

# MODEL RISK AND DETERMINATION OF ECONOMIC CAPITAL IN THE SOLVENCY 2 PROJECT

Frédéric PLANCHET\*

Pierre-E. THÉROND<sup>α</sup>

ISFA – Laboratory SAF

University Claude Bernard Lyon 1 <sup>β</sup>

WINTER & Associés <sup>γ</sup>

## ABSTRACT

This paper investigates the robustness of the Solvency Capital Requirement (SCR) when a lognormal reference model is slightly disturbed by the heaviness of its tail distribution. It is shown that situations with “almost” lognormal data and a rather important variation between the “disturbed” SCR and the reference SCR can be built. The consequences of the estimation errors on the level of the SCR are studied too.

KEYWORDS: Solvency, extreme values.

## RÉSUMÉ

Le présent article s'intéresse à la robustesse du capital de solvabilité (SCR) lorsqu'un modèle de référence log-normal est perturbé légèrement par l'alourdissement de sa queue de distribution. On montre que l'on peut construire des situations avec des données « presque » log-normales et une variation pourtant importante entre le SCR « perturbé » et le SCR de référence. On s'intéresse également aux conséquences des erreurs d'estimation sur le niveau du SCR.

MOTS-CLEFS : Solvabilité, valeurs extrêmes.

---

\* Frédéric Planchet is an associate professor of Finance and Insurance at ISFA (Université Claude Bernard Lyon 1 – France) and an associate actuary for WINTER & Associés. Contact : [fplanchet@winter-associes.fr](mailto:fplanchet@winter-associes.fr)

<sup>α</sup> Pierre ThéronD is a Ph.D. student, lecturer at ISFA and actuary for WINTER & Associés. Contact : [ptherond@winter-associes.fr](mailto:ptherond@winter-associes.fr)

<sup>β</sup> Institut de Science Financière et d'Assurances (ISFA) - 50 avenue Tony Garnier - 69366 Lyon Cedex 07.

<sup>γ</sup> WINTER & Associés – 43-47 avenue de la Grande Armée - 75116 Paris et 18 avenue Félix Faure - 69007 Lyon.

**TABLE OF CONTENTS**

1. Introduction ..... 3

2. Description of the model ..... 4

    2.1. Presentation ..... 4

    2.2. Specific case of the lognormal distribution ..... 6

        2.2.1. Calculation of the SCR ..... 6

    2.3. Estimation of the model parameters ..... 7

    2.4. Issue on the level of the capital of the parameter estimation ..... 9

        2.4.1. Case of the lognormal model ..... 9

        2.4.2. Case of the blended model ..... 10

    2.5. Numerical application ..... 10

        2.5.1. Simulation of the blended distribution ..... 11

        2.5.2. Results ..... 11

        2.5.3. Identification of the extreme values ..... 12

        2.5.4. Adjustment of the blended model ..... 14

3. Conclusion ..... 15

## 1. INTRODUCTION

The Solvency 2 project (see COMMISSION EUROPÉENNE [2003], [2004] and AAI [2004]) that is still in development modifies deeply the fixing rules of the level of equity in insurance. This project introduces as explicit criterion the control of total risk supported by the company. This risk will have to be quantified through the ruin probability at a time horizon of one year.

This Solvency Capital Requirement or SCR will be obtained by a common standard formula for all insurers, and will be built according to a modular approach<sup>1</sup> of risks. Another option will consist to use of an internal model that will be more adapted to the risk effectively supported by insurer. In these two situations, the intention remains the same: to establish the level of resources which insurer must need at this present day in order to not be on the road to ruin in one year in one case out of 200.

The retained level of 99.5% implies the requirement to assess suitably a high-order quantile of the interest distribution (generally and in our case, the excess distribution or the distribution of the asset-liability margin). This problematical point is widely built up in the financial literature that is confronted with these questions since the Basel II accords in the banking area. For instance, we can quote ROBERT [1998] or GAUTHIER and PISTRE [2000].

