VARIABLE STRIKE OPTIONS IN LIFE INSURANCE GUARANTEES

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Abstract

Variable strike options offer a flexible hedging tool in finance and they give new models for guarantees in life insurance policies. An example is given by the α-quantile options: indeed, the strike is strictly related with VaR and we are led to guarantees that are straight measures of the risk level.

Usually, either only one risk is considered or its final impact on the management is summarized. But the risk appears under several categories and we need consider multivariate quantiles. Since these tools are very cumbersome and we are able to measure the relative weight of the different kinds of risks, a directional quantile is introduced. Therefore, we can apply the classical VaR theory also to multiple risks and offer wider guarantees to life insurance policies.

Moreover, according to the need of a detailed analysis of the several kinds of risks, that is promoted by Solvency II, we introduce an easy-to-practice tool.

Key Words

Options, Value-at-Risk, Insurance guarantees

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1. Variable strike options

The increasing development of derivative financial products is linked with speculation and hedging strategies, overcoming the traditional risk profile. The innovation process leads to new kinds of exotic options and it modifies the typical elements such as the underlying, the strike price, the maturity.

In this note, we are especially interested with the development of the strike price role towards guarantees in life insurance policies.

Among the wide variety of exotic options we recognize two main families, the path dependent options and the correlation options, while leaving the remaining ones in a third general class. The payoff at maturity of a standard option depends only on the difference between the price of the underlying and the strike price, with no reference to the time evolution. But the process leading to the final value has to be taken explicitly into account: then we face the family of the path dependent options.

A flexible family of path dependent options is the one of lookback, i.e. no regret options: the payoff depends not only on the final price S_T, but also on the minimum (maximum) price that has been registered in an assigned time period. The strike price may be fixed, but it is more realistic to adjust the buying (selling) price of the underlying to the minimum (maximum) price observed in the given period. The variable strike price of a lookback option depends on the frequency of registration for the underlying price.

Another family of flexible path dependent options is offered by the cliquet options, that are also called resetting strike options. Indeed, the strike price is updated on a set of intermediate dates, in order to better represent the market behaviour. Consider a call option with a one year maturity and a strike price assigned at the beginning of each period, according to the market value: the holder takes advantage of the corresponding quarter performance of the underlying asset. This remark shows that a cliquet option is a strip of one beginning option and then forward options: the strike is given as a spot current price and then as the forward market prices.

We mention the cliquet options that adjust the strike price at given dates and the ladder options, assigning some reference levels to be compared with the previous strike price.

At an extreme side, we find the shout options, for which the holder can declare the strike at any level and at any date he judges convenient. It can be proved that the final value can be split into two components, the first one linked with the difference of the underlying values $S_T - S_t$ and the other one with the difference between the current value S_t and the original strike price $S_t - K$, that is

$$\max\{0, S_T - S_t\} + (S_t - K).$$

In the previous cases the strike price is absolutely random. It takes a functional form in the Asian options. These options are path dependent, because the strike price takes the form of a suitable mean over the preassigned set of observations. For the geometric mean a closed form expression for the value is available, while we use the approximated relation between the geometric and the arithmetic means. We can distinguish:

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- the average price options, when the strike price is given, while the final price of the underlying asset is a suitable mean of the observed prices;
- the average strike options, that apply a strike price calculated as the chosen mean.

Denote M any mean. Then the payoff at maturity of an average price option is given by $\max\{0, \varphi M - \varphi K\}$, φ being an indicator function, $\varphi = 1$ for the call options, $\varphi = -1$ for the put options.

Alternatively, the final payoff for the average strike options is, $\max\{0, \varphi S_T - \varphi M\}$. Let us assume that an arithmetic mean is suitably approximated by the half sum of the initial and final prices of the underlying asset, $A \cong \frac{1}{2}(S_0 + S_T)$. Then the expected value of an Asian option at maturity is equal to the half expected value of a standard option issued at the money,

$$E\left[\max\{0, S_T - A\}\right] = \frac{1}{2} E\left[\max\{0, S_T - S_0\}\right].$$

Starting from the classical put-call parity, some symmetry results have been proved for options with the same strike price. Bates (1991) proves a relation for European put and call options with different strike prices, when the underlying follows an exponential Brownian motion.

