For example, we will need the conditional expectation of $PL_r$. Due to the independence of mortality and interest, the following holds:

$$E[PL_r | L_r, \delta(r)] = \sum_{j=0}^{n-r} E[PC_j^r | L_r] \cdot E[e^{-I(r,r+j)} | \delta(r)].$$

(6.2)

To calculate the conditional expected value of $PC_j^r$, the following facts can be used:

$$\{L_{r+j} | L_r = m_r\} \sim \text{binomial}(m_r, j p_{x+r})$$

(6.3)

and

$$\{D_{r+j} | L_r = m_r\} \sim \text{binomial}(m_r, j-1 q_{x+r})$$

(6.4)

7 Surplus

In general, we define insurance surplus to be the difference between assets and liabilities at a given valuation date. Recall that the retrospective gain is the accumulated value of past premiums collected net of past benefits paid and, thus, in our context, it can be interpreted as the value of assets. In turn, the liabilities associated with a portfolio of life policies are based on the prospective loss, which is the discounted value of future obligations net of future premiums. So, the liabilities can simply be represented by the prospective loss random variable. In this case the surplus is referred to as the net stochastic surplus or just stochastic surplus and is denoted $S_{r}^{\text{stoch}}$.

$$S_{r}^{\text{stoch}} = RG_r - PL_r.$$  

(7.1)

In practice, at each valuation date, an insurer is required to set aside an actuarial reserve based on the current situation including the number of in-force policies as well as the prevailing rate of return. This reserve is a liability item on the balance sheet of the insurance company. So, an alternative definition of the surplus is the difference between the value of assets and the actuarial reserve, in which case we call it the accounting surplus, denoted $S_{r}^{\text{acct}}$.

$$S_{r}^{\text{acct}} = RG_r - r V(L_r, \delta(r)),$$

(7.2)

where $r V(L_r, \delta(r))$ is the actuarial reserve for the valuation at time $r$.

Although the name may suggest a deterministic nature of the accounting surplus, in fact, it is a stochastic quantity for $r > 0$, since, when viewed from time 0, there is an uncertainty about both the number of in-force policies and the force of interest at time $r$.

The reserve is intended to cover the net future liabilities of the insurer. Therefore, the amount needed to be set as a reserve at time $r$ should be at least the expected value of $PL_r$ conditional on the number of in-force policies in the portfolio $L_r$ and the force of interest $\delta(r)$. If, instead, it is required to have a conservative reserve that will cover future obligations with a high probability, one can use a $p$th percentile of the prospective loss random variable with $p$ between 70% and 95%, for example. However, this reserve calculation can be fairly difficult to incorporate into the model, since we need to know the distribution function of $PL_r$, which is not
easy to obtain. Alternatively, a reserve could be set equal to the expected value plus a multiple of the standard deviation of $PL_r$, as suggested in Norberg (1993).

In the rest of this section, assume that

$$rV(L_r, \delta(r)) = E[PL_r | L_r, \delta(r)]. \quad (7.3)$$

It is easy to show that under our particular choice of the reserve

$$E[S^\text{acct}_r] = E[S^\text{stoch}_r] = E[RG_r] - E[PL_r]. \quad (7.4)$$

The variance of the accounting surplus can be calculated using the following result.

**Result 7.1.**

$$\text{Var}[S^\text{acct}_r] = \left( \text{Var}_{\delta(r)} E[PL_r | \delta(r)] + E_{\delta(r)} \text{Var}_{L_r} (E[PL_r | L_r, \delta(r)]) \right)$$

$$+ \text{Var}[RG_r] - 2 \text{Cov}(RG_r, PL_r),$$

where

$$\text{Var}_{\delta(r)} E[PL_r | \delta(r)] + E_{\delta(r)} \text{Var}_{L_r} (E[PL_r | L_r, \delta(r)]) =$$

$$= \sum_{i=0}^{n-r} \sum_{j=0}^{n-r} \left\{ E_{\delta(r)} \left( E[PC^r_i | L_r] \cdot E[PC^r_j | L_r] \right) \right\} \cdot E_{\delta(r)} \left( E[e^{-I(r,r+i)} | \delta(r)] \cdot E[e^{-I(r,r+j)} | \delta(r)] \right) - \left( E[PL_r] \right)^2$$

and

$$\text{Cov}[RG_r, PL_r] = E[RG_r \cdot PL_r] - E[RG_r] \cdot E[PL_r]$$

$$= \sum_{j=0}^{n-r} \sum_{i=0}^{n-r} E[RC^r_j \cdot PC^r_i] \cdot E[e^{I(j,r) - I(r,r+i)}] - E[RG_r] \cdot E[PL_r].$$

A proof of Result 7.1 is given in Appendix A.

The variance calculation for the stochastic surplus is straightforward as

$$\text{Var}[S^\text{stoch}_r] = \text{Var}[RG_r - PL_r]$$

$$= \text{Var}[RG_r] + \text{Var}[PL_r] - 2 \cdot \text{Cov}[RG_r, PL_r].$$

We also consider a portfolio with the number of policies approaching infinity, which we refer to as the limiting portfolio. Although the limiting portfolio is an abstract concept and not achievable in practice, its characteristics such as variability can serve as benchmarks for portfolios of finite sizes and can provide some useful information for insurance risk managers.