In this new insurance context, the classic asset/liability modellings that accredit a limited attention at the tail distribution modelling can be proved a penalizing point, because they lead at a low-level representation of extreme values. For instance, this point is illustrated for the modellings of asset in BALLOTTA [2004] in case of hidden options in life insurance contracts, and in PLANCHET and THEROND [2005] in the framework of mono-periodic simplified model in non-life insurance for the determination of the target capital and asset allocation. THÉROND and PLANCHET [2007] draw the intention to the extent of extremes in the determination of Solvency Capital Requirement (SCR).

In this present article, we develop this point of view in disturbing a model of simple reference in making heavy its tail distribution. It is shown that is possible to obtain situations in which the basic model underestimates significantly the Solvency Capital Requirement, while being

---

<sup>1</sup> The quantitative impact study 3 carried out by CEIOPS (<http://www.ceiops.org>) gives a proficient idea of which will be the standard formula when Solvency 2 guidelines are adapted.

not easily discernible statistically with the disturbed model if a detailed attention is not paid to the extreme values.

We suggest an empirical approach in order to decide if modellings of extreme values type must be carry out on the basis of an observed sample.

## 2. DESCRIPTION OF THE MODEL

### 2.1. PRESENTATION

We consider a probability distribution described by its survival function  $S_0$ ; more precisely we suppose the positive random  $X$  is defined by the following survival function:

$$S_X(x) = \begin{cases} S_0(x) & x \leq m \\ \left(\frac{x}{m}\right)^{-\alpha} S_0(m) & x > m \end{cases}.$$

In other words,  $X$  is distributed according to the distribution  $S_0$  until the threshold  $m$ , and then according to a Pareto distribution with the parameters  $(m, \alpha)$ . In particular,  $P(X > m) = S_X(m) = S_0(m)$ . In this situation, we will not reconsider motivations which lead to retain the Pareto distribution, but we encourage the reader to consult EMBRECHTS and al. [1997] for theoretical aspects of the question and ZAJDENWEBER [2000] for a practical perspective.

We verify that the above equality defines a decreasing, continuous function if  $S_0$  is continuous, such as  $S_X(0) = 1$  and  $S_X(0) = 1$ . So,  $S_X$  defines a survival function.

The existence of moments of  $S_X$  depends on the existence of moments of the same order for the Pareto distribution with parameters  $(m, \alpha)$ . So the  $k$ -order moment exists only for  $k < \alpha$ .

In this present context, we will choose the threshold  $m$  so that it corresponds to a high quantile of the distribution  $S_0$ , for instance such as  $S_0(m = 1.5 \%)$ . The “blended” model in this

precise case, behaves “almost” like the basis model associated with  $S_0$  (for the portion  $1 - S_0(m)$  of observations), but differs beyond this threshold. From this definition of  $S_X$ , it may be deduced that:

$$P(X > x | X > m) = \frac{P(X > x)}{P(X > m)} = \frac{S_X(x)}{S_X(m)} = \left(\frac{x}{m}\right)^{-\alpha},$$

which means that the distribution of  $X$  conditionally to the fact that the threshold  $m$  is exceeded, is a Pareto distribution with parameters  $(m, \alpha)$ . Symmetrically, we find:

$$P(X > x | X \leq m) = \frac{P(x < X \leq m)}{P(X \leq m)} = \frac{S_X(x) - S_X(m)}{1 - S_X(m)} = \frac{S_0(x) - S_0(m)}{1 - S_0(m)}.$$

The quantile function of  $X$ , for values of  $p$  lower than  $1 - S_0(m)$  is simply given by:

$$x_p = m \times \left(\frac{1-p}{S_0(m)}\right)^{-1/\alpha}.$$

This expression is simply obtained with the equality  $1 - p = \left(\frac{x}{m}\right)^{-\alpha} S_0(m)$ , valid for  $x > m$ .

Logically we have:  $x_{1-S_0(m)} = m$ .

We wish to compare the case where the risk  $X$  is distributed simply like  $S_0$  and the case where the tail distribution is weighed as above (“blended distribution”). More precisely, we wish to compare the quantile functions in the two situations, for high-order quantiles. From a practical point of view, we desire to compare the Solvency Capital Requirement in the two situations.

