An Axiomatic Approach to the Valuation of Cash Flows

Fredrik Armerin
Hälsingegatan 3, 113 23 Stockholm, Sweden*

Abstract

We model a stream cash flow as an optional stochastic process, and value the cash flows by using a continuous and strictly positive linear functional. By applying a representation theorem from the general theory of processes we are able to study this valuation principle, as well as properties of the stochastic discount factor it implies. This approach to valuation is useful in the non-presence of a financial market, as is often the case when valuing cash flows arising from insurance contracts and in the application of real-options.

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*Email: fredrik@framkonsulter.se
1 Introduction

In this note we give a framework for valuing cash flows – a framework where there is no need for an underlying financial market to exist. The standard way of valuing risky cash flows is to start with a model of a financial market. Arguing that a market in equilibrium should not allow any possibilities of making something out of nothing (i.e. an arbitrage opportunity), one can derive the existence of a risk-neutral probability measure (for a formal treatment see e.g. Delbaen & Schachermayer [5]). The problem when applying this standard approach in an insurance or real-options context, is that there may not be a natural underlying financial market. One should note that an arbitrage opportunity is defined as a cleverly chosen portfolio strategy. Without a financial market, there is no possibility of forming portfolios, and hence there is possibility of defining an arbitrage opportunity. Instead of focusing on the financial market, we let the cash flows be the central object of study. The basic problem is the following: given a stream of cash flows, we would like to assign a real number that represents the value today of the cash flow stream.

An approach similar to the one presented here, but in a deterministic setting, is given in Norberg [14]. The use of general semimartingale models in an insurance context includes Devolder [7], Dietz [8], Carrière [3] and [4], Norberg & Steffensen [15], Smith [20] and Jarvis et al [10]. Recent models of cash flow valuation reminding of the approach taken here include Jobert & Rogers [12] and Aase [1]. See Ross [17] and Rubinstein [18] for two early papers using modern finance methods.

2 Preliminaries

2.1 The abstract cash flow model

Inspired by the work of Kreps [13] we let the set of cash flows be a topological vector space $\mathbb{X}$. On this space we define the set $\mathbb{P}$ of positive elements of $\mathbb{X}$. $\mathbb{P}$ is assumed to be a closed convex cone. An element belonging to $\mathbb{P}\setminus\{0\}$ is said to be strictly positive.

We value cash flows using a linear and continuous functional. This means in practise that we assume valuation to frictionless (short-selling is allowed, there are no taxes or transaction costs, the cash flows are infinitely divisible, etc.). A functional $v$ is said to be positive if $v(x) \geq 0$ for every $x \in \mathbb{P}$, and a positive functional $v$ is strictly positive if $v(x) > 0$ for every $x \in \mathbb{P}\setminus\{0\}$.

2.2 The probabilistic model

To build models in this paper, we will consider a fixed filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ which we assume to be complete and fulfilling the usual assumptions ($\mathcal{F}_0$ contains the zero sets of $\mathcal{F}$, and $(\mathcal{F}_t)$ is right-continuous). We let $\mathcal{P}$ and $\mathcal{O}$ denote the predictable and optional $\sigma$-algebra respectively. If $X$ is a positive
$\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$-measurable process, then there exists an optional process $^oX$ such that
\[ E[X_\tau 1(\tau < \infty)|\mathcal{F}_\tau] = ^oX_\tau \]
for every stopping time $\tau$ (Dellacherie & Meyer [6], p. 103). The process $^oX$ is called the optional projection of $X$. If $X$ and $Y$ are two semimartingales, then
\[ d(X_tY_t) = X_t dY_t + Y_t dX_t + d[X,Y]_t \]
(the integration by parts formula). If $Y$ is of finite variation, then
\[ d(X_tY_t) = X_t dY_t + Y_t dX_t. \]