If the variance of the surplus per policy for a given portfolio is much larger than the corresponding variance for the limiting portfolio, then it can be concluded that a large portion of the total risk is due to the insurance risk. In other words, there is a great uncertainty about future cash flows. One implication of this is that, if the insurer decides to hedge the financial risk, for instance, by purchasing bonds whose cash flows will match those of the portfolio’s liabilities, this strategy will not be very efficient and the cost incurred to implement it might not
be justified. In this case, selling more policies, sharing the mortality risk or buying reinsurance are better strategies to mitigate the risk.

For a limiting portfolio, the calculation of the moments is done similarly to the case when the size of the portfolio is finite, except that the random cash flows per policy, $RC_j^r/m$ and $PC_j^r/m$, are replaced by their expected values.

8 Distribution of Accounting Surplus

In this section, we describe a method for calculating the distribution function of the accounting surplus.

8.1 Portfolio of finite size

Recall that the accounting surplus at time $r$ is defined as

$$S^{\text{acct}}_r = RG_r - rV(L_r, \delta(r)),$$

where $rV(L_r, \delta(r))$ is the reserve at time $r$.

Notice that, given the values of $L_r$ and $\delta(r)$, $rV(L_r, \delta(r))$ is constant. Therefore, we can obtain the distribution function (df) of $\{S^{\text{acct}}_r \mid L_r = m_r, \delta(r) = \delta_r\}$ from the df of $\{RG_r \mid L_r = m_r, \delta(r) = \delta_r\}$ using

$$P[S^{\text{acct}}_r \leq \xi \mid L_r = m_r, \delta(r) = \delta_r] = P[RG_r \leq \xi + rV(m_r, \delta_r) \mid L_r = m_r, \delta(r) = \delta_r].$$

Since it is not straightforward to get the distribution function of $\{RG_r \mid L_r, \delta(r)\}$ directly, we propose a recursive approach. This approach is based on the ideas presented in Parker(1998).

For the valuation at time $r$, let $G_t = \sum_{j=0}^t RC_j^r \cdot e^{I(j,t)}$ denote the accumulated value to time $t$ of the retrospective cash inflows that occurred up to and including time $t$, $0 \leq t \leq r$. Observe that $G_r$ is equal to $RG_r$.

We can relate $G_t$ and $G_{t-1}$ as follows:

$$\begin{align*}
G_t &= \sum_{j=0}^t RC_j^r \cdot e^{I(j,t)} \\
    &= \sum_{j=0}^{t-1} RC_j^r \cdot e^{I(j,t-1)+I(t-1,t)} + RC_t^r \cdot e^{I(t,t)} = \left(\sum_{j=0}^{t-1} RC_j^r \cdot e^{I(j,t-1)}\right) \cdot e^{\delta(t)} + RC_t^r \\
    &= G_{t-1} \cdot e^{\delta(t)} + RC_t^r.
\end{align*}$$

Equation (8.1) can be used to build up the df of $G_t$ from the df of $G_{t-1}$ and thus the df of $RG_r$ recursively from $G_t$ for $t = 0, 1, \ldots, r - 1$.

Note that $f(\cdot)$ denotes the probability density function (pdf). Under our assumption for the rates of return, $f_{\delta(t)}(\cdot)$ is the pdf of a normal random variable with mean $E[\delta(t) \mid \delta(0) = \delta_0] = \delta_0$ and variance $\text{Var}[\delta(t) \mid \delta(0) = \delta_0]$, and $f_{\delta(t)}(\cdot \mid \mathcal{A})$, where $\mathcal{A} \equiv \{\delta(t-1) = \delta_{t-1}\}$, is the pdf of a normal random variable with mean $E[\delta(t) \mid \delta(0) = \delta_0, \mathcal{A}] = \delta_0$ and variance $\text{Var}[\delta(t) \mid \delta(0) = \delta_0, \mathcal{A}]$. 

3
Using Equation (8.1), which implies that

\[ \mathbb{P}[G_t \leq \lambda | \mathcal{L}_t = m_t, \delta(t) = \delta_t] = \frac{\mathbb{P}[\mathcal{L}_t = m_t, \delta(t) = \delta_t | G_t \leq \lambda] \cdot \mathbb{P}[G_t \leq \lambda]}{\mathbb{P}[\mathcal{L}_t = m_t, \delta(t) = \delta_t]} \]

and motivated by Equation (8.2).

\[ \mathbb{P}[\mathcal{L}_t = m_t | G_t \leq \lambda] \cdot f_{\delta(t)}(\delta_t | G_t \leq \lambda) \cdot \mathbb{P}[G_t \leq \lambda] \]

where the last line follows from the independence of \( \mathcal{L}_t \) and \( \delta(t) \).

Next we consider a function \( g_t(\lambda, m_t, \delta_t) \) given by

\[ g_t(\lambda, m_t, \delta_t) = \mathbb{P}[G_t \leq \lambda | \mathcal{L}_t = m_t, \delta(t) = \delta_t] \cdot \mathbb{P}[\mathcal{L}_t = m_t] \cdot f_{\delta(t)}(\delta_t) \tag{8.3} \]

and motivated by Equation (8.2).

The following result gives a way for calculating \( g_t \) from \( g_{t-1} \), \( 1 < t \leq r \leq n \).