Delbaen-Yor (1999) give the equivalence between a passport option and a fixed strike lookback option. In this section we give explicit mention of a symmetry result for exotic options with fixed and variable strike. Indeed, let us exchange the roles of the interest rate r and of the dividend δ. Then an Asian call option with variable strike exhibits the same price as a put option with fixed strike, provided that $K = \lambda S_0$,

$$c_f(S_0, \lambda, r, \delta, 0, T) = e^{-rt} E\left[\max\{0, \lambda S_T - A_T\}\right] =$$
$$= p_x(K, S_0, \delta, r, 0, T) = e^{-\delta t} E\left[\max\{0, K - A_T\}\right]$$

An analogous symmetry relation holds for a call option with fixed strike and a put option with variable strike.

Correlation options are devoted to enlarge the number of the underlying assets, that characterize the option price. We can distinguish a first order correlation, if it directly influences options' payoffs, and a second order correlation, if it merely modifies the options' payoffs.

Spread options are built on the difference among indexes, prices and rates: then the correlation is of first order. An out-performance option can use both the correlation orders. Think for instance of an outperformance option on two different indexes A and B, denominated in currency C: then we have to consider not only the direct covariance between A and B, as a first order effect, but also the covariances between A and C, B and C, thus using a second order effect.

Among correlation options we recognize basket options for portfolios of underlying assets, for instance several currencies.

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Exchange options, that have been proposed by Margrabe (1978) allow us to exchange the underlying asset with another one at maturity. Consider for instance a U.S. investor who buys yen in exchange for Australian dollars, that is, he exchanges one foreign currency for another one. These options exhibit a random strike: indeed, an exchange option can be considered as:

-   both a call option on a first underlying asset, with strike price represented by the future price of the second asset at maturity;

-   and as a put option on the second underlying asset, with strike price given by the future price of the first asset at maturity.

Exchange options can be used to model several other types of exotic options: for instance, the rainbow options are written either on the maximum or on the minimum of two underlying assets.

Interest rate spread options are contingent claims on the difference of two interest rates, typically a short rate and a long rate, that allow us to control the risk due to changes in the shape of the yield curve. An example of pricing is given in Fu (1996).

2. Random guarantees and the insurer risk

The Value-at-Risk measurement is one of the main issues in life insurance both on the side of the insurer and on the side of the insured. Take for instance the Equity Linked policies. The benefit is related with either the market value of mutual funds or suitable indices and it is subject to a high level risk.

In the Italian market of Unit Linked policies the whole risk is charged to the insured.

In the Equity Linked policies a suitable guarantee is added to weaken the insured's risk (CIA, 2001).

The most common guarantee models are:

-   Guaranteed Minimum Death Benefit, GMDB

The benefit at the insured's death is the maximum between the value of the fund quotas and the already paid premia: sometimes a roll up mechanism is added and the premia are revalued according to the length of period between the stipulation of the contract and the insured's death.

-   Guaranteed Minimum Maturity Benefit, GMMB

A suitable quota from 75% up to 100% of the already paid premia is guaranteed: sometimes a revaluation mechanism is added according to either a suitable law (roll up) or to the market behaviour (ratchet).

-   Guaranteed Minimum Accumulation Benefit, GMAB

This guarantee refers to the level of the already paid premia as in the GMMB, but it adds some dates when the benefit is consolidated.

-   Guaranteed Minimum Surrender Benefit, GMSB

In this case the GMMB is recognized also to the surrender.

The guarantee's price is found under the option model.

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While in the actuarial approach the guarantee is set at a fixed value and it can be covered by a suitable portfolio of risk-free bonds, in the financial approach the insurer has to build a suitable portfolio replicating the European put options' payoff.

Let us introduce the following notations:

- F_t is the market value at time t of the investment fund related with the contract. Management fees are deduced by the fund at the beginning of each month. The notations F_{t-} and F_{t+} indicate the fund value at the end of month t before and after the subtraction of the management fees and represent the cost of the guarantees;

- S_t denotes the value at time t of the investment underlying the investment fund, with $S_0 = 1$;

- G_t gives the guarantee level per unit of invested capital;

- m is the managenent fees amount, that is deducted every month;

- m_C gives the margin offset, that is the quota of the management fees devoted to the guarantee's funding;

- M_t represents the time t gain that is obtained from the margin offset applied to the fund value;

- C_t represents the cash flow of the insured's gains and losses from the guarantee.

If the fund dynamics is valued starting from the origin, it is described by the following expression

$$F_{t+} = F_{0-} \frac{S_t}{S_0} (1-m)^t \,;$$

and the revenue is given by

$$M_t = F_{t-} m_C = m_C F_{0-} \frac{S_t}{S_0} (1-m)^t \,.$$

These expressions allow us to give a mathematical formula for gains and losses of the guarantees: for the GMMB guarantee, we are analysing, the cash flow is given by

$$C_t = -{}_t p_x^\tau M_t, \quad t = 0,1,...,n-1$$

$$C_n = {}_n p_x^\tau (G - F_{n+})^+, \quad t = n \,,$$

where we assume that the negative flows C_t represent an income, the positive flows, giving the guarantee's payment, represent a cost.