3 Bounded cash flows

3.1 The model

In this section we take
\[ \mathbf{X} = \{X \in b\mathcal{O}|X(0-) = 0\}; \]
the bounded $\mathcal{O}$-measurable process which are 0 at 0−, and let $\mathbf{P}$ be the set of bounded non-negative processes in $\mathbf{X}$. In this model, an element $C \in \mathbf{X}$ represents the cumulative cash flows, i.e. $C(t)$ is the net amount of cash flows up to and including time $t$. Every linear and continuous functional on $b\mathcal{O}$ can be written
\[ v(C) = E \left[ \int_{[0,\infty)} C(s) dA(s) \right] \]
for some unique (up to indistinguishability) process $A$ of finite variation. If $v$ is positive (i.e. if $v(C) \geq 0$ for every $C \in \mathbf{P}$), then $A$ is increasing. We use the normalization
\[ v(1) = 1, \]
which simply means that the value of one unit of currency today ($t = 0$) is equal to 1.

3.2 Cash flows of finite variation

Theorem 3.1 If $C \in \mathbf{X}$ is of finite variation, then
\[ v(C) = E \left[ \int_{[0,\infty)} \Lambda(s) dC(s) \right], \]
where
\[ \Lambda(s) = E[A_{\infty-} |\mathcal{F}_s] - A_{s-}. \]
Proof. Integration by parts yields

\[ v(X) = E \left[ \int_{[0,\infty)} C_s dA_s \right] \]

\[ = E \left[ A_\infty - C_\infty - \int_{(0,\infty)} A_s - dC_s \right] \]

\[ = E \left[ \int_{[0,\infty)} (A_\infty - A_s) dC_s \right] \]

\[ = E \left[ \int_{[0,\infty)} (\omega(A_\infty - A_s) dC_s \right] \]

\[ = E \left[ \int_{[0,\infty)} (\text{E}[A_\infty - |F_s] - A_s) dC_s \right] \]

\[ = E \left[ \int_{[0,\infty)} \Lambda_s dC_s \right] \]

We call \( \Lambda \) the state-price deflator or the stochastic discount factor. The normalization \( v(1) = 1 \) implies that \( \Lambda(0) = 1 \).

### 3.3 The state-price deflator \( \Lambda \)

We recall that the state-price deflator is given by

\[ \Lambda_t = E [A_\infty - |F_t] - A_t =: M_t - A_t, \quad t \geq 0. \]

The process \( \Lambda \) is a strictly positive supermartingale tending to 0 a.s. as \( t \to \infty \). Since \( A \) is increasing and of integrable variation, it further follows that \( \Lambda \) is of Class (D). \( \Lambda \) is only lagad, and need not be cadlag. Hence, it need not be a semimartingale, but it is a strong optional supermartingale (Dellacherie & Meyer [6] p. 393-394). Introducing the process \( B = A \) we have \( \Lambda_t = E [B_\infty - B_t|F_t] \).

Following result is due to Mertens (see Dellacherie & Meyer [6] p. 414).

**Theorem 3.2** Let \( Z \) be a positive optional strong supermartingale of class (D) (with the convention \( Z_0^- = Z_0, Z_\infty = 0 \)). Then there exists a predictable process \( A \) with increasing (but not necessarily right or left continuous) paths such that

\[ E[A_\infty] < \infty, \quad Z_T = E[A_\infty|F_T] - A_T \text{ for every stopping time } T. \]

\( A \) is unique and the following hold

\[ \Delta_- A = A - A_\infty = Z_\infty - \omega Z; \quad \Delta_+ A = A_\infty - A = Z - \omega(Z_+) = Z - Z_+ \]

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(the last equality is true under the usual assumptions). In particular, $A$ is right continuous if and only if $Z$ is right continuous and left continuous if and only if $Z_- = Z$ (i.e. if $Z$ is regular).