**Result 8.1.1.**

\[
g_t(\lambda, m_t, \delta_t) = \sum_{m_{t-1}=m_t}^m \mathbb{P}[\mathcal{L}_t = m_t | \mathcal{L}_{t-1} = m_{t-1}] \times \\
\times \int_{-\infty}^{\infty} g_{t-1} \left( \frac{\lambda - \eta_t}{e^\delta t}, m_{t-1}, \delta_{t-1} \right) \cdot f_{\delta(t)}(\delta_t | \delta(t-1) = \delta_{t-1}) \, d\delta_{t-1},
\]

where \( \eta_t \) is the realization of \( RC_t^\pi \) for given values of \( m_{t-1} \) and \( m_t \),

\[
\eta_t = \begin{cases} 
\pi \cdot m_t - b \cdot (m_{t-1} - m_t), & 1 \leq t \leq r - 1, \\
-b \cdot (m_{t-1} - m_t), & t = r,
\end{cases}
\]

with the starting value for \( g_t \)

\[
g_1(\lambda, m_1, \delta_1) = \begin{cases} 
\mathbb{P}[\mathcal{L}_1 = m_1] \cdot f_{\delta(1)}(\delta_1) & \text{if } G_1 \leq \lambda, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof:**

From Equation (8.2) we have

\[
g_t(\lambda, m_t, \delta_t) = \mathbb{P}[\mathcal{L}_t = m_t, \delta(t) = \delta_t | G_t \leq \lambda] \cdot \mathbb{P}[G_t \leq \lambda]
\]

\[
= \sum_{m_{t-1}=m_t}^m \mathbb{P}[\mathcal{L}_{t-1} = m_{t-1}, \mathcal{L}_t = m_t, \delta(t) = \delta_t | G_t \leq \lambda] \cdot \mathbb{P}[G_t \leq \lambda]
\]

\[
= \sum_{m_{t-1}=m_t}^m \mathbb{P}[G_t \leq \lambda | \mathcal{L}_{t-1} = m_{t-1}, \mathcal{L}_t = m_t, \delta(t) = \delta_t] \times
\]

\[
\times \mathbb{P}[\mathcal{L}_{t-1} = m_{t-1}, \mathcal{L}_t = m_t, \delta(t) = \delta_t] \quad \text{(by the Bayes' rule)}.
\]

Using Equation (8.1), which implies that \( \{G_t \leq \lambda\} \equiv \left\{G_{t-1} \leq \lambda \frac{RC_t}{e^{\delta(t)}}\right\} \), and the assumption.
of independence of $\mathcal{L}_{t-1}$ and $\mathcal{L}_t$ from $\delta(t)$, we get

$$g_t(\lambda, m_t, \delta_t) = \sum_{m_{t-1}=m_t}^{m} P[G_{t-1} \leq \frac{\lambda - \eta t}{e^{\delta t}} \mid \mathcal{L}_{t-1} = m_{t-1}, \mathcal{L}_t = m_t, \delta(t) = \delta_t] \times$$

$$\times P[\mathcal{L}_{t-1} = m_{t-1}, \mathcal{L}_t = m_t] \cdot f_{\delta(t)}(\delta_t)$$

$$= \sum_{m_{t-1}=m_t}^{m} P[\mathcal{L}_{t-1} = m_{t-1}, \mathcal{L}_t = m_t \mid G_{t-1} \leq \frac{\lambda - \eta t}{e^{\delta t}}] \cdot f_{\delta(t)}(\delta_t) \cdot P[G_{t-1} \leq \frac{\lambda - \eta t}{e^{\delta t}}] \times$$

$$\times P[G_{t-1} \leq \frac{\lambda - \eta t}{e^{\delta t}}] \times$$

$$\times \int_{-\infty}^{\infty} f_{\delta(t)}(\delta_t) \delta(t-1) = \delta_{t-1}, G_{t-1} \leq \frac{\lambda - \eta t}{e^{\delta t}} \mid P[\mathcal{L}_{t-1} = m_{t-1}, \mathcal{L}_t = m_t, \delta(t-1) = \delta_{t-1}] \mid \delta(t-1) = \delta_{t-1}] \mid d\delta_{t-1}.$$

By the Markovian property of $\mathcal{L}_t$ and $\delta(t)$ and the definition of $g_{t-1}(\frac{\lambda - m}{e^{\delta t}}, m_{t-1}, \delta_{t-1})$, $g_t(\lambda, m_t, \delta_t)$ becomes

$$g_t(\lambda, m_t, \delta_t) = \sum_{m_{t-1}=m_t}^{m} P[\mathcal{L}_t = m_t \mid \mathcal{L}_{t-1} = m_{t-1}] \times$$

$$\times \int_{-\infty}^{\infty} g_{t-1}(\frac{\lambda - m}{e^{\delta t}}, m_{t-1}, \delta_{t-1}) \cdot f_{\delta(t)}(\delta_t \mid \delta(t-1) = \delta_{t-1}) \mid d\delta_{t-1}. \quad \Box$$

Once $g_r(\lambda, m_r, \delta_r)$ is obtained using Result 8.1.1, the cumulative distribution function of $S^{\text{act}}_r$ can be calculated as follows:

$$P[S^{\text{act}}_r \leq \xi] = \int_{-\infty}^{\infty} \sum_{m_r=0}^{m} P[S^{\text{act}}_r \leq \xi \mid \mathcal{L}_r = m_r, \delta(r) = \delta_r] \cdot P[\mathcal{L}_r = m_r] \cdot f_{\delta(r)}(\delta_r) \mid d\delta_r.$$

Since $S^{\text{act}}_r = R \mathcal{L}_r - r V(\mathcal{L}_r, \delta(r))$ and $G_r = RG_r$, we get

$$P[S^{\text{act}}_r \leq \xi] =$$

$$= \int_{-\infty}^{\infty} \sum_{m_r=0}^{m} P[G_r \leq \xi + r V(m_r, \delta_r) \mid \mathcal{L}_r = m_r, \delta(r) = \delta_r] \cdot P[\mathcal{L}_r = m_r] \cdot f_{\delta(r)}(\delta_r) \mid d\delta_r$$

$$= \int_{-\infty}^{\infty} \sum_{m_r=0}^{m} g_r(\xi + r V(m_r, \delta_r), m_r, \delta_r) \mid d\delta_r. \quad (8.4)$$

### 8.2 Limiting Portfolio

For a very large insurance portfolio, the actual mortality experience follows very closely the life table. In this case, we can approximate the true distribution of the surplus per policy by the distribution of the surplus for the limiting portfolio, which takes into account the investment risk but treats cash flows as given and equal to their expected values.
This distribution for the limiting portfolio can be derived similarly to the case of random cash flows.