In the case where  $X$  is distributed according to  $S_0$ , the quantile function is by definition  $x_p = S_0^{-1}(1-p)$ . In this case, we still have of course  $x_{1-S_0(m)} = m$ .

In the continuation of this work, we consider that the distribution of reference is lognormal, at the same time because of its simplicity of use and its very major use in the insurance.

## 2.2. SPECIFIC CASE OF THE LOGNORMAL DISTRIBUTION

### 2.2.1. Calculation of the SCR

From now on, we consider that the basis risk is lognormal, and so:

$$x_p = VaR_p(X) = S_0(1-p) = F_0^{-1}(p) = \mathbf{exp}\left(\mu + \sigma\phi^{-1}(p)\right).$$

We have:  $S_0(m) = P\left(Z > \frac{\ln(m) - \mu}{\sigma}\right) = 1 - \phi\left(\frac{\ln(m) - \mu}{\sigma}\right)$ . It may be deduced the explicit expression of the quantile function in the case of the blended model:

$$x_p = m \times \left( \frac{1-p}{1 - \phi\left(\frac{\ln(m) - \mu}{\sigma}\right)} \right)^{-1/\alpha}.$$

In the applications, we fix  $m$  while controlling  $1 - S_0(m)$  on a rather large level but lower than  $p$ ; typically in the Solvency 2 context  $p = 99.5\%$  and we will choose  $S_0(m) = 2\%$  or  $S_0(m) = 1\%$ .

We note  $p_0 = 1 - S_0(m)$ , the selected level, so that  $x_p = S_0^{-1}(1 - p_0) \times \left(\frac{1-p}{1-p_0}\right)^{-1/\alpha}$ .

In the case of lognormal reference distribution, we obtain in consequence for the blended model:

$$x_p^{MEL} = \mathbf{exp}\left(\mu + \sigma\phi^{-1}(p_0)\right) \times \left(\frac{1-p}{1-p_0}\right)^{-1/\alpha},$$

this formula has to be compared with the version obtained from the lognormal direct model:

$$x_p^{LN} = \mathbf{exp}\left(\mu + \sigma\phi^{-1}(p)\right).$$

The ratio of two quantiles gives:

$$r(\alpha) = \mathbf{exp}\left(\sigma\left(\phi^{-1}(p_0) - \phi^{-1}(p)\right)\right) \times \left(\frac{1-p}{1-p_0}\right)^{-1/\alpha}.$$

By the way, we can notice necessary that this ratio does not depend on the parameter  $\mu$ .  $r(\alpha)$  is a decreasing function of  $\alpha$ : when  $\alpha$  decreases, the risk associated with the blended distribution increases and as a consequence the capital requirement to cover it too.

We will be confronted with the situation of model risk in the case where despite a value  $r(\alpha) \gg 1$ , a sample derived from the blended model would be difficult to differentiate with a lognormal sample. The lognormal model is very widespread in insurance and in particular, it is on this model that were gauged a part of parameters of the standard formula described in QIS 3. We are going to pay particular attention to examine this situation in the continuation of this paper.

### 2.3. ESTIMATION OF THE MODEL PARAMATERS

The estimation of parameters can be performed by the maximum likelihood method. Indeed, the log-likelihood can be written, while noting  $(x_{(1)}, \dots, x_{(n)})$  the order statistic associated with the sample  $(x_1, \dots, x_n)$  and  $k$  the smallest index such as  $x_{(k)} \geq m$ :

$$l(x_1, \dots, x_n; \mu, \sigma, \alpha) = \sum_{i=1}^{k-1} \ln \left( \frac{1}{\sigma x_{(i)} \sqrt{2\pi}} \mathbf{exp} \left[ -\frac{1}{2} \left( \frac{\ln(x_{(i)} - \mu)}{\sigma} \right)^2 \right] \right) + \sum_{i=k}^n \ln \left( \alpha m^\alpha S_0(m) x_{(i)}^{-\alpha-1} \right)$$

which leads after simplification to:

$$\begin{aligned}
 l(x_1, \dots, x_n; \mu, \sigma, m, \alpha) &= cste - (k-1) \ln(\sigma) - \frac{1}{2} \sum_{i=1}^{k-1} \left( \frac{\ln(x_{(i)}) - \mu}{\sigma} \right)^2 \\
 &+ (n-k+1) \ln(\alpha) + \alpha(n-k+1) \ln(m) - \alpha \sum_{i=k}^n \ln(x_{(i)}) + (n-k+1) S_0(m)
 \end{aligned}$$