The Black-Scholes formula is applied, when the underlying is the investment fund, reduced of the annual charge m.

The GMMB option price is

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$$P_0 = e^{-rT} E_Q \left[\left(G - F_T \right)^+ \right] = e^{-rT} E_Q \left[\left(G - S_T (1-m)^T \right)^+ \right].$$

Then, if $S_0(1-m)$ denotes the guarantee's price at $t = 0$ after the management fees, we obtain

$$P_0 = G e^{-rT} \Phi \left(-d_2 \right) - S_0 (1-m)^T \Phi \left(-d_1 \right)$$

with

$$d_1 = \frac{1}{\sigma \sqrt{T}} \left\{ \log \left[S_0 (1-m)^T / G \right] + (r + \sigma^2 / 2)T \right\}$$

$$d_2 = d_1 - \sigma \sqrt{T} \ .$$

Withdrawals due to death and surrender are added as $P_0 \times {}_T p_x^{\tau}$.

The risks affecting the insurer due to the Equity Linked policies are different from the classical insurance risks, that can be eliminated by applying the mutuality and diversification principles. Indeed, the financial side of risk cannot be eliminated and we look for a suitable measure.

Denote L_0 the present value of the financial gains and losses flow

$$L_0 = \sum_{t=1}^n C_t \left(1 + i \right)^{-t} .$$

The quantile risk measure

$$V_\alpha = \inf \left\{ V : \Pr \left(L_0 \le V \right) \ge \alpha \right\},$$

represents the smallest amount V to be invested in risk free assets, so that the insurer has enough money to pay the guarantee G with probability at least α.

Let us confine our attention to a GMMB guarantee.

Suppose the assets' return exhibits a lognormal distribution

$$LN \left(n \left(\mu + \ln (1-m) \right), n\sigma^2 \right),$$

then the explicit expression of V_α becomes

$$V_\alpha = \left\{ G - F_0 \exp \left\{ -z_\alpha \sqrt{n} \sigma + n \left(\mu + \ln (1-m) \right) \right\} \right\} e^{-rn},$$

with $z_\alpha = \Phi^{-1}(\alpha)$ and Φ^{-1} the standard normal distribution function.

Such a measure is not coherent and we need consider the Expected Shortfall, ES,

$$ES_\alpha \left(L_0 \right) = E \left[L_0 : L_0 > V_\alpha \right],$$

that is, the expected loss conditioned to the upper tail $(1 - \alpha)$ of the distribution.

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The explicit formula for a GMMB guarantee is

$$ES_\alpha = E\left[(G - F_n)e^{-rn} : F_n < \left(G - V_\alpha e^{rn}\right)\right].$$

Let us consider an Equity Linked policy with underlying the EUROSTOXX 50. The subscriber is 50 yeas old, the maturity is 10 years and the risk free rate is $i = 3,5\%$.

A 100% GMMB guarantee leads to a 20% probability of loss.

Quantile options and guarantees

Up to now we have followed the insurer's point of view. But the flexibility of the guarantee can be linked to the VaR measure also for the insured's side, by modelling the guarantee itself as a quantile option.

An intermediate way between the Unit Linked policies with the whole risk charged to the insured and the Equity Linked policies is offered by the introduction of guarantees using the Value-at-Risk (Baione-Menzietti, 2005).

Let us suppose that the guarantee is related with the risk aversion degree of the insured and it is modelled as a quantile of the underlying distribution at maturity.

Denote $S(t)$ the underlying, either an index or a basket of indices, whose evolution is described by a geometric Brownian motion,

$$dS(t) = \mu S(t)dt + \sigma S(t)dZ(t).$$

Therefore the reference distribution is the normal one and the α-quantile has an explicit expression

$$S_\alpha(T) = S(0)\exp\{N^{-1}(\alpha)\} = S(0)f(\alpha,\mu,\sigma,T),$$

$N^{-1}(\alpha)$ being the inverse normal distribution function: it results to be function of the level α, of the distribution parameters μ,σ and of the maturity T.