Define $G_t = \sum_{j=0}^{t} E[RC^r_j/m] \cdot e^{\delta t}$. It can easily be shown that $G_t = G_{t-1} \cdot e^{\delta t} + E[RC^r_t/m]$ (cf. Equation (8.1)).

Now, let $h_t(\lambda, \delta_t) = P[G_t \leq \lambda | \delta(t) = \delta_t] \cdot f_{\delta(t)}(\delta_t)$. This function can be used to calculate the df of $G_t$ recursively similar to the way $g_t(\lambda, m_t, \delta_t)$ was used for obtaining the df of $G_t$. A recursive relation for $h_t(\lambda, \delta_t)$ is given by the following result.

**Result 8.2.1.**

$$h_t(\lambda, \delta_t) = \int_{-\infty}^{\infty} f_{\delta(t)}(\delta_t | \delta(t-1) = \delta(t-1)) \cdot h_{t-1}\left(\frac{\lambda - E[RC^r_t/m]}{e^{\delta t}}, \delta(t-1)\right) d\delta_{t-1}$$

with the starting value for $h_t$

$$h_1(\lambda, \delta_1) = \begin{cases} f_{\delta(1)}(\delta_1) & \text{if } G_1 \leq \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\lim_{m \to \infty} P[S_{r\text{act}}/m \leq \xi | \delta(r) = \delta_r] = P[G_r \leq \xi + r\mathcal{V}(\delta(r)) | \delta(r) = \delta_r],$

$$\lim_{m \to \infty} P[S_{r\text{act}}/m \leq \xi] = \int_{-\infty}^{\infty} h_r(\xi + r\mathcal{V}(\delta(r)), \delta_r) d\delta_r, \quad (8.5)$$

where $r\mathcal{V}(\delta(r))$ denotes the benefit reserve at time $r$ per policy for the limiting portfolio.

**9 Numerical Illustrations**

We use the following arbitrarily chosen values of the parameters for the conditional AR(1) process: $\phi = 0.90; \sigma = 0.01; \delta = 0.06$ and $\delta_0 = 0.08$.

A nonparametric life table is used to determine the distribution of $K_x$. In our examples we use the Canada 1991, age nearest birthday (ANB), male, aggregate, population mortality table; see Appendix for mortality rates.

We also assume that the actuarial reserves are set equal to the conditional expected value of the prospective loss. That is,

$r\mathcal{V}(\mathcal{L}_r, \delta(r)) = E[PL_r | \mathcal{L}_r, \delta(r)],$ and $r\mathcal{V}(\delta(r)) = E[PL_r/m | \delta(r)].$

In our examples, we compare results for different premium rates. A $\theta$% premium loading is applied to the benefit premium determined under the equivalence principle; i.e. the premium such that $E[PL_0]=0$ is satisfied.

Here, a company with a negative surplus in any year is considered insolvent.
9.1 Example 1: Portfolio of endowment policies

Consider a portfolio of 100 10-year endowment life insurance policies with $1000 death and endowment benefits issued to a group of people aged 30 with the same risk characteristics.

Tables 1 and 2 provide the first three moments of the accounting surplus per policy represented by the expected value, standard deviation and coefficient of skewness for the portfolio of 100 endowment contracts and the corresponding limiting portfolio, respectively. The estimates of the probability of insolvency based on the accounting surplus are given in Table 3.

First of all, one can see from both the moments and the probability estimates that the difference between the two portfolios is not very large. In this case, approximation of the distribution of the surplus per policy for portfolios of moderate size by the distribution of the surplus per policy for the corresponding limiting portfolio may be justified. An insignificant loss in the accuracy of the estimates is well compensated by a gain in computing time.

When charging the benefit premium \(\theta = 0\%\), probabilities for all values of \(r\) are slightly less than 50%. This can be expected since no profit or contingency margin is built into the premium when pricing is done under the equivalence principle. The fact that these probabilities are not exactly 50% is due to the asymmetry of the discount function when a Gaussian model for the rates of return is assumed. With a 10% loading factor, the probability of insolvency sharply decreases compared to the case of \(\theta = 0\%\) in the first few years but this reduction is not as large in the later years of the contract. The probability that the accounting surplus falls below zero increases from 0.23% at \(r = 1\) to 14.57% at \(r = 10\) (for \(m = 100\)). A 20% loading factor appears to be sufficient to ensure that the probability of insolvency in any given year is less than 5% for both portfolios.

Table 1: The first three moments of the accounting surplus per policy for a portfolio of 100 10-year endowment policies.