It follows from this that $\Lambda$ is right continuous if and only if $A$ is right continuous, and (since $A$ is right continuous) the latter holds if and only if $A$ is continuous. Hence, a necessary and sufficient condition for $\Lambda$ to be a semimartingale is that $A$ is continuous. Note that this implies that $\Lambda$ is a semimartingale if and only if it is a special semimartingale. Let

$$T(\omega) = \inf\{t|\Lambda(t, \omega) = 0 \text{ or } A(t^-, \omega) = 0\}.$$ 

One can show that for almost all $\omega$, $\Lambda(\cdot, \omega) = 0$ on $[T(\omega), 0)$. The following proposition is obvious.

**Proposition 3.3** The positive functional $v$ is strictly positive if and only if $T = \infty$ $P$-a.s.

From now on we make the following assumption.

**Assumption 3.4** The pricing functional $v$ is strictly positive.

Now recall from Jacod [9] (Lemma 6.19) that every strictly positive supermartingale $X$ that is also a special semimartingale can be written

$$X_t = X_0 Z_t V_t,$$

where $Z$ is a strictly positive local martingale with $Z_0 = 1$ and $V$ is a strictly positive decreasing process with $V_0 = 1$. Since $\Lambda_0 = 1$, there exists a local martingale $Z$ and an increasing process $D$ with $D_0 = 1$ such that

$$\Lambda_t = e^{-D_t} Z_t.$$

If $\Lambda$ is a semimartingale, Itô’s formula implies

$$d\Lambda_t = -\Lambda_t dD_t + e^{-D_t} dZ_t = -\Lambda_t - \left( dD_t + \frac{dZ_t}{Z_t} \right).$$

Comparing this with the representation $\Lambda_t = M_t - A_t$ yields (since $\Lambda$ is a special martingale)

$$A_t = \int_0^t \Lambda_s dD_s \quad \text{and} \quad M_t = 1 + \int_0^t e^{-D_s} dZ_s.$$ 

Note that the first relation implies that $D$ is continuous when $\Lambda$ is a semimartingale. Defining

$$dR_t = dD_t - \frac{dZ_t}{Z_t} = dD_t - \mathcal{L}(Z)_t,$$

which is minus the stochastic logarithm of the state-price deflator, we can write

$$\Lambda_t = \mathcal{E}(-R)_t.$$
The process \( R \) is the (integrated) discount rate which takes account of both time and risk. In the case in which there is no risk and a constant risk free rate \( r \) we have \( R_t = rt \). Note that

\[
\Lambda_t = E \left[ \int_t^\infty \Lambda_s dD_s \middle| \mathcal{F}_t \right] \quad \Leftrightarrow \quad 1 = E \left[ \int_t^\infty \frac{\Lambda_s}{\Lambda_t} dD_s \middle| \mathcal{F}_t \right] \quad (\ast).
\]

This shows that we can interpret \( D \) as the integrated short rate:

\[
D_t = \int_0^t r(s) ds
\]

if there exists a short rate in the model. If \( D_t \to \infty \) when \( t \to \infty \), then a slight modification of the proof of Proposition 3.9 in Armerin [2] shows that \( \Lambda \) fulfills \((\ast)\) if and only if it can be written

\[
\Lambda_t = e^{-D_t} \left( 1 + \int_0^t e^{D_s} dN_s \right) = e^{-D_t} + \int_0^t e^{-(D_t - D_s)} dN_s
\]

for some uniformly integrable martingale \( N \). Using the uniqueness of the multiplicative decomposition it is immediate that

\[
Z_t = 1 + \int_0^t e^{D_s} dN_s.
\]

### 3.4 Semimartingale cash flows

If the cash flow process is a general semimartingale in \( b \mathcal{O} \) we have the following representation of the value.