Because of the presence of  $k = \mathbf{min} \{i; x_{(i)} \geq m\}$ , the expression of log-likelihood is not easily usable in this form. Nevertheless, we can break up the problem of maximization by noticing that:

$$\mathbf{max}_{(\mu, \sigma, m, \alpha)} l(x_1, \dots, x_n; \mu, \sigma, m, \alpha) = \mathbf{max}_m \mathbf{max}_{(\mu, \sigma, \alpha)} l(x_1, \dots, x_n; \mu, \sigma, m, \alpha).$$

We notice in which time  $m$  is fixed, the expressions of partial derivatives of the log-likelihood are the classic expressions of two subjacent distributions, on the ranges of data with regard to them. The estimators of  $\mu$  and  $\sigma$  are like this classic empirical estimators for the gaussian sample  $(\ln x_{(i)}; i = 1, \dots, k-1)$ :

$$\hat{\mu} = \frac{1}{k-1} \sum_{i=1}^{k-1} \ln x_{(i)} \quad \text{et} \quad \hat{\sigma} = \sqrt{\frac{1}{k-1} \sum_{i=1}^{k-1} (\ln x_{(i)} - \hat{\mu})^2}.$$

The estimator of tail parameter  $\alpha$  is given by the following expression:

$$\hat{\alpha} = \frac{n-k+1}{\sum_{i=k}^n \ln \left( \frac{x_{(i)}}{m} \right)}.$$

It remains to eliminate  $m$ , unknown, in the above equation. In practice we can proceed in the following way:

- we fix  $k$  ( while starting for example by  $k = 95\% \times n$  );
- we calculate  $\hat{\mu}$  and  $\hat{\sigma}$  ;
- we calculate  $\hat{m} = \mathbf{exp} \left( \hat{\mu} + \hat{\sigma} \phi^{-1}(p_0) \right)$  ;

- the estimator (pseudo maximum likelihood) of tail parameter  $\alpha$  is given by the

expression: 
$$\hat{\alpha} = \frac{n - k + 1}{\sum_{i=k}^n \ln \left( \frac{x_{(i)}}{\hat{m}} \right)}$$

We obtain a value  $l(k)$  of log-likelihood; we restart with  $k' > k$  and we retain the estimation of parameters associated with the maximal value of the sequence  $l(k)$  thus obtained.

In principle, we will notice that the above estimators are skewed (even if as estimators of the maximum likelihood they are asymptotically without skew).

#### 2.4. ISSUE ON THE LEVEL OF THE CAPITAL OF THE PARAMETER ESTIMATION

BOYLE and WINDCLIFF [2004] underline the importance of the phase of parameters estimation, because of the loss of information on this level, in the relevance of the results provided by an theoretical model. As in this case, we have closed formulas for the quantile function in each model, the level of Solvency Capital Requirement will be simply estimate, in the blended model by:

$$\hat{x}_p^{MEL} = \mathbf{exp} \left( \hat{\mu} + \hat{\sigma} \phi^{-1}(p_0) \right) \times \left( \frac{1-p}{1-p_0} \right)^{-1/\hat{\alpha}},$$

and in the lognormal model, by:

$$\hat{x}_p^{LN} = \mathbf{exp} \left( \hat{\mu} + \hat{\sigma} \phi^{-1}(p) \right).$$

##### 2.4.1. Case of the lognormal model

We verify easily that the function  $f_a(x, y) = \mathbf{exp}(x + ay)$  is convex and we deduce with the Jensen's inequality (DACUNHA-CASTELLE and DUFLO [1982]) that:

$$E\left(\hat{x}_p^{LN}\right) = E \mathbf{exp}\left(\hat{\mu} + \hat{\sigma}\phi^{-1}(p)\right) \geq \mathbf{exp}\left(E(\hat{\mu}) + E(\hat{\sigma})\phi^{-1}(p)\right).$$