<table>
<thead>
<tr>
<th>(r)</th>
<th>(E\left[\frac{S_r}{m}\right])</th>
<th>(sd\left[\frac{S_r}{m}\right])</th>
<th>(sk\left[\frac{S_r}{m}\right])</th>
<th>(E\left[\frac{S_r}{m}\right])</th>
<th>(sd\left[\frac{S_r}{m}\right])</th>
<th>(sk\left[\frac{S_r}{m}\right])</th>
<th>(E\left[\frac{S_r}{m}\right])</th>
<th>(sd\left[\frac{S_r}{m}\right])</th>
<th>(sk\left[\frac{S_r}{m}\right])</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.181</td>
<td>18.445</td>
<td>-0.197</td>
<td>53.758</td>
<td>17.384</td>
<td>-0.205</td>
<td>107.336</td>
<td>16.332</td>
<td>-0.216</td>
</tr>
<tr>
<td>2</td>
<td>0.580</td>
<td>28.985</td>
<td>-0.211</td>
<td>58.379</td>
<td>27.960</td>
<td>-0.220</td>
<td>116.179</td>
<td>26.941</td>
<td>-0.218</td>
</tr>
<tr>
<td>3</td>
<td>1.228</td>
<td>38.938</td>
<td>-0.192</td>
<td>63.485</td>
<td>38.322</td>
<td>-0.188</td>
<td>125.742</td>
<td>37.710</td>
<td>-0.185</td>
</tr>
<tr>
<td>4</td>
<td>2.153</td>
<td>48.726</td>
<td>-0.151</td>
<td>69.123</td>
<td>48.817</td>
<td>-0.145</td>
<td>136.093</td>
<td>48.914</td>
<td>-0.138</td>
</tr>
<tr>
<td>5</td>
<td>3.375</td>
<td>58.366</td>
<td>-0.095</td>
<td>75.336</td>
<td>59.443</td>
<td>-0.086</td>
<td>147.296</td>
<td>60.529</td>
<td>-0.077</td>
</tr>
<tr>
<td>6</td>
<td>4.904</td>
<td>67.757</td>
<td>-0.025</td>
<td>82.157</td>
<td>70.095</td>
<td>-0.015</td>
<td>159.410</td>
<td>72.444</td>
<td>-0.005</td>
</tr>
<tr>
<td>7</td>
<td>6.740</td>
<td>76.759</td>
<td>0.056</td>
<td>89.612</td>
<td>80.637</td>
<td>0.068</td>
<td>172.483</td>
<td>84.527</td>
<td>0.077</td>
</tr>
<tr>
<td>8</td>
<td>8.867</td>
<td>85.244</td>
<td>0.147</td>
<td>97.708</td>
<td>90.946</td>
<td>0.157</td>
<td>186.549</td>
<td>96.660</td>
<td>0.164</td>
</tr>
<tr>
<td>9</td>
<td>11.250</td>
<td>93.165</td>
<td>0.246</td>
<td>106.438</td>
<td>100.974</td>
<td>0.252</td>
<td>201.627</td>
<td>108.790</td>
<td>0.256</td>
</tr>
<tr>
<td>10</td>
<td>13.830</td>
<td>102.477</td>
<td>0.346</td>
<td>115.772</td>
<td>112.530</td>
<td>0.349</td>
<td>217.714</td>
<td>122.605</td>
<td>0.349</td>
</tr>
</tbody>
</table>
Table 2: The first three moments of the accounting surplus per policy for the limiting portfolio of 10-year endowment policies.

<table>
<thead>
<tr>
<th>r</th>
<th>$E\left[\frac{S_m}{m}\right]$</th>
<th>sd $\left[\frac{S_m}{m}\right]$</th>
<th>sk $\left[\frac{S_m}{m}\right]$</th>
<th>$E\left[\frac{S_m}{m}\right]$</th>
<th>sd $\left[\frac{S_m}{m}\right]$</th>
<th>sk $\left[\frac{S_m}{m}\right]$</th>
<th>$E\left[\frac{S_m}{m}\right]$</th>
<th>sd $\left[\frac{S_m}{m}\right]$</th>
<th>sk $\left[\frac{S_m}{m}\right]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.181</td>
<td>18.139</td>
<td>-0.194</td>
<td>53.758</td>
<td>17.026</td>
<td>-0.199</td>
<td>107.336</td>
<td>15.912</td>
<td>-0.198</td>
</tr>
<tr>
<td>2</td>
<td>0.580</td>
<td>28.592</td>
<td>-0.216</td>
<td>58.379</td>
<td>27.511</td>
<td>-0.217</td>
<td>116.179</td>
<td>26.430</td>
<td>-0.218</td>
</tr>
<tr>
<td>3</td>
<td>1.228</td>
<td>38.495</td>
<td>-0.199</td>
<td>63.485</td>
<td>37.825</td>
<td>-0.196</td>
<td>125.742</td>
<td>37.156</td>
<td>-0.193</td>
</tr>
<tr>
<td>4</td>
<td>2.153</td>
<td>48.247</td>
<td>-0.160</td>
<td>69.123</td>
<td>48.290</td>
<td>-0.153</td>
<td>136.093</td>
<td>48.337</td>
<td>-0.147</td>
</tr>
<tr>
<td>5</td>
<td>3.375</td>
<td>57.859</td>
<td>-0.103</td>
<td>75.336</td>
<td>58.895</td>
<td>-0.094</td>
<td>147.296</td>
<td>59.937</td>
<td>-0.086</td>
</tr>
<tr>
<td>6</td>
<td>4.904</td>
<td>67.224</td>
<td>-0.033</td>
<td>82.157</td>
<td>69.528</td>
<td>-0.023</td>
<td>159.410</td>
<td>71.842</td>
<td>-0.013</td>
</tr>
<tr>
<td>7</td>
<td>6.740</td>
<td>76.199</td>
<td>0.048</td>
<td>89.612</td>
<td>80.050</td>
<td>0.059</td>
<td>172.483</td>
<td>83.914</td>
<td>0.069</td>
</tr>
<tr>
<td>8</td>
<td>8.867</td>
<td>84.653</td>
<td>0.140</td>
<td>97.708</td>
<td>90.337</td>
<td>0.149</td>
<td>186.549</td>
<td>96.032</td>
<td>0.157</td>
</tr>
<tr>
<td>9</td>
<td>11.250</td>
<td>92.537</td>
<td>0.239</td>
<td>106.438</td>
<td>100.337</td>
<td>0.244</td>
<td>201.627</td>
<td>108.142</td>
<td>0.249</td>
</tr>
<tr>
<td>10</td>
<td>13.830</td>
<td>99.998</td>
<td>0.342</td>
<td>115.772</td>
<td>110.167</td>
<td>0.342</td>
<td>217.714</td>
<td>120.336</td>
<td>0.342</td>
</tr>
</tbody>
</table>