**Theorem 3.5** If \( \Lambda \) is a semimartingale, \( C \in b \mathcal{O} \) is a semimartingale and the local martingale \( \int_0^t C_s dM_s \) is a uniformly integrable martingale, then

\[
v(C) = E \left[ \int_{[0, \infty)} \Lambda(s-)dC(s) + [\Lambda, C]|_{\infty} \right],
\]

where

\[
\Lambda(s) = E [A|_{\infty} - |\mathcal{F}_s| - A_s].
\]

**Proof.** The integration by parts relation yields

\[
\int_{[0, \infty)} C_s dA_s = A_{\infty} - C_{\infty} - \int_{[0, \infty)} A_s dC_s
\]

\[
= M_{\infty} - C_{\infty} - \int_{[0, \infty)} A_s dC_s
\]

\[
= \int_{[0, \infty)} C_s dM_s + \int_{[0, \infty)} M_s - dC_s + [M, C]|_{\infty} - \int_{[0, \infty)} A_s dC_s
\]

\[
= \int_{[0, \infty)} \Lambda_s - dC_s + [\Lambda, C]|_{\infty} + \int_{[0, \infty)} C_s dM_s
\]

Taking expectations ends the proof.
Note that if $\Lambda$ is a semimartingale and $A_\infty \in L^2$, then $M$ is an $L^2$-martingale, and since $C$ is (uniformly) bounded, the local martingale $\int_0^t C_s \, dM_s$ is in fact a uniformly martingale. See Jin & Glasserman [11] for a discussion of this in an interest rate modelling framework. Also note that

$$E[A_\infty^2] = 2E\left[\int_{[0,\infty)} \Lambda_s \, dA_s\right] = 2v(A)$$

holds (this is the energy formula for continuous $A$).

3.5 Dynamics of the value

Let $(S_t) \in bO$ be the price process of a non-dividend paying asset with cadlag paths. Getting one unit of the asset at time $t \in \mathbb{R}_+$ generates the cash flow $S_t$. The value at time 0 of getting this cash flow of $S_t$ at time $t$ is worth $S_0$; hence

$$S_0 = E[\Lambda_t S_t].$$

Now replace the deterministic time $t$ with a bounded stopping time $\tau$. The pricing relation will in this case be

$$S_0 = E[\Lambda_\tau S_\tau].$$

We recall the following:

**Proposition 3.6** Let $X$ be an adapted and cadlag stochastic process with $E|X_t| < \infty$ for $t \in \mathbb{R}_+$.

(a) $X$ is a martingale if and only if

$$E[X_\tau] = E[X_0]$$

for every bounded stopping time $\tau$.

(b) $X$ is a uniformly integrable martingale if and only if

$$E[X_\tau] = E[X_0]$$

for every stopping time $\tau$.

Since $\tau$ was a general bounded stopping time and $\Lambda_0 = 1$, this shows that

$$(\Lambda_t S_t)$$

is a martingale.

Adding dividends given by a dividend process $C$ of finite variation ($C(t)$ are the accumulated dividends during the time interval $[0, t]$) immediately gives

$$\Lambda_t S_t + \int_{[0, t]} \Lambda_s dC_s$$

is a martingale.
This means that for any $0 \leq t \leq T$ we have

$$\Lambda_t S_t + \int_{[0,t]} \Lambda_s dC_s = E \left[ \Lambda_T S_T + \int_{[0,T]} \Lambda_s dC_s \middle| \mathcal{F}_t \right]$$

$$S_t = E \left[ \int_{(t,T]} \frac{\Lambda_s}{\Lambda_t} dC_s + \frac{\Lambda_T}{\Lambda_t} S_T \middle| \mathcal{F}_t \right] = \frac{1}{\Lambda_t} \left( E \left[ \int_{(t,T]} \Lambda_s dC_s \middle| \mathcal{F}_t \right] + E \left[ \Lambda_T S_T \middle| \mathcal{F}_t \right] \right).$$

Since $\Lambda_T S_T \to 0$ a.s. when $T \to \infty$, assuming suitable integrability conditions we arrive at

$$S_t = E \left[ \int_{(t,\infty)} \frac{\Lambda_s}{\Lambda_t} dC_s \middle| \mathcal{F}_t \right].$$