Under the classical assumptions of the Black-Scholes model, the price for the α-quantile call and put options are

$$C\left(S(0), S_\alpha(T), T\right) = S(0)N(d_1) - S_\alpha(T)e^{-rT}N(d_2)$$

$$P\left(S(0), S_\alpha(T), T\right) = S_\alpha(T)e^{-rT}N(-d_2) - S(0)N(-d_1).$$

For instance, by setting $\alpha = 5\%$ the strike price for $T = 5$ is $S_\alpha(T) = 0,665012$ and for $T = 10$ $S_\alpha(T) = 0,610961$.

This approach allows us to value the exercise price no longer as an exogeneous parameter, but as a function both of the market and the insured parameters.

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3. Modern developments in the quantile theory

The quantile function is a fundamental tool in the modern theory of risk: indeed, a family of quantile risk measures is widely used in finance and insurance, under the classical assumption of normal distribution setting.

Not yet concluded is the discussion on the definition of a suitable quantile function for multivariate analysis. Several questions concern the comparison among the different proposals and the adherence with the probabilistic nature.

Let us remind only some problems. If we proceed from the univariate to the multivariate setting, it is worth orienting to the "center" of the underlying distribution, intended in "median" sense and defining "quantiles" as boundaries demarking inner and outer regions, having specified probabilities.

Let us start with the univariate case. Let $F^{-1}(\alpha)$ denote the usual α-quantile of a univariate cumulative distribution function F. The median M is given by $F^{-1}\left(\frac{1}{2}\right)$ and for $0 < \alpha < 1$, the values $F^{-1}\left(\frac{1-\alpha}{2}\right)$, $F^{-1}\left(1-\frac{1-\alpha}{2}\right)$ represent boundary points demarking lower and upper tail regions of equal probabilities totaling $1-\alpha$.

A corresponding "median oriented" α-quantile inner region having probability α is given by the closed interval

$$\left[Q(-1,\alpha) = F^{-1}\left(\frac{1-\alpha}{2}\right), Q(1,\alpha) = F^{-1}\left(1-\frac{1-\alpha}{2}\right) \right]$$

For $\alpha = \frac{1}{2}$ it represents an "interquantile region", for $\alpha = 0$ it reduces to the mean.

Still considering the univariate case a median oriented quantile can be set as $Q(u,\alpha)$ for $u = \pm 1$ denoting "direction from M", and $Q(u,0) = M$

The main properties to be preserved by the multivariate extension are:
- probabilistic interpretation
- directional monotonicity
- suitable set theoretic interpretation.

Let M be the multidimensional median. $S^{d-1}(M)$ denotes the set of directions u.

$A = \{A_\gamma : 0 < \gamma < \infty\}$ represents a family of regions nested about M with $A_0 = \{M\}$ and monotonic, $A_\gamma \subset A_{\gamma'}$ for $0 < \gamma < \gamma'$. Moreover, set $\gamma_\alpha = \inf\{\gamma : P(A_\gamma) > \alpha\}$ for $\alpha \in (0,1)$.

Then, $Q(u,\alpha)$ is given by the boundary point of A_γ in the direction u from M and denotes a α-quantile inner region. The boundaries ∂A_γ, which we may call "contours", have the effective interpretation as median-oriented quantile functions.

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Different notions of median M and different possible shapes for regions A_γ lead to different versions of the quantile functions.

Among the several proposals given in the literature, let us remind the "generalized" quantile approach by Einhmal-Mason (1992),

$$U(\alpha) = \inf\{\lambda(C) : P(C) \geq \alpha\}, 0 < \alpha < 1.$$

A kind of bivariate analysis is the one treating randomness and time evolution simultaneously, thus leading to a quantile theory for stochastic processes.

Let $\{B_t, t \geq 0\}$ be a one-dimensional Brownian motion starting from 0. Let $\sigma \in \Re_+$, $\mu \in \Re$ and define $X_t = \sigma B_t + \mu t$ a Brownian motion with drift.

In Dassios (1995) the α-quantile of X_t has been introduced in functional form. If we consider level α of the quantile function as a parameter, while time t plays the role of variable, the α-quantile of X_t is defined as

$$M(\alpha, t) = \inf\left\{ x : \int_0^t I(X_s \leq x)\,ds > \alpha t\right\}.$$

It should be noticed that

$$\lim_{\alpha \to 0} M(\alpha, t) = \inf\{X_s : 0 \leq s \leq t\}, \quad \lim_{\alpha \to 1} M(\alpha, t) = \sup\{X_s : 0 \leq s \leq t\}.$$

Let us set

$$Y_1 = \sup\{X_s : 0 \leq s \leq \alpha t\}, \quad Y_2 = \inf\{X_s : 0 \leq s \leq \alpha t\}$$

for $0 < \alpha < 1$, Y_1 and Y_2 independent random variables. Then the following equality in law does hold, $M(\alpha, t) = Y_1 + Y_2$.