Table 3: Estimates of the probability that accounting surplus per policy becomes negative for two portfolios of 10-year endowment policies.

<table>
<thead>
<tr>
<th>r</th>
<th>m = 100</th>
<th>m = ∞</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\theta = 0%$</td>
<td>$\pi = 67.90$</td>
</tr>
<tr>
<td>2</td>
<td>$\theta = 0%$</td>
<td>$\pi = 67.90$</td>
</tr>
<tr>
<td>3</td>
<td>$\theta = 0%$</td>
<td>$\pi = 67.90$</td>
</tr>
<tr>
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<td>$\theta = 0%$</td>
<td>$\pi = 67.90$</td>
</tr>
<tr>
<td>5</td>
<td>$\theta = 0%$</td>
<td>$\pi = 67.90$</td>
</tr>
<tr>
<td>6</td>
<td>$\theta = 0%$</td>
<td>$\pi = 67.90$</td>
</tr>
<tr>
<td>7</td>
<td>$\theta = 0%$</td>
<td>$\pi = 67.90$</td>
</tr>
<tr>
<td>8</td>
<td>$\theta = 0%$</td>
<td>$\pi = 67.90$</td>
</tr>
<tr>
<td>9</td>
<td>$\theta = 0%$</td>
<td>$\pi = 67.90$</td>
</tr>
<tr>
<td>10</td>
<td>$\theta = 0%$</td>
<td>$\pi = 67.90$</td>
</tr>
</tbody>
</table>
9.2 Example 2: Portfolio of temporary policies

In our next example, we study a homogeneous portfolio of 1000 5-year temporary insurance policies and the corresponding limiting portfolio with $1000 death benefit issued to people aged 30.

In contrast to the portfolio of endowment policies, the moments of the accounting surplus per policy for the 1000-policy portfolio and the limiting portfolio of temporary contracts (cf. Tables 4 and 5) are largely apart. This is confirmed by their distribution functions as well; see Table 6 for the estimates of the probabilities of insolvency. Premiums with 2% or 3% loading factors considerably decrease the probability of insolvency over the whole term of the contracts for the limiting portfolio but have essentially no impact on those probabilities for the 1000-policy portfolio. Even a 20% loading factor is not sufficient to reduce the probabilities of insolvency to a reasonably low level (e.g. 5-10%). An implication of this is that for portfolios of temporary insurances, an insurer either has to maintain a very large portfolio or use a large premium loading.

The distribution of the surplus for the 1000-policy portfolio remains negatively skewed for all values of $r$. In the case of the limiting portfolio, skewness coefficients are fairly small in magnitude and change from being negative for small values of $r$ to being positive for larger $r$.

In the case of the 1000-policy portfolio, note that the plots for small values of $r$ look more like plots of step functions for the df of a discrete random variable. This should not be a surprise. Remember that the surplus depends on the two random processes - a continuous one for the rates of return and a discrete one for the decrements. In the earlier years of the temporary contract, only a few deaths are likely to occur but each of them would have a relatively large impact on the surplus. This is reflected in the 'jumps' of the df of $S_{r}^{acct}/m$. The slightly upward sloped segments of the plots between any two 'jumps' indicate very small probabilities that the surplus realizes values in those regions. But in the later years the shape of the df gradually smooths out due to the fact that there are more possibilities for allocating death events over the past years.

Table 4: The first three moments of the accounting surplus per policy for a portfolio of 1000 5-year temporary policies.

<table>
<thead>
<tr>
<th>$\theta = 0%$, $\pi=1.27$</th>
<th>$\theta = 3%$, $\pi=1.31$</th>
<th>$\theta = 20%$, $\pi=1.52$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>$E\left[\frac{S_r}{m}\right]$</td>
<td>$sd\left[\frac{S_r}{m}\right]$</td>
</tr>
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<td>0.0005</td>
<td>1.1405</td>
</tr>
<tr>
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<td>0.0015</td>
<td>1.6834</td>
</tr>
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<td>0.0030</td>
<td>2.1560</td>
</tr>
<tr>
<td>4</td>
<td>0.0048</td>
<td>2.6059</td>
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Table 5: The first three moments of the accounting surplus per policy for the limiting portfolio of 5-year temporary policies.