### 3.6 Martingale measures

If $\Lambda$ is a semimartingale, and the local martingale $Z$ in the decomposition of $\Lambda$ is a uniformly integrable martingale, then we can represent the value of a cash flow process of finite variation as

$$v(C) = E \left[ \int_0^\infty \Lambda_s dC_s \right] = E \left[ \int_0^\infty Z_s e^{-D_s} dC_s \right] = E \left[ \int_0^\infty E \left[ Z_s \middle| \mathcal{F}_s \right] e^{-D_s} dC_s \right] = E \left[ \int_0^\infty \left( Z_s e^{-D_s} \right) dC_s \right] = E Q \left[ \int_0^\infty e^{-D_s} dC_s \right],$$

where $Q$ is the measure on $(\Omega, \mathcal{F}_\infty)$ defined by the Radon-Nikodym derivative $dQ/dP = Z_\infty$.

### 3.7 The return

Let us return to the relation

$$S_t \Lambda_t + \int_0^t \Lambda_s dC_s = M_t^S,$$
where $M^S$ is a martingale. We also assume that $\Lambda$ is a semimartingale. Differentiating this yields

$$S_t - d\Lambda_t + \Lambda_t - dS_t + [S + C, \Lambda]_t + \Lambda_t - dC_t = dM^S_t,$$

and dividing by $S_t - \Lambda_t$ results in

$$-dR_t + dR^S_t - [R^S, R]_t = \frac{1}{S_t - \Lambda_t} dM^S_t := dN^S_t,$$

where $dR^S_t = (dS_t + dC_t)/S_t$. The sharp bracket $\langle R^S, R \rangle$ exists if and only if $[R^S, R]$ is locally integrable. If this is the case, $R^S$ is a special semimartingale and the (unique) predictable part of it is given by

$$\mathbb{p} R^S_t = D_t + \langle R^S, R \rangle_t.$$

Both $R$ and $R^S$ are special semimartingales, and the drift $D$ of $R$ is continuous. Using the Kunita-Watanabe inequality for the sharp bracket we arrive at

$$\left| \mathbb{p} R^S_t - D_t \right| \leq \sqrt{\langle R^S \rangle_t}.$$

This is the Hansen-Jaganathan bound on "integrated form". If each of the (predictable) processes in this expression are absolutely continuous with respect to the Lebesgue measure, we arrive at the usual Hansen-Jaganathan bound. A non-dividend paying traded asset with price process $S$ also has value equal to $S$, and in this case the Hansen-Jaganathan upper bound is the greatest integrated return we can get. This means that it gives the slope of the efficient frontier, or it is the integrated Sharpe ratio. Independently of whether the bracket $\langle R^S, R^S \rangle$ is continuous or not, it always holds that

$$\langle R \rangle_t = \langle -\mathcal{L}(Z) \rangle_t = \langle \mathcal{L}(Z) \rangle_t = \int_0^t \frac{d\langle Z \rangle_s}{Z_s^2}.$$

See Schweizer [19] and references therein for similar results in a financial market context.

4 Conclusions and extensions

We have presented an approach in which it is possible to value cash flows without assuming an underlying financial market. The use of a representation theorem from the general theory of processes made it possible to analyse both the value as well as the state-price deflator. The space of bounded optional processes, which we chose as our space of cash flows, is quite restrictive, but it is possible to easily extend the results to the space $\mathcal{R}$ of adapted cadlag processes $X$ fulfilling

$$\sup_{t \in \mathbb{R}_+} \left\| X_t \right\|_p < \infty.$$
There is a representation theorem also for functionals defined on this space which is similar to the one used in this paper. It should also be noted that it is possible to build a theory of valuation using a model without a given probability measure, a direction which is the topic of future research.

References