As in the lognormal model the parameter  $\mu$  is estimated without skew, and that is possible to substitute  $\hat{\sigma}$  by its corrected version of skew  $\tilde{\sigma} = \sqrt{\frac{n}{n-1}}\hat{\sigma}$ , we conclude that:

$$E\left(\hat{x}_p^{LN}\right) \geq x_p^{LN} = \mathbf{exp}\left(\mu + \sigma\phi^{-1}(p)\right).$$

In other words, the estimation procedure of the Solvency Capital Requirement in lognormal model leads to overestimate it on average.

#### 2.4.2. Case of the blended model

In the case of the blended model, we must examine the behavior of  $f_{a,b}(x, y, z) = \mathbf{exp}\left(x + ay + \frac{b}{z}\right)$  with  $b = \ln\left(\frac{1-p_0}{1-p}\right) > 0$ . A simple matrix calculation makes it possible to verify the positivity of associated Hessian matrix and equally the convex nature of  $f_{a,b}$ . Unfortunately, it is not easy to deduce the meaning of the skew on the SCR estimation, because of the parameters is not anymore without skew.

The numerical simulations tend to highlight a negative skew, i.e. a underestimation of the SCR, which constitutes a penalizing point in practice (see below).

#### 2.5. NUMERICAL APPLICATION

From a practical point of view, the estimation of the SCR is not executed on observed data but on simulated values resulting from a model (the “internal model”); for instance we can consult THEROND and PLANCHET [2007]. The constraints of calculation make that it is not possible to dispose of an arbitrarily large number of achievements of the simulated asset-liability margin and that the estimation of the SCR will have to be effected on a modest size sample.

So, the modelling of the asset-liability margin is crucial about the determination of the level of capital.

### 2.5.1. Simulation of the blended distribution

The simulation of a sample resulting from the blended distribution can be obtained simply in the following way:

- drawing of a value  $u$  uniformly distributed on  $[0,1]$ ;
- if  $u > p_0$ , drawing of  $x$  in the Pareto distribution with parameters  $(m, \alpha)$ ;
- if  $u < p_0$ , drawing of  $x$  in distribution  $S(x) = \frac{S_0(x) - S_0(m)}{1 - S_0(m)}$ .

This last drawing can be carried out with a rejection method: we make a drawing in the lognormal distribution, and we refuse it if the obtained value is higher than  $m$ . Indeed, like:

$$P(X > x | X \leq m) = \frac{S_0(x) - S_0(m)}{1 - S_0(m)},$$

This leads exactly to the sought distribution.

### 2.5.2. Results

For the numerical application, we retain:

Threshold distribution ( $p_0$ )	98.50%
SCR threshold ( $p$ )	99.50%
$m$ (threshold distribution)	353.554

lognormal		Pareto	
$\mu =$	5	$\gamma =$	353.554
$\sigma =$	0.4	$\alpha =$	3.9

With these assumptions, the theoretical value of SCR in the blended model and the lognormal model reference is equal to 113%. In others words, to use the lognormal model leads to

underestimating the capital requirement of more than 10 % if the model, from which the data result, is the blended model.

So we generate 2 samples of 1000 achievements of each 2 models and we study the adequacy of the sample resulting from the “blended” distribution with a lognormal distribution. The following adjustment is obtained:

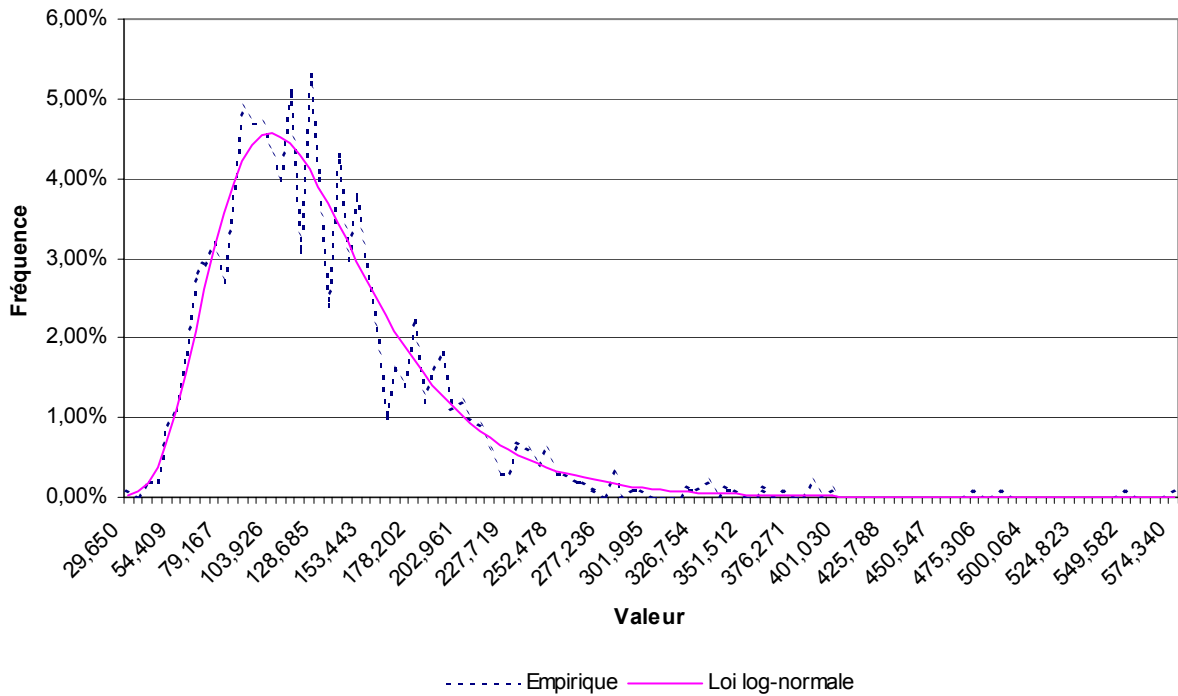


Fig. 1 : Adjustment of the lognormal distribution on a blended sample

The adjustment is widely accepted by a chi-square test. A too prompt analysis would lead to accept an inadequate adjustment with the reality of the data. It is necessary to examine the behavior of the tail distribution.

### 2.5.3. Identification of the extreme values

We notice that if we fix a probability  $p > p_0$ , then the probability that the p-order quantile of the lognormal distribution is exceeded in the blended distribution is:

$$\pi(p) = 1 - \left( \frac{\exp(\mu + \sigma\phi^{-1}(p))}{m} \right)^{-\alpha} S_0(m)$$

In our example, if  $p = 99.8\%$  then  $\pi(p) = 0.50\%$ ; as a consequence, on a sample of 1000 values, we will get on average two values which exceed  $S_0^{-1}(1 - 99.8\%)$ , whereas there will be 5 values which will exceed this threshold if the subjacent distribution is the blended distribution. As the number of values  $N_u$  exceeding a threshold  $u$  is approximately normal we obtain:

$$P(N_u \geq k) \approx 1 - \phi\left(\frac{k - nS(u)}{\sqrt{nS(u)(1 - S(u))}}\right).$$

This provides a test to reject the assumption that the subjacent distribution is lognormal by counting the number of excesses of the threshold  $S_0^{-1}(1 - 99.8\%)$  in the sample. For instance, in this application, at the confidence threshold of 10% this rule leads to reject the null assumption as soon as  $k \geq 4$ . On the sample presented on the above graph we notice thus that 4 points are in this situation:

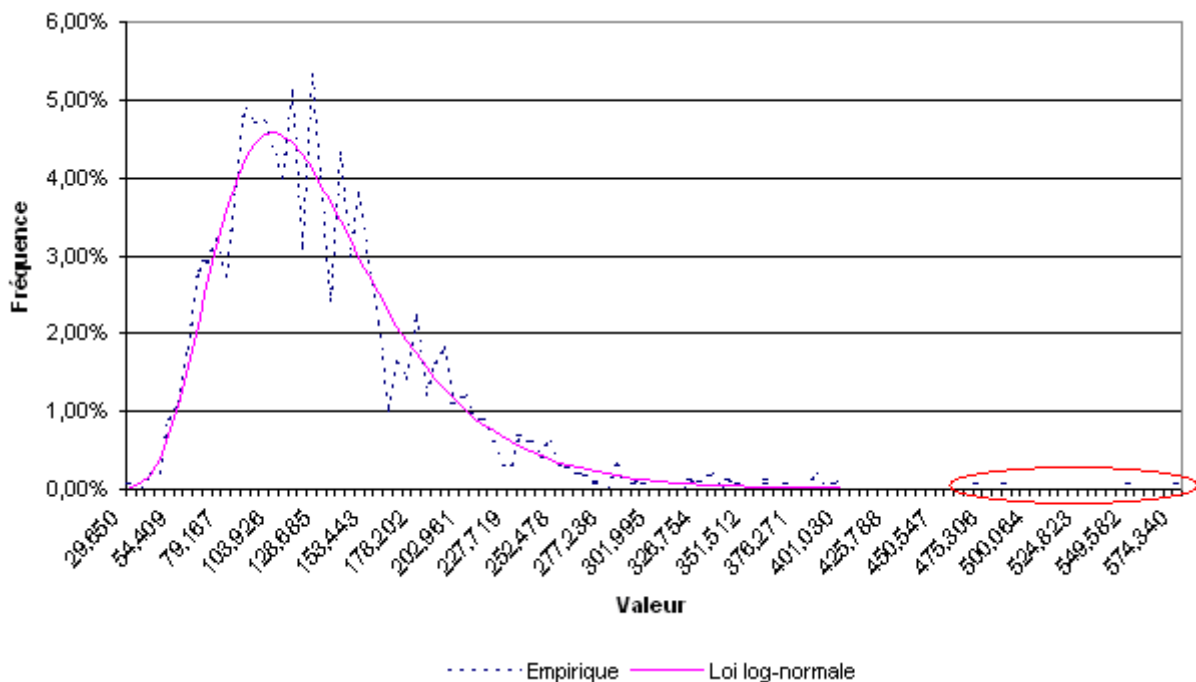


Fig. 2 : Identification of extreme value

So we would be led to reject the lognormal adjustment and to use a model taking into consideration the presence of these extreme values.

### 2.5.4. Adjustment of the blended model

The adjustment by maximum likelihood of blended model does not present a practical difficulty. Indeed, the iterative calculation of log-likelihood performed by various values of  $k$  reveals a brutal change of slope when  $\frac{k}{n} \approx p_0 = 1 - S_0(m)$ , as the graph shows it below :

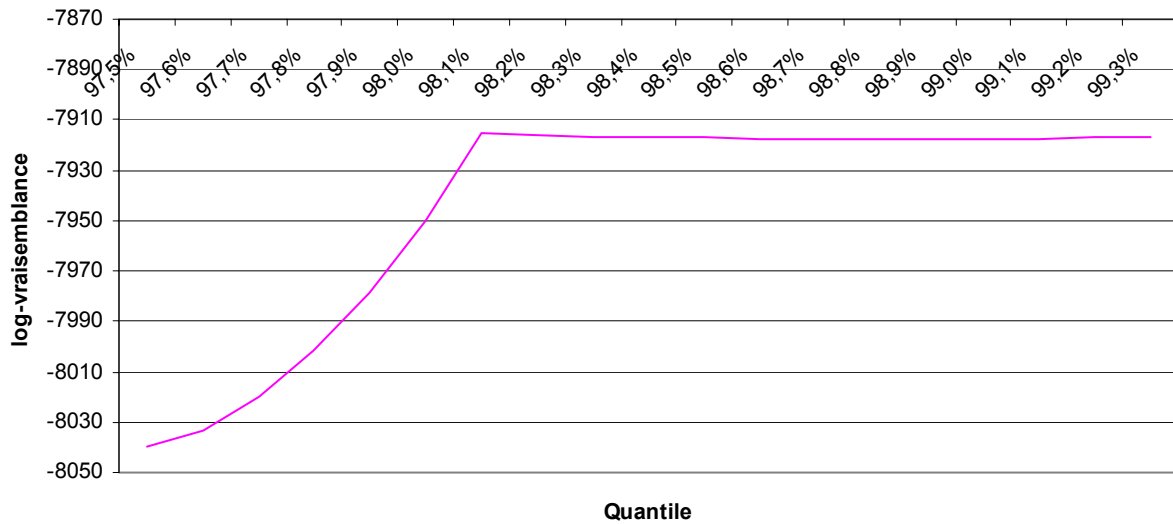


Fig. 3 : Calculation of maximum likelihood : identification of  $m$

The values obtained on a "typical" sample arise in the following way:

	Theoretical	Estimation	
$\mu$	4.958	5	
$\sigma$	0.386	0.4	
$\gamma$	317.097799	353.553971	
$\alpha$	3.475	3.9	
<b>Estimated ratio =</b>	117%	113%	
<b>Solvency capital requirement</b>			
SCR LN	416.00	415.85	0.0%
SCR mélangé	451.29	468.59	-3.7%

The estimation of SCR in lognormal sample is relatively robust in the case of a sample of size 1000. However, we observe an underestimation of the capital in the case of the blended model. In the end, we can retain if the data result from the blended model, the fact of considering that they are really issued from a lognormal sample leads to an important

underestimation of the capital requirement. Moreover within the framework of the well-specified model, the estimation still leads to a light underestimation.

This example underlines the importance of an appropriate tail distribution modelling.

### 3. CONCLUSION

The results presented here, within a very simplified framework, underline once again the lack of robustness that is inherent in the criterion of fixing of the Solvency Capital Requirement in the project Solvency 2 project.

So it seems essential to us that the implementation methods of the ruin probability criterion are clarified in the long term and notably that the constraints on the modelling of the tail distribution are specified within the framework of an internal model. These constraints must be expressed on three levels: for the asset modelling, for the liability modelling, and finally within the framework of the exploitation of the empirical distribution of a asset-liability margin simulated from "way out" of the model.

### BIBLIOGRAPHY

AAI [2004] *A global framework for insurer solvency assessment*, <http://www.actuaires.org>.

BALLOTTA L. [2004] « Alternative framework for the fair valuation of participating life insurance contracts ». *Proceedings of the 14<sup>th</sup> AFIR Colloquium 1*, 337-67.

BOYLE P., WINDCLIFF H. [2004] « The 1/n pension investment puzzle », *North American Actuarial Journal* 8.

COMMISSION EUROPÉENNE [2003] « Conception d'un futur système de contrôle prudentiel applicable dans l'Union européenne - Recommandation des services de la Commission ». Document de travail, MARKT/2509/03.

COMMISSION EUROPÉENNE [2004] « Solvency II - Organisation of work, discussion on pillar I work areas and suggestions of further work on pillar II for CEIOPS », Document de travail, MARKT/2543/03.

DACUNHA-CASTELLE D., DUFLO M. [1982] *Probabilités et statistiques : problèmes à temps fixe*, Paris : Masson.

EMBRECHTS P., KLUPPELBERG C., MIKOSCH T. [1997] *Modelling extremal events*, Berlin : Springer Verlag.

GAUTHIER C., PISTRE N. [2000] « Événements extrêmes sur les spreads de crédit », Working Paper ENSAE.

PLANCHET F., THÉRON P.E. [2004] « Allocation d'actifs d'un régime de rentes en cours de service ». *Proceedings of the 14<sup>th</sup> AFIR Colloquium 1*, 111-34.

PLANCHET F., THÉRON P.E. [2005] « L'impact de la prise en compte des sauts boursiers dans les problématiques d'assurance », *Proceedings of the 15<sup>th</sup> AFIR Colloquium*.

ROBERT C. [1998] « Mouvements extrêmes des séries financières haute fréquence », *Finance* 19, 221-71.

THÉRON P.E., PLANCHET F. [2007] « Provisions techniques et capital de solvabilité d'une compagnie d'assurance : méthodologie d'utilisation de Value-at-Risk », *Assurances et gestion des risques* 74 (4), 533-63.

ZAJDENWEBER D. [2000] *Économie des extrêmes*, Paris : Flammarion