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<th>$E\left[\frac{S_r}{m}\right]$</th>
<th>sd$\left[\frac{S_r}{m}\right]$</th>
<th>sk$\left[\frac{S_r}{m}\right]$</th>
<th>$E\left[\frac{S_r}{m}\right]$</th>
<th>sd$\left[\frac{S_r}{m}\right]$</th>
<th>sk$\left[\frac{S_r}{m}\right]$</th>
<th>$E\left[\frac{S_r}{m}\right]$</th>
<th>sd$\left[\frac{S_r}{m}\right]$</th>
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Table 6: Estimates of the probability that accounting surplus per policy becomes negative for two portfolios of 5-year temporary policies.

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</table>
10 Conclusions

The paper examined the insurance surplus at different future valuation times viewed from the point of the contracts’ initiation. An advantage of this framework is that it allows to assess adequacy of, for example, initial surplus level, pricing and reserving method before the block of business is launched, and so any necessary modifications to the terms of the contract can be made.

It was suggested to distinguish between two types of insurance surplus, namely stochastic and accounting surpluses, each serving a slightly different purpose in addressing insurer’s solvency. The accounting surpluses represent the financial results as they will be seen by shareholders and regulators at future valuation dates. When studying the stochastic surplus, one considers the range of possible portfolio values measured at a given valuation date that could become reality once all contracts in the portfolio have matured. The distribution function of the accounting surplus was numerically obtained by applying the proposed recursive formula.

The analysis of the probabilities of insolvency based on the accounting surplus was used to comment on the adequacy of premium rates. In fact, the probability of insolvency can be used as a risk measure. For example, from the numerical illustrations, we saw that the premium loading required to ensure a sufficiently small probability of insolvency is much larger for a small portfolio than it is for a very large portfolio in the case portfolios of temporary policies.

The recursive formula for the distribution of the accounting surplus took advantage of conditionally constant liability. Obtaining the distribution of the stochastic surplus is a much harder problem since one needs to take into account both random assets and liabilities.

Another interesting question would be to calculate the probability of solvency over a time horizon, finite or infinite. The model can be made more realistic by including expenses and lapses.

11 Acknowledgements

Financial support from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

References


A Proof of Result 7.1

In the proof, we use the formula for computing expectation by conditioning and the conditional variance formula (e.g., see Equation 3.3 p.106 and Proposition 3.1 p.118 in Ross (2003)).

\[
\text{Var}[S^\text{acct}_r] = \text{Var}_\delta(r) \cdot E[S^\text{acct}_r | \delta(r)] + \text{E}_\delta(r) \cdot \text{Var}[S^\text{acct}_r | \delta(r)]
\]

\[
= \text{Var}_\delta(r) \left( E_{L_r} \left[ E[S^\text{acct}_r | L_r, \delta(r)] \right] \right) + \\
+ \text{E}_\delta(r) \left( \text{Var}_{L_r} \left[ E[S^\text{acct}_r | L_r, \delta(r)] \right] + \text{E}_{L_r} \left[ \text{Var}[S^\text{acct}_r | L_r, \delta(r)] \right] \right)
\]

\[
= \text{Var}_\delta(r) \left( E[RG_r | L_r, \delta(r)] - E[PL_r | L_r, \delta(r)] \right) + \\
+ \text{E}_\delta(r) \text{Var}_{L_r} \left( E(RG_r - E[PL_r | L_r, \delta(r)] | L_r, \delta(r)) \right) + \\
+ \text{E}_\delta(r) \text{E}_{L_r} \left[ \text{Var}(RG_r - E[PL_r | L_r, \delta(r)] | L_r, \delta(r)) \right]
\]

\[
= \text{Var}_\delta(r) \left( E[RG_r | \delta(r)] - E[PL_r | \delta(r)] \right) + \\
+ \text{E}_\delta(r) \left( \text{Var}_{L_r} \left( E[RG_r | L_r, \delta(r)] \right) + \text{Var}_{L_r} \left( E[PL_r | L_r, \delta(r)] \right) - \\
-2 \text{Cov}_{L_r} \left( E[RG_r | L_r, \delta(r)], E[PL_r | L_r, \delta(r)] \right) \right) + \\
+ \text{E}_\delta(r) \text{E}_{L_r} \left[ \text{Var}(RG_r | L_r, \delta(r)) \right]
\]

Next, we simplify expressions (A.1), (A.2) and (A.3).

Applying the conditional variance formula twice, Expression (A.1) becomes

\[
\text{Var}_\delta(r) \left( E[RG_r | \delta(r)] \right) + \text{E}_\delta(r) \left( \text{Var}_{L_r} \left( E[RG_r | L_r, \delta(r)] \right) + \\
- \text{E}_{L_r} \left[ \text{Var}(RG_r | L_r, \delta(r)) \right] \right)
\]

\[
= \text{Var}_\delta(r) \left( E[RG_r | \delta(r)] \right) + \text{E}_\delta(r) \left( \text{Var}[RG_r | \delta(r)] \right)
\]