It is well known that the normal distribution does not describe the behaviour of asset returns in a very realistic way. Then, it is worth overcoming the classical Wiener process in the geometric Brownian motion: take for instance some general Lévy process. Let us denote $L(t)$ a general exponential Lévy process and consider its representation in the finite variation case

$$L(t) = \gamma(t) + \beta W(t) + \sum_{0 \leq s \leq t} \Delta L(s), t \geq 0,$$

that is sum of a drift term $\gamma(t)$, a Gaussian component $\beta W(t)$ and a pure jump part in terms of ΔL.

Let $r \in \Re$ be the riskless interest rate and $\sigma = (\sigma_{ij})$ the variance-covariance matrix, $b \in \Re^d$ a suitable vector such that each stock has the desired appreciation rate. The prices follow the dynamic evolution

$$\frac{dP_i(t)}{P_i(t)} = b_i dt + d\hat{L}_i(t) ,$$

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where $\hat{L}_i$ is such that $\exp\left(\sum_{j=1}^{d}\sigma_{ij}L_j(t)\right)=e(\hat{L}_i)$ and e denotes the stochastic exponential of the process.

$\pi(t)\in\Re^d$ is an admissible portfolio process and $\pi_i(t)$ represents the fraction of the wealth $X^\pi(t)$, which is invested in asset i; $s=(1,...,1)'$ is the summation vector. Then the wealth process $\left(X^\pi(t)\right)_{t\geq0}$ follows the dynamic behaviour

$$\frac{dX^\pi(t)}{X^\pi(t-)}=\left((1-\pi's)r+\pi'b\right)dt+\pi'd\hat{L}(t),\ \text{with}\ t>0,\ X^\pi(0)=x\,.$$

In the Emmer-Kluppelberg (2003) the capital at risk , CaR, is used as a risk measure of the portfolio strategy to be optimized,

$$CaR(x,\pi,T)=xe^{rT}\left(1-z_\alpha\exp(\pi'(b-rs)T)\right)$$

The calculation of CaR requires the knowledge of the quantile z_α of the stochastic exponential $e\left(\pi\hat{L}(T)\right)$ of process $\pi\hat{L}(T)$ at maturity T: indeed, we have

$$VaR=\inf\left\{z\in\Re:P\left(X^\pi(T)\leq z\right)\geq\alpha\right\}=xz_\alpha\exp\left\{\left(\pi'(b-rs)+r\right)T\right\}$$

 But this is a quite complicated object according to the mentioned reference.

4. Directional quantile

The operational difficulty to calculate quantiles for stochastic processes and the parameter independence on time suggests to lead back to a one dimensional definition.

Suppose that time and uncertainty have different weights in the stochastic analysis. It is worth applying a directional analysis.

Then pair (ω,t) is considered as a unified entity and the stochastic process

$$X(t,\omega)=X(u_0+\xi d)=X(\xi)$$

is read as a random variable with respect to ξ, along direction d. In correspondence, statement

$$Prob\left(X(\xi)\leq x\right)=\alpha$$

means that we look for a pair $u^*=u_0+\xi^*d=\left(\omega^*,t^*\right)$, such that the probability not exceeding x is equal to parameter α. That is, time and randomness are set simultaneously. More generally, expression

$$Q_\alpha=\inf\left\{u:\mathrm{Pr}ob\left(X(\xi)\leq x\right)>\alpha\right\}$$

defines the lowest event ω^* and the first time t^* such that the probability level is blocked.

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But the approach is even more relevant, when we face several risks. Indeed, denote $X = (X_1,...,X_n)$ the risk vector. Since we know the weight of each variable, we can develop a unidimensional research along ray $X(\xi) = X_0 + \xi D$, where $X_0 = (X_1^0,...,X_n^0)$ and D represents the direction induced by the actual risk weights. Then we obtain the directional VaR

$$VaR = \inf\{\xi : P(X_0 + \xi D \leq z) \geq \alpha\}$$

and a corresponding coherent directional measure

$$ES_\alpha = E\left[X(\xi) : X(\xi) > VaR_\alpha\right].$$

The quite detailed approach to risk promoted by Solvency II allows us to recognize the quotas of the several kinds of risks. Therefore the VaR measure should be no longer applied to an unidentified SURPLUS, but rather to the direction of risks, induced by their variety. The directional approach allows us to give an immediate and easy-to-practice approach to unify the risks towards Solvency II.

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