\[
= \text{Var}[RG_r].
\]
To numerically evaluate Expression (A.2) it can be rewritten as
\[
\text{Var}_{\delta(r)} \left( E[PL_r | \delta(r)] \right) + E_{\delta(r)} \text{Var}_{\mathcal{L}_r} \left( E[PL_r | \mathcal{L}_r, \delta(r)] \right)
\]
\[
= \text{Var}_{\delta(r)} \left( \sum_{j=0}^{n-r} E[PC_j^r] \cdot E[e^{-I(r,r+j)} | \delta(r)] \right) + \\
+ E_{\delta(r)} \text{Var}_{\mathcal{L}_r} \left( \sum_{j=0}^{n-r} E[PC_j^r | \mathcal{L}_r] \cdot E[e^{-I(r,r+j)} | \delta(r)] \right)
\]
\[
= \sum_{i=0}^{n-r} \sum_{j=0}^{n-r} E[PC_i^r] \cdot E[PC_j^r] \cdot \text{Cov}_{\delta(r)} \left( E[e^{-I(r,r+i)} | \delta(r)], E[e^{-I(r,r+j)} | \delta(r)] \right) + \\
+ \sum_{i=0}^{n-r} \sum_{j=0}^{n-r} \text{Cov}_{\mathcal{L}_r} \left( E[PC_i^r | \mathcal{L}_r], E[PC_j^r | \mathcal{L}_r] \right) \times \\
\times E_{\delta(r)} \left( E[e^{-I(r,r+i)} | \delta(r)] \cdot E[e^{-I(r,r+j)} | \delta(r)] \right) - \\
- \sum_{i=0}^{n-r} \sum_{j=0}^{n-r} E[PC_i^r] \cdot E[PC_j^r] \cdot E[e^{-I(r,r+i)}] \cdot E[e^{-I(r,r+j)}]
\]
\[
= \sum_{i=0}^{n-r} \sum_{j=0}^{n-r} E_{\mathcal{L}_r} \left( E[PC_i^r | \mathcal{L}_r] \cdot E[PC_j^r | \mathcal{L}_r] \right) \times \\
\times E_{\delta(r)} \left( E[e^{-I(r,r+i)} | \delta(r)] \cdot E[e^{-I(r,r+j)} | \delta(r)] \right) - \left( E[PL_r] \right)^2.
\]  (A.5)

Finally, Expression (A.3) simplifies to
\[
E_{\delta(r)} \left( E[RG_r | \delta(r)] \cdot E[PL_r | \delta(r)] \right) - E[RG_r] \cdot E[PL_r] + \\
+ E_{\delta(r)} \left[ E_{\mathcal{L}_r} \left( E[RG_r | \mathcal{L}_r, \delta(r)] \cdot E[PL_r | \mathcal{L}_r, \delta(r)] \right) \right] - \\
- E_{\delta(r)} \left( E[RG_r | \delta(r)] \cdot E[PL_r | \delta(r)] \right)
\]
\[
= \sum_{j=0}^{r} \sum_{i=0}^{n-r} E_{\mathcal{L}_r} \left( E[RC_j^r | \mathcal{L}_r] \cdot E[PC_i^r | \mathcal{L}_r] \right) \cdot E_{\delta(r)} \left( E[e^{I(j,r)} | \delta(r)] \cdot E[e^{-I(r,r+i)} | \delta(r)] \right) - E[RG_r] \cdot E[PL_r]
\]
\[
= \sum_{j=0}^{r} \sum_{i=0}^{n-r} E_{\mathcal{L}_r} \left( E[RC_j^r \cdot PC_i^r | \mathcal{L}_r] \right) \cdot E_{\delta(r)} \left( E[e^{I(j,r)-I(r,r+i)} | \delta(r)] \right) - E[RG_r] \cdot E[PL_r]
\]
\[
= \sum_{j=0}^{r} \sum_{i=0}^{n-r} E[RC_j^r \cdot PC_i^r] \cdot E[e^{I(j,r)-I(r,r+i)}] - E[RG_r] \cdot E[PL_r]
\]
\[
= E[RG_r \cdot PL_r] - E[RG_r] \cdot E[PL_r] = \text{Cov}(RG_r, PL_r).
\]  (A.6)
In the above derivation we use the fact that \( \{RC_j^r \mid L_r \} \) and \( \{PC_i^r \mid L_r \} \) are uncorrelated as well as \( \{e^{I(j,r)} \mid \delta(r) \} \) and \( \{e^{-I(r,r+i)} \mid \delta(r) \} \), which can be shown as follows:

\[
\begin{align*}
E[e^{I(j,r)} \cdot e^{-I(r,r+i)} \mid \delta(r)] &= E_{I(j,r)} \left[ E[e^{I(j,r)} \cdot e^{-I(r,r+i)} \mid I(j, r), \delta(r)] \right] \\
&= E_{I(j,r)} \left[ e^{I(j,r)} \cdot E[e^{-I(r,r+i)} \mid I(j, r), \delta(r)] \right] \\
&= E_{I(j,r)} \left[ e^{I(j,r)} \cdot E[e^{-I(r,r+i)} \mid \delta(r)] \right] \quad \because \text{Markovian property} \\
&= E[e^{I(j,r)} \mid \delta(r)] \cdot E[e^{-I(r,r+i)} \mid \delta(r)],
\end{align*}
\]

\( \therefore \) \( \text{Cov}(e^{I(j,r)}, e^{-I(r,r+i)} \mid \delta(r)) = 0. \)

Similarly,

\[
\begin{align*}
E[RC_j^r \cdot PC_i^r \mid L_r] &= E_{RC_j^r} \left[ E[RC_j^r \cdot PC_i^r \mid RC_j^r, L_r] \right] \\
&= E_{RC_j^r} \left[ RC_j^r \cdot E[PC_i^r \mid RC_j^r, L_r] \right] \\
&= E_{RC_j^r} \left[ RC_j^r \cdot E[PC_i^r \mid L_r] \right] \quad \because \text{Markovian property} \\
&= E[RC_j^r \mid L_r] \cdot E[PC_i^r \mid L_r],
\end{align*}
\]

\( \therefore \) \( \text{Cov}(RC_j^r, PC_i^r \mid L_r) = 0. \)

Now, replacing (A.1)-(A.3) with (A.4)-(A.6) proves the result. \( \square \)
### B  Mortality Table

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